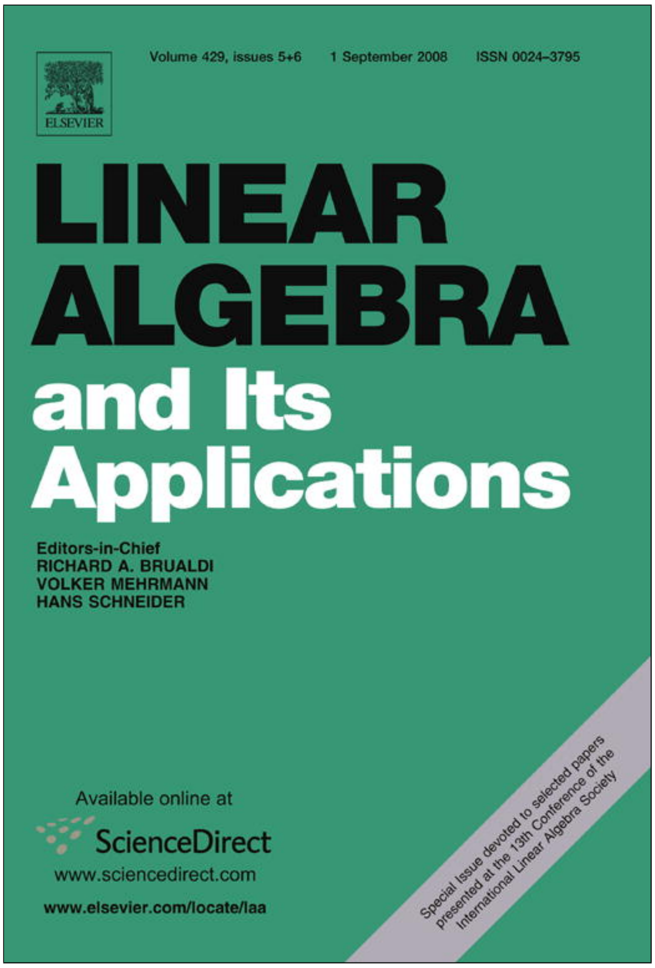
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Geometric matrix algebra

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Abstract

Matrix multiplication was first introduced by Arthur Cayley in 1855 in agreement with the composition of linear transformations. We explore an underlying geometric framework in which matrix multiplication naturally arises from the product of numbers in a geometric (Clifford) algebra. Consequently, all invariants of a linear operator become geometric invariants of the multivectors that they represent. Two different kinds of bases for matrices emerge, a spectral basis of idempotents and nilpotents, and a standard basis of scalars, vectors, bivectors, and higher order k-vectors. The Kronecker product of matrices naturally arises when considering the block structure of a matrix. Conformal geometry of $\mathbb{R}^3$ is expressed in terms of the concept of an h-twistor, which is a generalization of a Penrose twistor.

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Keywords: Clifford algebra; Conformal transformation; Geometric algebra; Horosphere; Kronecker product; Möbius transformation; Quaternions; Twistor

1. Introduction

Great advances in mathematics have been made by repeated extensions of the concept of number. By extending the real number system to include new *anticommutating* square roots of ± 1, we obtain a geometric basis for matrices. The resulting *geometric matrix algebra* seamlessly knits together two powerful currents of modern mathematics geometry, and linear algebra and matrix theory. Section 2 discusses the extension of numbers to include the concept of direction, giving the complex numbers, hyperbolic numbers, quaternions, and higher dimensional analogs.

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Section 3 shows how the (Clifford) geometric numbers of the Euclidean plane and Euclidean space provide a natural geometric basis for real and complex 2×2 matrices, respectively. Whereas algebra isomorphisms between geometric algebras and matrix algebras are well known [10], we show how the matrix representation can be simply derived in terms of inner automorphisms of a geometric algebra. Section 4 looks at higher dimensional algebras by utilizing the Kronecker product of matrices, and explores the block structure of a matrix in terms of geometric numbers. Section 5 employs 2×2 matrices over elements of the geometric algebra of Euclidean space, and the concept of an h-twistor to define the conformal geometry of $\mathbb{R}^3$ in the five-dimensional homogeneous space of the horosphere. Whereas Möbius transformations have been studied intensively in the complex number plane, h-twistors provide the algebraic machinery to efficiently define conformal transformations in higher dimensional pseudoeuclidean spaces of arbitrary signature [9,13].

2. Geometric extension of number

The real number system $\mathbb{R}$ has a long and august history spanning a host of civilizations over a period of many centuries [3]. It may be considered the rock upon which many other mathematical systems are constructed, and at the same time, serves as a model of desirable properties that any extension of the real numbers should have. A property which the real number system does not have, the *closure property* for the solution of any real polynomial equation, historically provided the most important reason for their extension to the *complex numbers* $\mathbb{C}$. The complex numbers enjoy all the algebraic properties of the reals, but in addition are algebraically closed. Any complex number $z \in \mathbb{C}$ can be expressed in the *standard basis* $\{1, i\}_{\mathbb{R}}$ as $z = x + iy$ where $i^2 = -1$ and $x, y \in \mathbb{R}$.

The complex number plane has been studied and a rich *complex analysis* has been developed over the last 150 years to the point of where they have been fully incorporated into the mathematical toolbox of every mathematician and practitioners of mathematics from the engineering and scientific communities [16]. The *Euler formula* $z = r \exp(i\theta)$ makes clear the geometric significance of the multiplication of complex numbers. Whereas the complex numbers have enjoyed universal acceptance and admiration, other extensions of the real and complex numbers have met with greater resistance and have found only limited acceptance and utility. For example, the extension of the complex numbers to Hamilton's quaternions has been more divisive in its effects upon the mathematical community [2], one reason being the necessity of giving up universal commutativity, and another the lack of a unique, clear geometric interpretation. Here, however, I would like to consider the extension of the real numbers to include a *new* square root of +1.

Perhaps the extension of the real numbers to include a new square root $u = \sqrt{1} \notin \mathbb{R}$ has never really been seriously considered because people were happy with the status quo that $\sqrt{1} = \pm 1$, and because such considerations were before the advent of Einstein's *theory of special relativity* and the study of *non-Euclidean* geometries. Extending the real number system $\mathbb{R}$ to include a new square root $u = \sqrt{1} \notin \mathbb{R}$ leads to the concept of the *hyperbolic number plane* $\mathbb{H}$, which in many ways is analogous to the complex number plane $\mathbb{C}$. Understanding the hyperbolic numbers is key to understanding even more general geometric extensions of the real numbers [5].

A *hyperbolic number* $w \in \mathbb{H}$, in the *standard basis* $\{1, u\}$, has the form $w = x + uy$ for $x, y \in \mathbb{R}$. The hyperbolic numbers $\mathbb{H}$ enjoy all the properties of the real numbers $\mathbb{R}$, except that $\mathbb{H}$ has zero divisors. In the *spectral basis* $\{u_+, u_-\}$ of $\mathbb{H}$,

$$w = w_+ u_+ + w_- u_-,$$

where $w_+ = x + y$ and $w_- = x - y$, and $u_+ = \frac{1}{2}(1 + u)$ and $u_- = \frac{1}{2}(1 - u)$. Note the properties that $u_+ + u_- = 1$, $u_+^2 = u_+$, $u_-^2 = u_-$, $u_+u_- = 0$. The real hyperbolic numbers $\mathbb{H}$ have the structure of a commutative ring, but are not algebraically closed. It is interesting to note that the hyperbolic numbers, just like the complex numbers, can be used to derive the not-so-well known formula for the zeros of a real cubic polynomial [15].

The *Euler forms* of a hyperbolic number $w = x + uy \in \mathbb{H}$ are $w = \pm\rho \exp u\phi$ or $w = \pm\rho u \exp u\phi$ for $\rho = \sqrt{|x^2 - y^2|}$ and $\phi = \tanh^{-1}(y/x)$ or $\phi = \tanh^{-1}(x/y)$, respectively, corresponding to the four branches of the unit hyperbola $x^2 - y^2 = \pm 1$. The Euler forms facilitate the geometric interpretation of the multiplication of hyperbolic numbers. The *hyperbolic distance* between $w_1, w_2 \in \mathbb{H}$ is defined by $|w_1 - w_2| = \sqrt{|(x_1 - x_2)^2 - (y_1 - y_2)^2|}$, and the equation of the hyperbola with hyperbolic radius ρ is $|ww^-| = |x^2 - y^2| = \rho^2$, where $w^- := x - yu$.

The defect that the hyperbolic numbers are not algebraically closed can be remedied by introducing the four-dimensional *complex hyperbolic numbers*. But instead, we will consider higher order extensions of the real numbers obtained by introducing an arbitrary number of *anticommuting* square roots of ± 1.

3. Geometric numbers of the $\mathbb{R}^2$ and $\mathbb{R}^3$

3.1. Geometric numbers of $\mathbb{R}^2$

To obtain the geometric algebra $\mathbb{G}_2$ of the plane $\mathbb{R}^2$, we extend the real numbers $\mathbb{R}$ to include two new *anticommuting* square roots e_1, e_2 of +1, so that

$$\mathbb{G}_2 = \text{span}_{\mathbb{R}}\{1, e_1, e_2, e_{12}\}, \tag{1}$$

where $e_1^2 = e_2^2 = 1$ and $e_{12} = e_1e_2 = -e_2e_1 = -e_{21}$. We give e_1 and e_2 the geometric interpretation of *orthonormal vectors* along the x and y axis of $\mathbb{R}^2$. The *imaginary* e_{12} has the geometric interpretation of a *unit bivector* in the xy-plane, and

$$e_{12}^2 = (e_1e_2)(e_1e_2) = -e_1(e_1e_2)e_2 = -e_1^2e_2^2 = -1.$$

The concept of a bivector, dating back to the ideas of Hermann Grassmann, characterizes the direction of a *plane segment* in the same manner that a vector characterizes the direction of a *line segment*.

The most general geometric number $g \in \mathbb{G}_2$ has the form

$$g = (\alpha + \beta e_{12}) + (xe_1 + ye_2), \tag{2}$$

where $\alpha, \beta, x, y \in \mathbb{R}$, in the basis (1) of $\mathbb{G}_2$. The geometric algebra $\mathbb{G}_2$ obeys all the algebraic rules of the real numbers $\mathbb{R}$, except that $\mathbb{G}_2$ has zero divisors and is not universally commutative.

The algebraic rules satisfied by the elements of $\mathbb{G}_2$ are completely compatible with the rules of matrix algebra and, as we now show, provide a natural *geometric basis* for matrices. For the orthonormal basis (1) of $\mathbb{G}_2$, the property that $e_1e_2 = -e_2e_1$ is equivalent to the property that the *inner product* of two vectors $a, b \in \mathbb{R}^2$

$$a \cdot b := \frac{1}{2}(ab + ba) = 0$$

if and only if $ab = -ba$.

By the *spectral basis* of $\mathbb{G}_2$ generated by the orthonormal vectors e_1, e_2, we mean

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u_+ & e_1u_- \\ e_1u_+ & u_- \end{pmatrix}, \tag{3}$$

where $u_\pm = \frac{1}{2}(1 \pm e_2)$ are mutually annihiliating idempotents. Noting that

$$\begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix}1\\ e_1\end{pmatrix} = u_+ + e_1 u_+ e_1 = u_+ + u_- = 1,$$

and using (2), we find that

$$\begin{aligned} g &= \begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix}1\\ e_1\end{pmatrix} g \begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix}1\\ e_1\end{pmatrix} \\ &= \begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix} g & g e_1\\ e_1 g & e_1 g e_1\end{pmatrix} u_+ \begin{pmatrix}1\\ e_1\end{pmatrix} \\ &= \begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix}\alpha + y & x - \beta\\ x + \beta & \alpha - y\end{pmatrix} \begin{pmatrix}1\\ e_1\end{pmatrix}. \end{aligned} \tag{4}$$

The real matrix $[g] := \begin{pmatrix}\alpha + y & x - \beta\\ x + \beta & \alpha - y\end{pmatrix}$ is called the *matrix* of g with respect to the spectral basis (3).

By the e_1-*conjugate* g^{e_1} of $g \in \mathbb{G}_2$, with respect to the unit vector e_1, we mean the operation

$$g^{e_1} := e_1 g e_1, \tag{5}$$

which is an *inner automorphism* on $\mathbb{G}_2$. Using the e_1-conjugate, we can explicitly solve for the matrix $[g]$ of g in Eq. (4). Multiplying this equation on the left and right by $\begin{pmatrix}1\\ e_1\end{pmatrix}$ and $\begin{pmatrix}1 & e_1\end{pmatrix}$, respectively, we obtain

$$\begin{pmatrix}1\\ e_1\end{pmatrix} g \begin{pmatrix}1 & e_1\end{pmatrix} = \begin{pmatrix}1 & e_1\\ e_1 & 1\end{pmatrix} u_+ [g] \begin{pmatrix}1 & e_1\\ e_1 & 1\end{pmatrix}$$

and

$$u_+ \begin{pmatrix}1\\ e_1\end{pmatrix} g \begin{pmatrix}1 & e_1\end{pmatrix} u_+ = u_+ \begin{pmatrix}1 & e_1\\ e_1 & 1\end{pmatrix} u_+ [g] \begin{pmatrix}1 & e_1\\ e_1 & 1\end{pmatrix} u_+ = u_+[g].$$

Taking the e_1-conjugate of this equation then gives

$$u_- \begin{pmatrix}1\\ e_1\end{pmatrix} g^{e_1} \begin{pmatrix}1 & e_1\end{pmatrix} u_- = u_-[g].$$

Adding the last two expressions gives the desired result that

$$[g] = u_+ \begin{pmatrix} g & g e_1\\ e_1 g & e_1 g e_1\end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g\\ g e_1 & g\end{pmatrix} u_-. \tag{6}$$

3.2. Geometric numbers of $\mathbb{R}^3$

We now introduce the geometric numbers of three-dimensional Euclidean space $\mathbb{R}^3$.

$$\mathbb{G}_3 := \text{span}_{\mathbb{R}}\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_{123}\}, \tag{7}$$

where $e_1^2 = e_2^2 = e_3^2 = 1$ have the geometric interpretation of orthonormal unit vectors along the x, y, z axes in $\mathbb{R}^3$, and $e_{ij} := e_i e_j = -e_j e_i$ for $i \neq j = 1, 2, 3$ are *unit bivectors* in the xy, xz and yz planes, respectively. The unit pseudoscalar element $i := e_{123}$ is a *unit trivector*, the *directed volume element* of $\mathbb{R}^3$, and

$$i^2 = (e_1e_2e_3)(e_1e_2e_3) = (e_1e_2)^2 e_3^2 = (-1)(1) = -1.$$

Because $e_k i = i e_k$ for $k = 1, 2, 3$, it follows that $i \in Z(\mathbb{G}_3)$, the *center* of the algebra $\mathbb{G}_3$. Algebraically, the center $Z(\mathbb{G}_3)$ of $\mathbb{G}_3$ is isomorphic to the complex numbers $\mathbb{C}$,

$$Z(\mathbb{G}_3) = \text{span}\{1, i\} \cong \mathbb{C}.$$

The geometric algebra $\mathbb{G}_3$ is algebraically closed with $i = e_{123} \in Z(\mathbb{G}_3)$.

Noting that the basis elements (7) of $\mathbb{G}_3$ are related to the basis elements (1) of $\mathbb{G}_2$ by

$$\mathbb{G}_3 = \mathbb{G}_2 + i\mathbb{G}_2,$$

it follows that the elements of $\mathbb{G}_3$ can be considered to be the *complexification* of the elements of the geometric algebra $\mathbb{G}_2$ over the center $Z(\mathbb{G}_3)$ of $\mathbb{G}_3$. Thus, each element $g \in \mathbb{G}_3$ can be written in the form $g = a + ib$ where $a, b \in \mathbb{G}_2$. Eqs. (1)–(4) and (6) derived for $\mathbb{G}_2$ remain valid for geometric numbers in $\mathbb{G}_3$; we need only replace the real scalars in (2) by scalars $\alpha, \beta, x, y \in Z(\mathbb{G}_3)$. For the basis vectors e_1, e_2 and $e_3 = -ie_{12}$, we get the matrices

$$[e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [e_3] = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

known as the *Pauli matrices* [7, p. 51].

The geometric algebra $\mathbb{G}_3$ has three involutions which are related to complex conjugation [7,11]. For the geometric number $g = \sum_{\mu=0}^{3} \alpha_\mu e_\mu \in \mathbb{G}_3$ where $\alpha_\mu \in Z(\mathbb{G}_3)$ and $e_0 := 1$, the *main involution* is obtained by changing the sign of all vectors,

$$g^* := \bar{\alpha}_0 - \bar{\alpha}_1 e_1 - \bar{\alpha}_2 e_2 - \bar{\alpha}_3 e_3, \tag{8}$$

where $\bar{\alpha}_\mu$ represents ordinary complex conjugation of $\alpha_\mu \in Z(\mathbb{G}_3)$. *Reversion* is obtained by reversing the order of the products of vectors,

$$g^\dagger := \bar{\alpha}_0 + \bar{\alpha}_1 e_1 + \bar{\alpha}_2 e_2 + \bar{\alpha}_3 e_3, \tag{9}$$

and *Clifford conjugation* is obtained by combining the above two operations,

$$\tilde{g} := (g^*)^\dagger = \alpha_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3. \tag{10}$$

In the spectral basis (3), using (6), we find that the matrix $[g]$ of g is given by

$$[g] = \begin{pmatrix} \alpha_0 + \alpha_2 & \alpha_1 + i\alpha_3 \\ \alpha_1 - i\alpha_3 & \alpha_0 - \alpha_2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \tag{11}$$

where $z_j \in Z(\mathbb{G}_3)$ for $1 \leqslant j \leqslant 4$. Using (11), (8)–(10) we also find that

$$[g]^* := [g^*] = \begin{pmatrix} \bar{\alpha}_0 - \bar{\alpha}_2 & -\bar{\alpha}_1 - i\bar{\alpha}_3 \\ -\bar{\alpha}_1 + i\bar{\alpha}_3 & \bar{\alpha}_0 + \bar{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_4 & -\bar{z}_3 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},$$

$$[g]^\dagger := [g^\dagger] = \begin{pmatrix} \bar{\alpha}_0 + \bar{\alpha}_2 & \bar{\alpha}_1 + i\bar{\alpha}_3 \\ \bar{\alpha}_1 - i\bar{\alpha}_3 & \bar{\alpha}_0 - \bar{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 & \bar{z}_3 \\ \bar{z}_2 & \bar{z}_4 \end{pmatrix},$$

$$\widetilde{[g]} := [\tilde{g}] = \begin{pmatrix} \alpha_0 - \alpha_2 & -\alpha_1 - i\alpha_3 \\ -\alpha_1 + i\alpha_3 & \alpha_0 + \alpha_2 \end{pmatrix} = \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix}.$$

Since $g\tilde{g} = \det[g]$, it follows that $g^{-1} = \frac{\tilde{g}}{\det[g]}$.

4. Higher dimensional geometric algebras

Higher dimensional geometric algebras, and their corresponding matrix algebras, can be generated by extending G_3 by additional anticommuting square roots of ± 1. For example, the geometric algebra $\mathbb{G}_{4,1} = \text{gen}_{\mathbb{R}}\{\mathbb{G}_3, \sigma, \gamma\}$ where

$$\sigma^2 = 1 = -\gamma^2 \quad \text{and} \quad \sigma\gamma = -\gamma\sigma.$$

Elements $h \in \mathbb{G}_{4,1}$ can be represented by 2×2 matrices over $\mathbb{G}_3$. To see this, first note that the *unit pseudoscalar* $I = e_1e_2e_3\sigma\gamma = iv$, is in the center $Z_{4,1} := Z(\mathbb{G}_{4,1})$, where $i = e_1e_2e_3$ and $v = \sigma\gamma$, and therefore commutes with all the elements of $\mathbb{G}_{4,1}$. Note also that $I^2 = i^2v^2 = -1$, so that the geometric algebra $\mathbb{G}_{4,1}$ is algebraically closed. It follows that

$$\mathbb{G}_{4,1} = \text{gen}_{Z(\mathbb{G}_{4,1})}\{\mathbb{G}_3, \sigma\}.$$

A general element $h \in \mathbb{G}_{4,1}$ can be expressed in the form

$$h = (g_1 + g_2v) + (g_3 + g_4v)\sigma, \tag{12}$$

where $g_j \in \mathbb{G}_3$ for $j = 1, 2, 3, 4$. A *spectral basis* for $\mathbb{G}_{4,1}$ over $\mathbb{G}_3$ is

$$\begin{pmatrix} 1 \\ \sigma \end{pmatrix} v_+ \begin{pmatrix} 1 & \sigma \end{pmatrix} = \begin{pmatrix} v_+ & \sigma v_- \\ \sigma v_+ & v_- \end{pmatrix},$$

where $v_\pm = \frac{1}{2}(1 \pm v)$ are mutually annihiliating idempotents.

Noting that

$$\begin{pmatrix} 1 & \sigma \end{pmatrix} v_+ \begin{pmatrix} 1 \\ \sigma \end{pmatrix} = v_+ + \sigma v_+ \sigma = v_+ + v_- = 1,$$

we find that

$$\begin{aligned} h &= \begin{pmatrix} 1 & \sigma \end{pmatrix} v_+ \begin{pmatrix} 1 \\ \sigma \end{pmatrix} h \begin{pmatrix} 1 & \sigma \end{pmatrix} v_+ \begin{pmatrix} 1 \\ \sigma \end{pmatrix} \\ &= \begin{pmatrix} 1 & \sigma \end{pmatrix} v_+ \begin{pmatrix} h & h\sigma \\ \sigma h & \sigma h \sigma \end{pmatrix} v_+ \begin{pmatrix} 1 \\ \sigma \end{pmatrix} \\ &= \begin{pmatrix} 1 & \sigma \end{pmatrix} v_+ \begin{pmatrix} g_1 + g_2 & g_3 + g_4 \\ g_3^\sigma - g_4^\sigma & g_1^\sigma - g_2^\sigma \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \end{pmatrix}, \end{aligned}$$

where for $h \in \mathbb{G}_{4,1}$, $h^\sigma := \sigma h \sigma$ is the inner automorphism defined by σ. When restricted to elements $g \in \mathbb{G}_3 \subset \mathbb{G}_{4,1}$ the inner automorphism g^σ of g reduces to the main involution g^* of $g \in \mathbb{G}_3$.

Solving this last equation for the $\mathbb{G}_3$-matrix $[h]$ of the element $h \in \mathbb{G}_{4,1}$, we find

$$\begin{aligned} [h] &= v_+ \begin{pmatrix} h & h\sigma \\ \sigma h & \sigma h \sigma \end{pmatrix} v_+ + v_- \begin{pmatrix} \sigma h \sigma & \sigma h \\ h\sigma & h \end{pmatrix} v_- \\ &= \begin{pmatrix} g_1 + g_2 & g_3 + g_4 \\ g_3^* - g_4^* & g_1^* - g_2^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned} \tag{13}$$

where $a, b, c, d \in \mathbb{G}_3$. We also find that

$$\begin{aligned} [h]^* &:= [h^*] = \begin{pmatrix} a^* & -b^* \\ -c^* & d^* \end{pmatrix}, \\ [h]^\dagger &:= [h^\dagger] = \begin{pmatrix} \tilde{d} & \tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}, \\ \widetilde{[h]} &:= [\tilde{h}] = \begin{pmatrix} d^\dagger & -b^\dagger \\ -c^\dagger & a^\dagger \end{pmatrix}. \end{aligned}$$

We can also represent h as a complex scalar 4×4 matrix by employing the Kronecker product of matrices:

$$\begin{pmatrix}1 & \sigma\end{pmatrix}\begin{pmatrix}1 & e_1\end{pmatrix} u_+ v_+ \begin{pmatrix}1\\ e_1\end{pmatrix}\begin{pmatrix}1\\ \sigma\end{pmatrix} := \begin{pmatrix}1 & e_1 & \sigma & \sigma e_1\end{pmatrix} u_+ v_+ \begin{pmatrix}1\\ e_1\\ \sigma\\ e_1\sigma\end{pmatrix} = 1.$$

We find that

$$\begin{aligned} h &= \begin{pmatrix}1 & \sigma\end{pmatrix} v_+ \begin{pmatrix}1 & e_1\end{pmatrix} u_+ \begin{pmatrix}[g_1+g_2] & [g_3+g_4]\\ [g_3^*-g_4^*] & [g_1^*-g_2^*]\end{pmatrix}\begin{pmatrix}1\\ e_1\end{pmatrix}\begin{pmatrix}1\\ \sigma\end{pmatrix}\\ &= \begin{pmatrix}1 & e_1 & \sigma & \sigma e_1\end{pmatrix} u_+ v_+ [h] \begin{pmatrix}1\\ e_1\\ \sigma\\ e_1\sigma\end{pmatrix}, \end{aligned}$$

where $[h] = \begin{pmatrix}[g_1+g_2] & [g_3+g_4]\\ [g_3^*-g_4^*] & [g_1^*-g_2^*]\end{pmatrix}$ is in block 2×2 matrix form.

Higher dimensional geometric and matrix algebras can be similarly treated. What we wish to emphasize here is their completely compatible and complementary structures. For example, we have the following geometric and matrix algebra isomorphisms:

$$\mathbb{G}_{n+1,n+2} \cong \mathrm{Mat}_{\mathbb{G}_{n,n+1}}(2 \times 2) \cong \mathrm{Mat}_{\mathbb{C}}(2^{n+1} \times 2^{n+1})$$

for $n = 0, 1, 2, \ldots$ The geometric algebras $\mathbb{G}_{n,n+1}$ are all algebraically closed [14].

5. Conformal geometric algebra

The geometric algebra $\mathbb{G}_{4,1}$ is algebraically isomorphic to 2×2 matrices over $\mathbb{G}_3$, *i.e.*, $\mathbb{G}_{4,1} \cong \mathrm{Mat}_{\mathbb{G}_3}(2 \times 2)$. Using the matrix representation found in the previous section, the matrix representation of the vector basis elements are easily found to be

$$[e_k] = \begin{pmatrix}e_k & 0\\ 0 & -e_k\end{pmatrix}, \quad k = 1, 2, 3$$

and

$$[\sigma] = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad [\gamma] = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Also note that

$$[v] = [\sigma\gamma] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad [v_+] = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad [v_-] = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Other elements are calculated by taking sums and products of the matrix representations of the vector basis elements. For example, the *dual* null vectors $e = \frac{1}{2}(\sigma + \gamma)$ and $\bar{e} = \sigma - \gamma$, satisfying the property that $e \cdot \bar{e} = 1$, have the matrices

$$[e] = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \qquad [\bar{e}] = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}.$$

A vector $x \in \mathbb{R}^{4,1}$ can be written

$$x = \mathbf{x} + \alpha_1 e + \tfrac{1}{2}\alpha_2 \bar{e},$$

so

$$[x] = \begin{pmatrix} \mathbf{x} & \alpha_2 \\ \alpha_1 & -\mathbf{x} \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Since the geometric algebra $\mathbb{G}_{4,1}$ is algebraically isomorphic to $M_{4\times 4}(\mathbb{C})$, the 4×4 matrix algebra over the complex numbers, it follows that the group $\mathbb{G}^*_{4,1}$ of all invertible elements of $\mathbb{G}_{4,1}$ is isomorphic to the general linear group $\mathrm{GL}_{\mathbb{C}}(4)$ of 4×4 invertible complex matrices. The *pseudodeterminant* of the matrix (13) of a general element $h \in \mathbb{G}_{4,1}$ can be expressed in the form

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\tilde{a}d\tilde{d} + b\tilde{b}c\tilde{c} - (\tilde{a}b\tilde{d}c + \tilde{c}d\tilde{b}a)$$

for $a, b, c, d \in \mathbb{G}_3$, [12, p. 194]. The *Lipschitz subgroup* $\Gamma_{4,1}$ of $\mathbb{G}^*_{4,1}$ consists of those elements in $\mathbb{G}^*_{4,1}$ for which $gxg^\dagger \in \mathbb{R}^{4,1}$ for all $x \in \mathbb{R}^{4,1}$ and is generated by the product of invertible vectors $x \in \mathbb{R}^{4,1}$, [7]. The condition that $gxg^\dagger \in \mathbb{R}^{4,1}$ for all $x \in \mathbb{R}^{4,1}$ guarantees that the elements of the Lipschitz subgroup preserves the grading of the algebra, taking vectors in $\mathbb{R}^{4,1}$ into vectors in $\mathbb{R}^{4,1}$.

We have

Definition 1. The *affine space* $\mathscr{A}_e(\mathbb{R}^3) := \{x_h = \mathbf{x} + e | \mathbf{x} \in \mathbb{R}^3\}$.

Note that $x_h \cdot \bar{e} = 1$ and $x_h^2 = x^2$ for all $x_h \in \mathscr{A}_e(\mathbb{R}^3)$.

Definition 2. The *horosphere*

$$\mathscr{H}(\mathbb{R}^3) := \{x_c = x_h + \beta\bar{e} | x_h \in \mathscr{A}_e(\mathbb{R}^3),\ x_c^2 = 0\}.$$

All points x_c on the horosphere are null vectors,

$$x_c^2 = x_h^2 + 2\beta\bar{e} \cdot x_h = \mathbf{x}^2 + 2\beta = 0$$

so $\beta = -\frac{1}{2}\mathbf{x}^2$. Thus,

$$\mathscr{H}(\mathbb{R}^3) = \left\{ x_c = \mathbf{x} - \frac{\mathbf{x}^2}{2}\bar{e} + e \,\middle|\, \mathbf{x} \in \mathbb{R}^3 \right\},$$

and since $x_c = \frac{\alpha x_c}{\bar{e}\cdot(\alpha x_c)} = \frac{\alpha x_c}{\alpha}$ for all $x_c \in \mathscr{H}(\mathbb{R}^3)$ and $\alpha \in \mathbb{R}^*$, the horosphere consists of *homogeneous* points x_c, or *rays* αx_c for $\alpha \in \mathbb{R}^*$. The matrix $[x_c]$ of x_c is found by

$$[x_c] = [x_h] - \left[\frac{\mathbf{x}^2}{2}\bar{e}\right] = \begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & -\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{x} \end{pmatrix}. \tag{14}$$

The horosphere has been rediscovered as a new computational tool by various authors [8,9,6,4,1].

5.1. h-Twistors

By the *column h-twistor* $[x_c]_t$, we mean

$$[x_c]_t = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix},$$

and by the space of *column h-twistors* $\mathscr{T}_{\mathbb{G}_3}$ of $\mathbb{G}_3$ we mean

$$\mathscr{T}_{\mathbb{G}_3} := \left\{ [w]_t = \begin{pmatrix} a \\ b \end{pmatrix} \middle| \; a, b \in \mathbb{G}_3 \right\}. \tag{15}$$

For the column h-twistor $[w]_t = \begin{pmatrix} a \\ b \end{pmatrix}$, we define a *conjugate row h-twistor* by

$$[w]_t^\dagger = \begin{pmatrix} \tilde{b} & \tilde{a} \end{pmatrix},$$

where $\tilde{x} = (x^*)^\dagger$ is the Clifford conjugation (10) of $x \in \mathbb{G}_3$. We can now express any point x_c on the horosphere by

$$[x_c] = [x_c]_t [x_c]_t^\dagger.$$

An *h-twistor inner product* is defined by

$$\langle [w_1]_t, [w_2] \rangle_t := \langle [w_1]_t^\dagger [w_2]_t \rangle_{0\oplus 3} = \langle \tilde{b}_1 a_2 + \tilde{a}_1 b_2 \rangle_{0\oplus 3} \in Z(G_3), \tag{16}$$

where $[w_1]_t = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $[w_2]_t = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ are h-twistors, and $\langle \arg \rangle_{0\oplus 3}$ denotes the operation of *grade projection* into the scalars and trivectors of the algebra $\mathbb{G}_3$.

Definition 3. The h-twistors $[w_1]_t$ and $[w_2]_t$ are *equivalent* $[w_1]_t \equiv [w_2]_t$ iff $[w_1]_t [w_1]_t^\dagger = [w_2]_t [w_2]_t^\dagger$, and they are *projectively equivalent* iff $[w_1]_t [w_1]_t^\dagger = \alpha [w_2]_t [w_2]_t^\dagger$ for $\alpha \in R^*$.

It follows that $[x_c]_t$ and $[x_c g]_t$ are projectively equivalent for all $g \in G_3$ such that $g\tilde{g} \in \mathbb{R}^*$. Thus points on the horosphere need only be defined up to a invertible multivector $g \in G_3$. The concept of an h-twistor cuts calculations on the horosphere in half and was first introduced in [13]. For example, for any $g \in G_{4,1}$ with $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$[g x_c g^\dagger] = [g][x_c][g]^\dagger = [g][x_c]_t ([g][x_c]_t)^\dagger.$$

Reflections on the horosphere have the form

$$S_{\mathbf{a}}(x_c) = \mathbf{a} I x_c (\mathbf{a} I)^\dagger = -\mathbf{a} x_c \mathbf{a}$$

for the unit vector $\mathbf{a} \in \mathbb{R}^3$ satisfying $\mathbf{a}^2 = 1$. Since we are dealing with a homogeneous representation, the condition that $\mathbf{a}^2 = 1$ can be omitted. In terms of the h-twistor representation, we have

$$\begin{aligned} [S_{\mathbf{a}}(x_c)] &= ([\mathbf{a}I][x_c]_t)([\mathbf{a}I][x_c]_t)^\dagger = -\mathbf{a} x_c \mathbf{a} \\ &= \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix}^\dagger \\ &= \begin{pmatrix} \mathbf{ax} \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} -\mathbf{a} & \mathbf{xa} \end{pmatrix} = \begin{pmatrix} -\mathbf{axa} & \mathbf{a}^2\mathbf{x}^2 \\ -\mathbf{a}^2 & \mathbf{axa} \end{pmatrix}. \end{aligned}$$

Rotations are the composition of two reflections. We find that

$$S_{\mathbf{b}} S_{\mathbf{a}}(x_c) = \mathbf{ba} x_c (\mathbf{ba})^\dagger = \mathbf{ba} x_c \mathbf{ab}.$$

In terms of the h-twistor construction, we find

$$\begin{aligned}[S_{\mathbf{c}}S_{\mathbf{a}}(x_c)] &= ([\mathbf{ba}][x_c]_t)([\mathbf{ba}][x_c]_t)^\dagger\\ &= \begin{pmatrix}\mathbf{ba} & 0\\ 0 & \mathbf{ba}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}^\dagger\begin{pmatrix}\mathbf{ba} & 0\\ 0 & \mathbf{ba}\end{pmatrix}^\dagger\\ &= \begin{pmatrix}\mathbf{bax}\\ \mathbf{ba}\end{pmatrix}\begin{pmatrix}\mathbf{ab} & -\mathbf{xab}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{baxab} & -\mathbf{a}^2\mathbf{b}^2\mathbf{x}^2\\ \mathbf{a}^2\mathbf{b}^2 & -\mathbf{baxab}\end{pmatrix}.\end{aligned}$$

We can also represent *translations* in the horosphere. For $\mathbf{a} \in R^3$,

$$T_{\mathbf{a}}(x_c) := \left(1 + \frac{\mathbf{a}\bar{e}}{2}\right)x_c\left(1 - \frac{\mathbf{a}\bar{e}}{2}\right).$$

In terms of the h-twistor construction,

$$\begin{aligned}[T_{\mathbf{a}}(x_c)] &= \left[1 + \frac{\mathbf{a}\bar{e}}{2}\right][x_c]_t\left(\left[1 + \frac{\mathbf{a}\bar{e}}{2}\right][x_c]_t\right)^\dagger\\ &= \begin{pmatrix}1 & \mathbf{a}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}^\dagger\begin{pmatrix}1 & \mathbf{a}\\ 0 & 1\end{pmatrix}^\dagger\\ &= \begin{pmatrix}\mathbf{x}+\mathbf{a}\\ 1\end{pmatrix}\begin{pmatrix}1 & -\mathbf{x}-\mathbf{a}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{x}+\mathbf{a} & -(\mathbf{x}+\mathbf{a})^2\\ 1 & -\mathbf{x}-\mathbf{a}\end{pmatrix}.\end{aligned}$$

For an element $g \in G^*_{4,1}$ with $[g] = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, the twistor transformation

$$[w(\mathbf{x})]_t := \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix}_t = \begin{pmatrix}a\mathbf{x}+b\\ c\mathbf{x}+d\end{pmatrix}_t,$$

leads to the general linear fraction *Möbius transformation* or conformal transformation

$$\mathbf{x}' = f(\mathbf{x}) = (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1} \in \mathbb{R}^3$$

at all points $\mathbf{x} \in \mathbb{R}^3$ where $(c\mathbf{x}+d)^{-1}$ is well defined and $\langle [w(\mathbf{x})]_t, [w(\mathbf{x})]_t\rangle = 0$. This is because of the projective equivalence of the h-twistors

$$\begin{pmatrix}a\mathbf{x}+b\\ c\mathbf{x}+d\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}(a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}\\ 1\end{pmatrix}.$$

More details of this approach can be found in [13].

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