Geometric Numbers:

The Language of Quantum Mechanics

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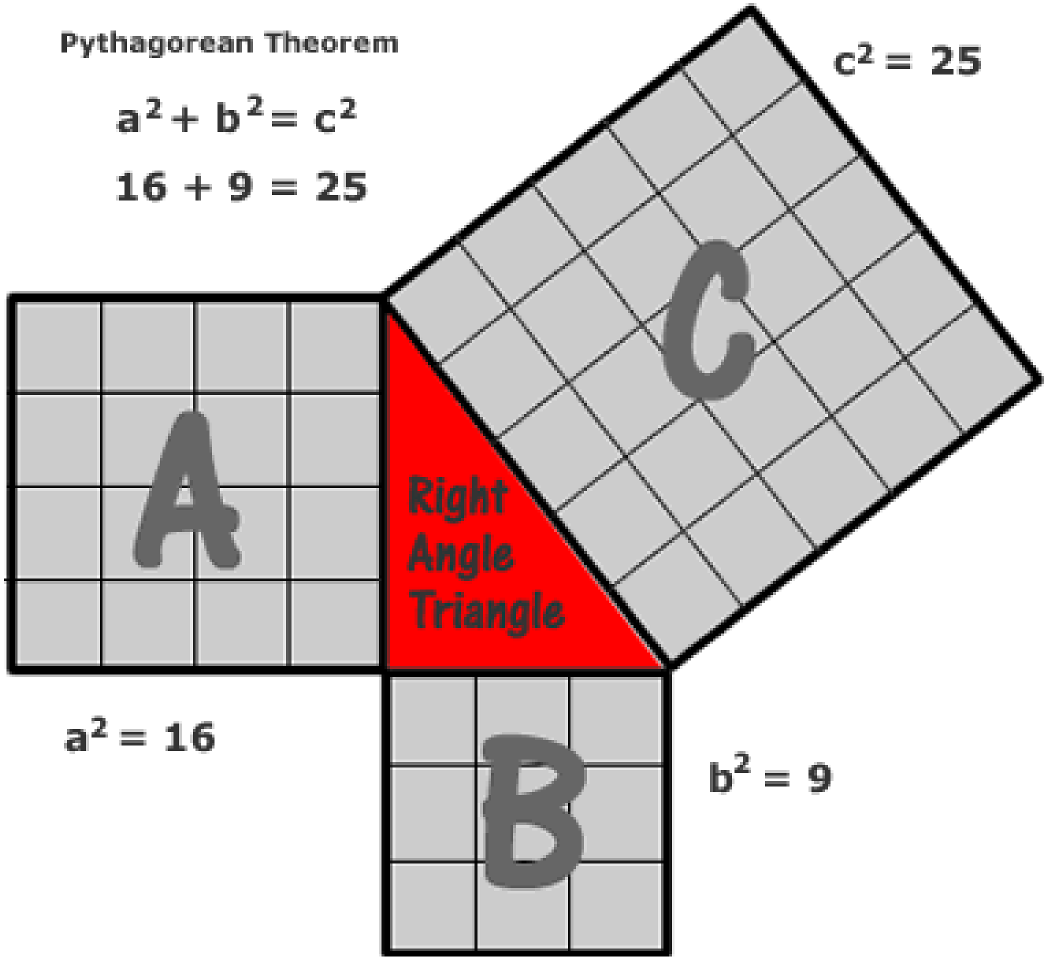
Pythagorean Theorem

- A simple proof relating the areas formed by the sides.
- Pythagorean School Philosophy

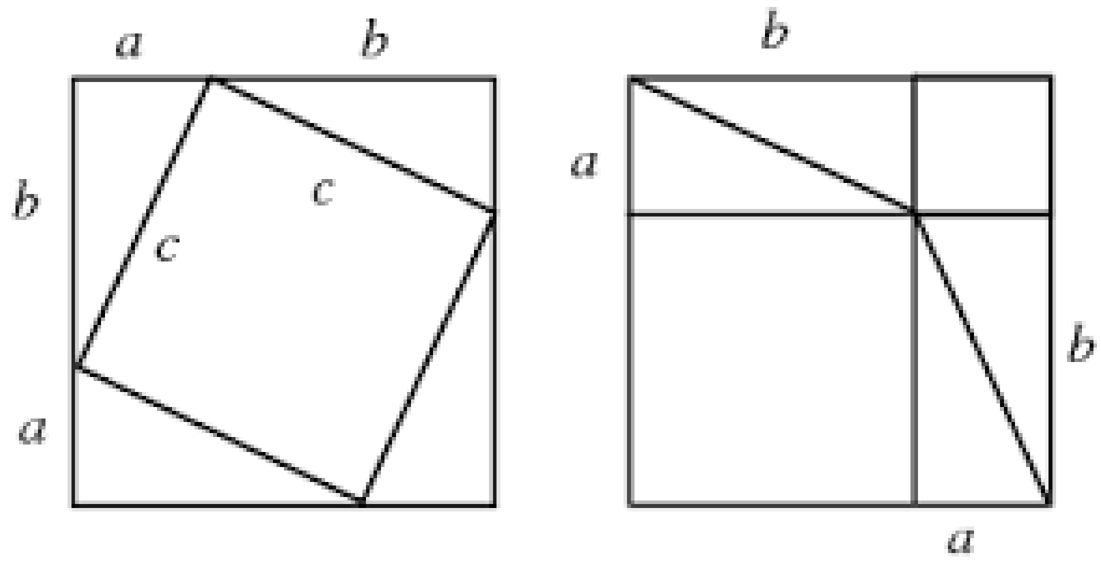
"Everything is numbers".

What about $\sqrt{2}$?



Another Proof

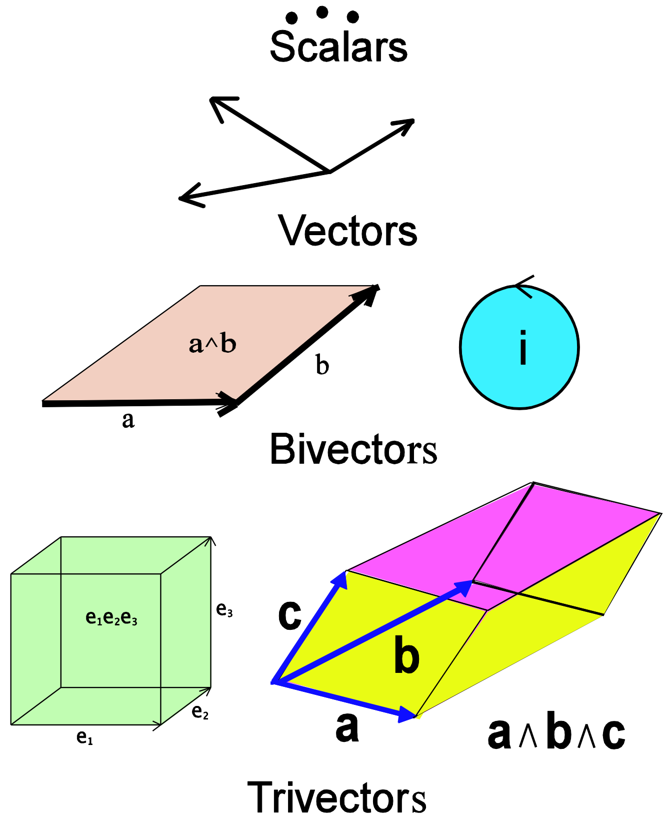
- There are more than a 100 "proofs" of the Pythagorean theorem.
- Which is the best?



What is a Geometric Number?

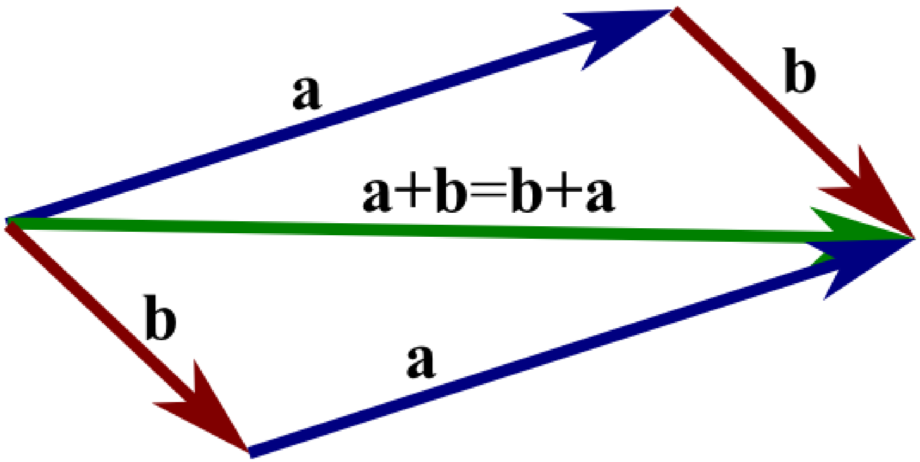
The geometric numbers of space are pictured on the right. They are obtained by extending the real number system to include new

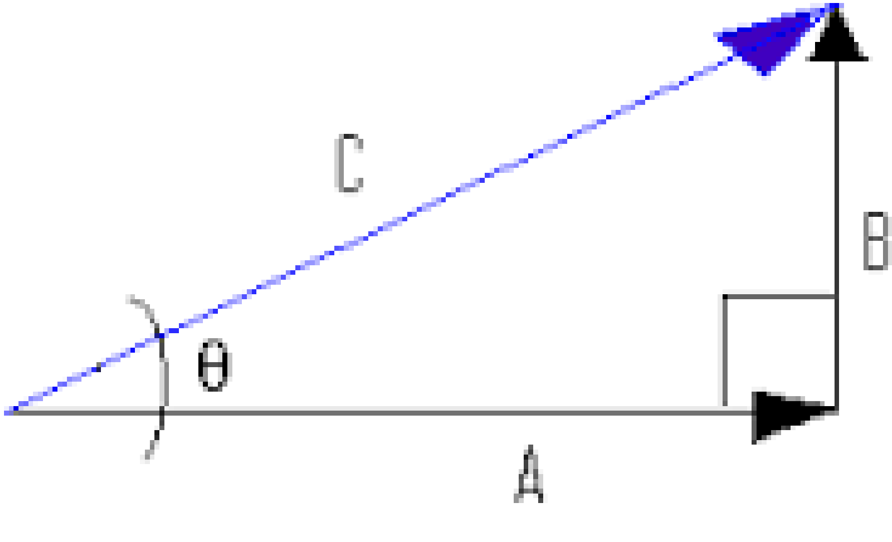
anti-commuting square roots of plus and minus one, each such root representing an orthogonal direction in successively higher dimensions.



Geometric numbers of Space

Rules of Geometric Numbers





Proof Using Geometric Numbers

Rules: 1) A vector: $\mathbf{a}^2 = \mathbf{aa} = |\mathbf{a}|^2$

2) Orthogonal vectors anti-commute:

$$\mathbf{a} \perp \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{ab} = -\mathbf{ba}.$$

Proof of Pythagorean theorem:

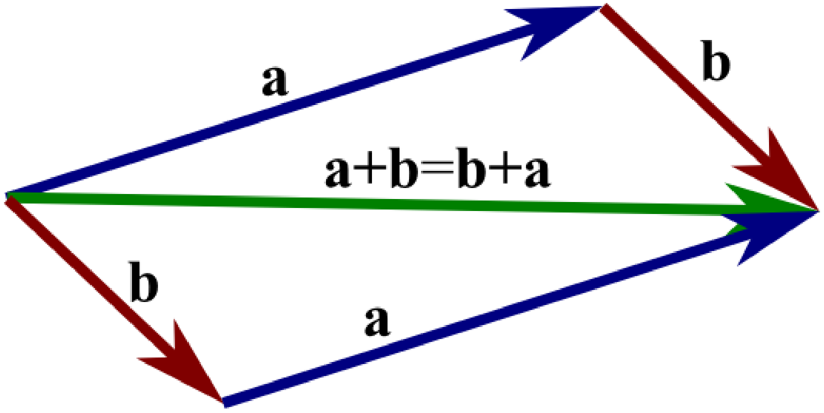
$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + b^2 = c^2.$$

Quote

"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them."

— Heinrich Hertz (1857 – 1894)

More generally



$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{ab} + \mathbf{ba} + \mathbf{b}^2$$

$$= \mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2$$

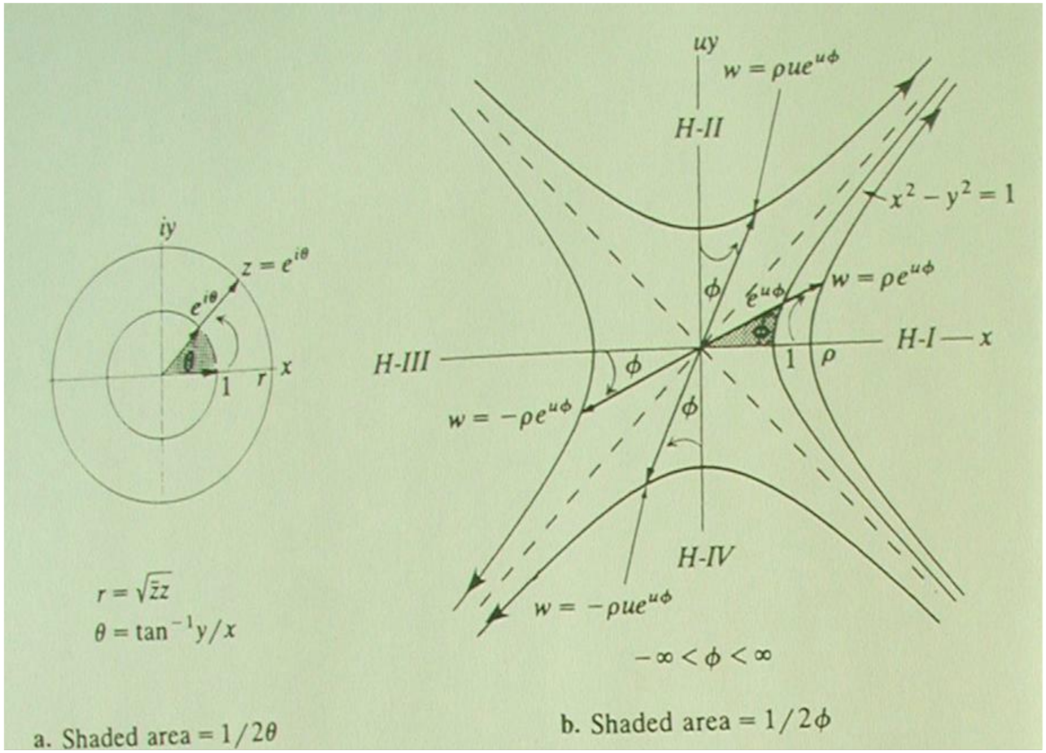
Geometric Product

Inner and Outer Products

$$\mathbf{a} \cdot \mathbf{b} \quad \text{and} \quad \mathbf{a} \wedge \mathbf{b}$$

$$\mathrm{ab} = \frac{1}{2}\left(\mathrm{ab} + \mathrm{ba}\right) + \frac{1}{2}\left(\mathrm{ab} - \mathrm{ba}\right)$$

Complex and Hyperpolic Numbers



$$u \notin R$$

$u^2 = 1$

$$z = x + iy = re^{i\theta}, \quad w = x + uy = \rho e^{u\phi}$$

$$i = \sqrt{-1}, \ i^2 = -1, \ u = \sqrt{+1}, \ u^2 = 1$$

Hyperbolic Numbers

$$w_1 = x_1 + uy_1,\ w_2 = x_2 + uy_2$$

$$w_1 w_2 = x_1 x_2 + y_1 y_2 + (x_1 y_2 + x_2 y_1)u$$

$$\text{Spectral Basis :}$$

$$w = x + yu = (x+y)u_+ + (x-y)u_-$$

$$\text{where } u_+ = \frac{1}{2}(1+u),\ u_- = \frac{1}{2}(1-u),$$

$$u_+ + u_- = 1,\ u_+ - u_- = u$$

$$u_+^2 = u_+,\ u_-^2 = u_-,\ u_+ u_- = 0.$$

:

Geometric Numbers G_2 of the Plane

Standard Basis of $G_2=\{1, e_1, e_2, e_{12}\}$.

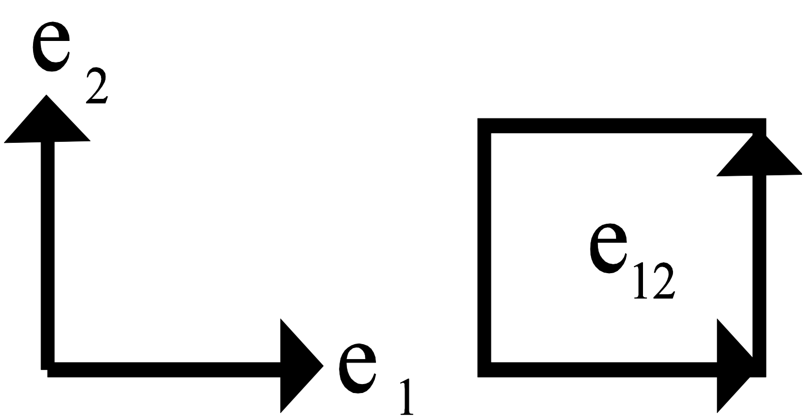
where $i=e_{12}$ is a unit bivector.

$$e_1^2 = e_2^2 = 1$$

$$e_{12} = e_1 e_2 = -e_2 e_1$$

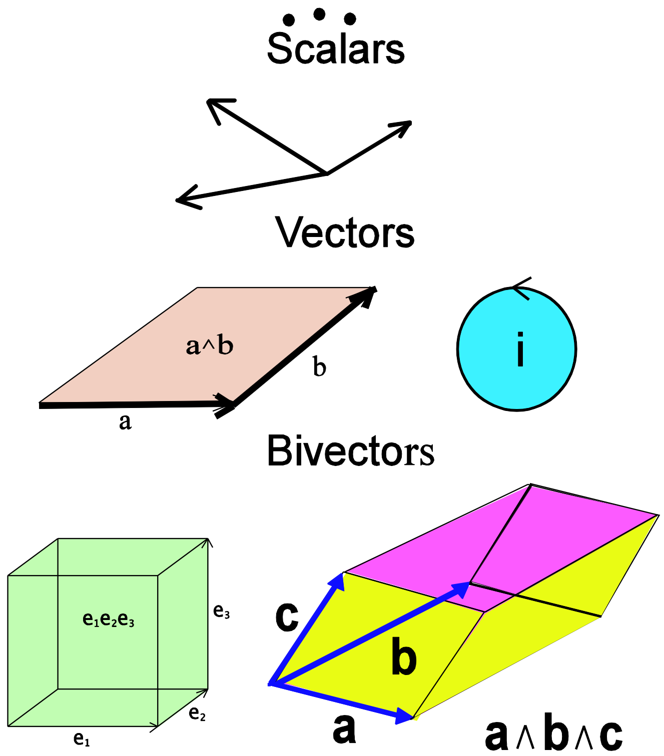
$$i^2 = (e_1 e_2)^2 = e_1 e_2 e_1 e_2$$

$$= -e_1 e_1 e_2 e_2 = -1$$



Geometric Numbers of 3-Space

$$\mathbf{G_3} = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}$$



Trivectors

- Real Numbers (scalars)
- Vectors
- Bivectors
- Trivectors

Geometric numbers of Space

$$g = \alpha_0 + \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$$

Cancellation Property

Let a, b, c be vectors in G_3, then

$$\mathbf{ab} = \mathbf{ac} \iff \mathbf{a}^2\mathbf{b} = \mathbf{a}^2\mathbf{c} \iff \mathbf{b} = \mathbf{c},$$

provided $\mathrm{a}^2 = |\mathrm{a}|^2 \neq 0.$

This is equivalent to:

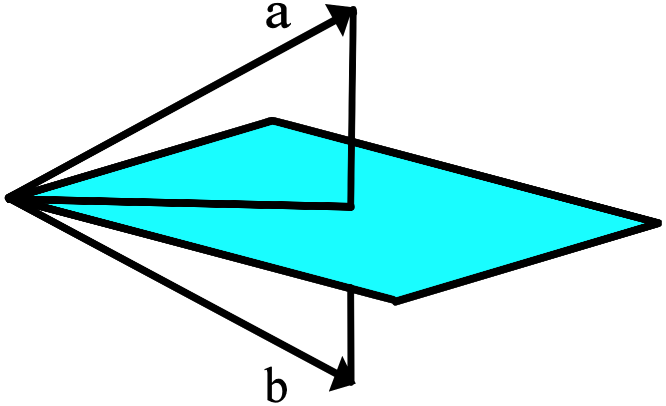
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$$

implies $\mathbf{a} = \mathbf{c}$

Reflections L(x) and Rotations R(x)

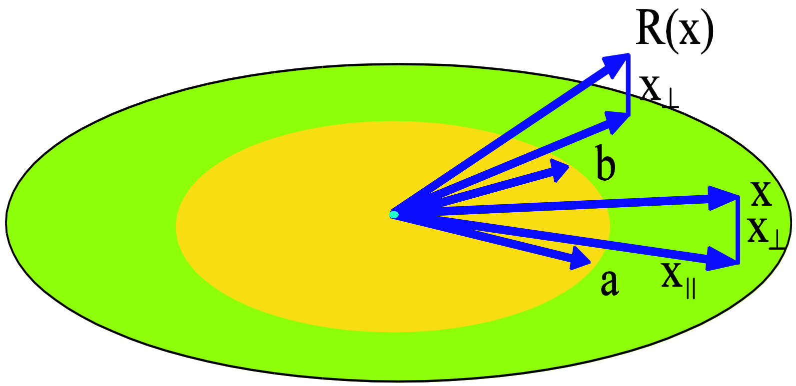
$$L(\mathrm{x}) = -(\mathrm{a}-\mathrm{b})\mathrm{x}(\mathrm{a}-\mathrm{b})^{-1}$$



$$R(\mathbf{x}) = \sqrt{\mathbf{ba}}\ \mathbf{x}\ \sqrt{\mathbf{ab}}$$

where |a|=|b|=1 and

$$\sqrt{\mathbf{ab}} = \pm\mathbf{a}\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$$



$$L(\mathrm{a}) = \mathrm{b}, \quad R(\mathrm{a}) = \mathrm{b}$$

The Spectral Basis of the Geometric Algebra G_3

$G_3 = \text{span}\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$.

By the spectral basis of G_3 we mean

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}$$

where $u_\pm = \frac{1}{2}(1 \pm \mathbf{e}_3)$

are mutually annihiliating idempotents.

Note that $e_1 u_+ = u_- e_1$.

For example, if $[\mathbf{g}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a, b, c, d \in C$

then the element g Ɛ G_3 is

$$\mathbf{g} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [\mathbf{g}] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = a u_+ + b \mathbf{e}_1 u_- + c \mathbf{e}_1 u_+ + d u_-$$

Pauli Matrices:

$$[1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [\mathbf{e}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and}$$

$$[\mathbf{e}_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, [\mathbf{e}_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

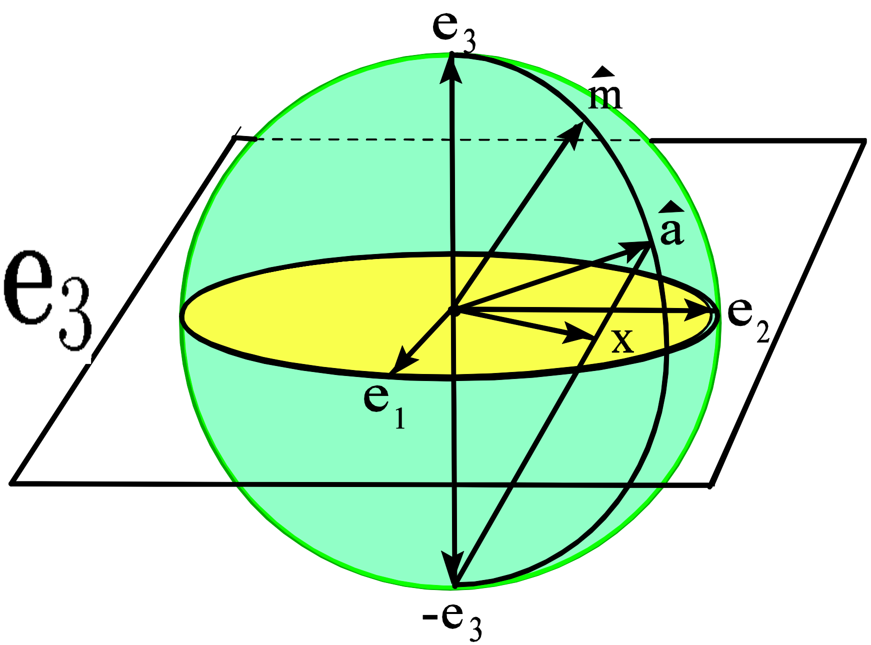
The Riemann sphere

Basic equation: $\mathbf{m} = \frac{2}{\hat{\mathbf{a}}+\mathbf{e}_3}$

where $\hat{\mathbf{a}}$ is a unit vector, and $\mathbf{m}$ is a vector.

$$\mathrm{m} = x\mathrm{e}_1 + y\mathrm{e}_2 + \mathrm{e}_3$$

$$\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}}$$



What is a Spinor?

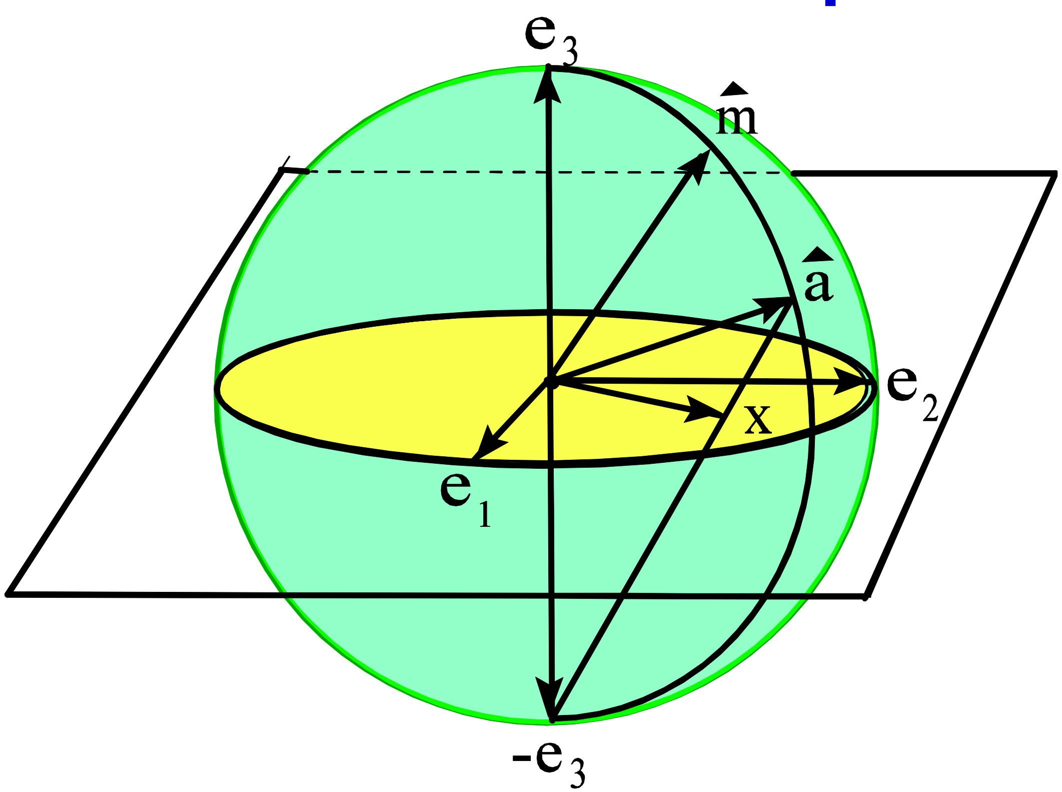
Pauli Spinor: $$|\alpha\rangle_p := \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

Also called a ket-vector.

Geometric Spinor: $$|\alpha\rangle := (\alpha_0 + \alpha_1 \mathbf{e}_1) u_+,$$

with matrix $$[|\alpha\rangle] := \begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix}$$

The Riemann Sphere



$$\mathbf{z} = \frac{\alpha_1}{\alpha_0}, \quad \mathbf{x} = P_{xy}(\mathbf{m}) = \frac{\mathbf{z}+\bar{\mathbf{z}}}{2}\mathbf{e}_1 + \frac{\mathbf{z}-\bar{\mathbf{z}}}{2\mathbf{i}}\mathbf{e}_2$$

$$\mathbf{m} = \mathbf{x} + \mathbf{e}_3$$

Norm of a Spinor

Sesquilinear inner product:

$$\langle \alpha | \beta \rangle := \alpha_0 \overline{\beta}_0 + \alpha_1 \overline{\beta}_1 .$$

A spinor is normalized if

$$\langle \alpha | \alpha \rangle := \alpha_0 \overline{\alpha}_0 + \alpha_1 \overline{\alpha}_1 = 1.$$

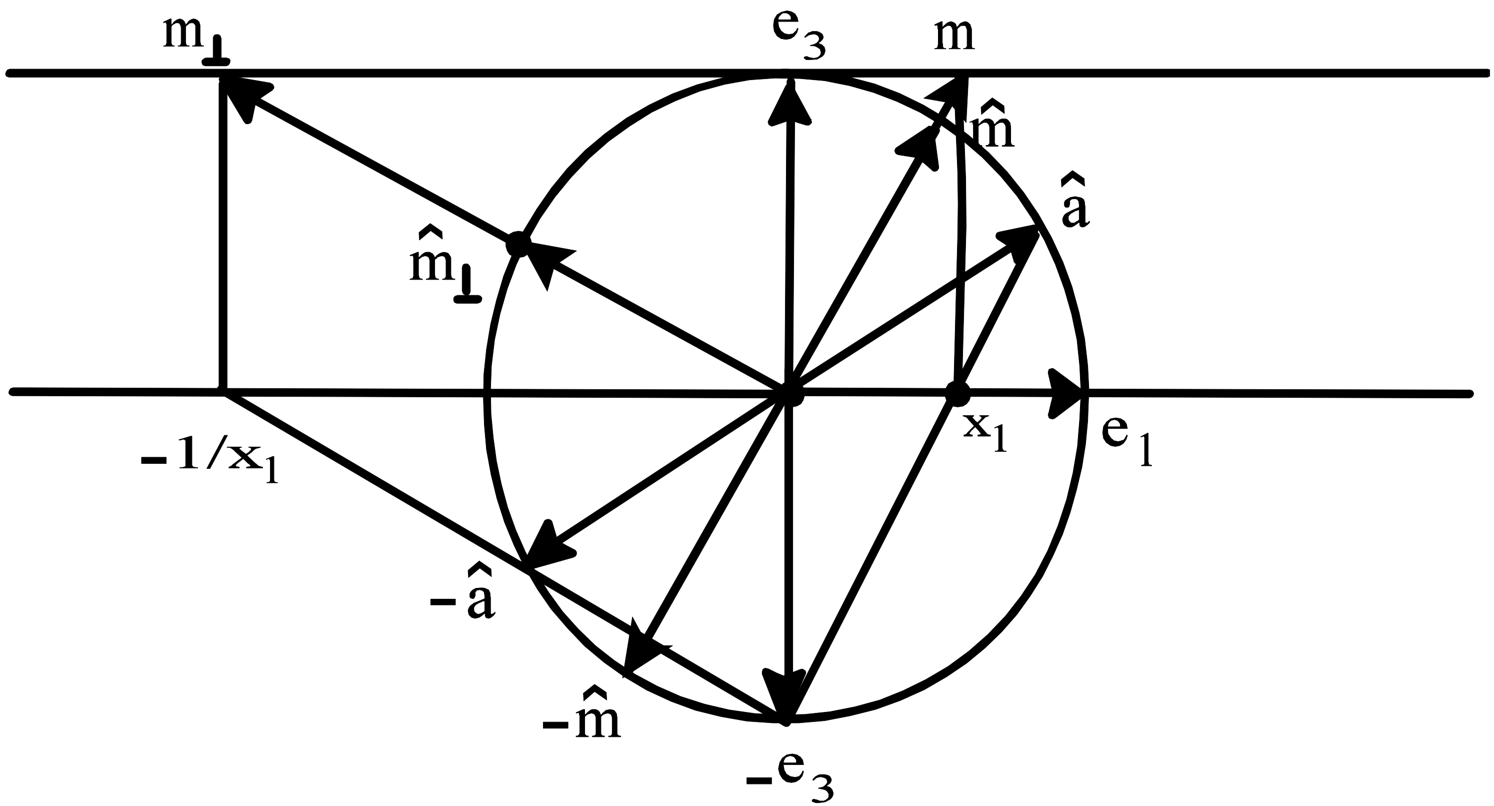
where $\langle \alpha | := | \alpha \rangle^{\dagger} = u_{+}(\overline{\alpha}_0 + \overline{\alpha}_1 \mathbf{e_1})$

Properties of Spinors

$$|\alpha\rangle\langle\alpha| = \hat{\mathbf{m}}\mathbf{u}_+\hat{\mathbf{m}} = \hat{\mathbf{a}}_+.$$

$$|\alpha\rangle = \alpha_0\mathbf{m}\mathbf{u}_+ = \rho e^{i\theta}\hat{\mathbf{m}}\mathbf{u}_+$$

Great Circle of Riemann Sphere



$$|\alpha\rangle = (1 + x_1\,\mathbf{e_1})u_+, \qquad |\alpha\rangle_\perp = (1 - \frac{1}{x_1}\mathbf{e_1})u_+$$

Magic of Quantum Mechanics

Observable: $$S = s_0 + \mathbf{s} \in \mathbf{G_3}$$

where $$\mathbf{s} = s_1\mathbf{e_1} + s_2\mathbf{e_2} + s_3\mathbf{e_3}.$$

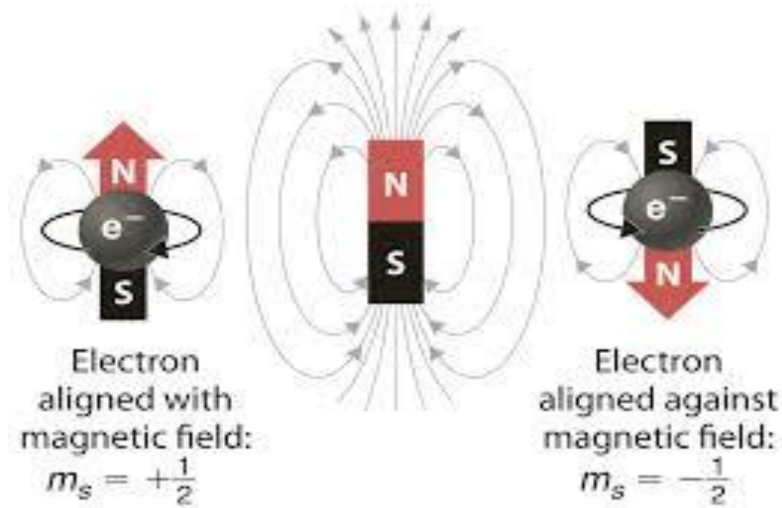
Note that $$[S] = \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix}$$

is an Hermitian matrix.

Probabilities of Spin

$$|\mathbf{a}\rangle = \mathbf{m}_a \mathbf{u}_+$$

$$|\mathbf{b}\rangle = \mathbf{m}_b \mathbf{u}_+$$

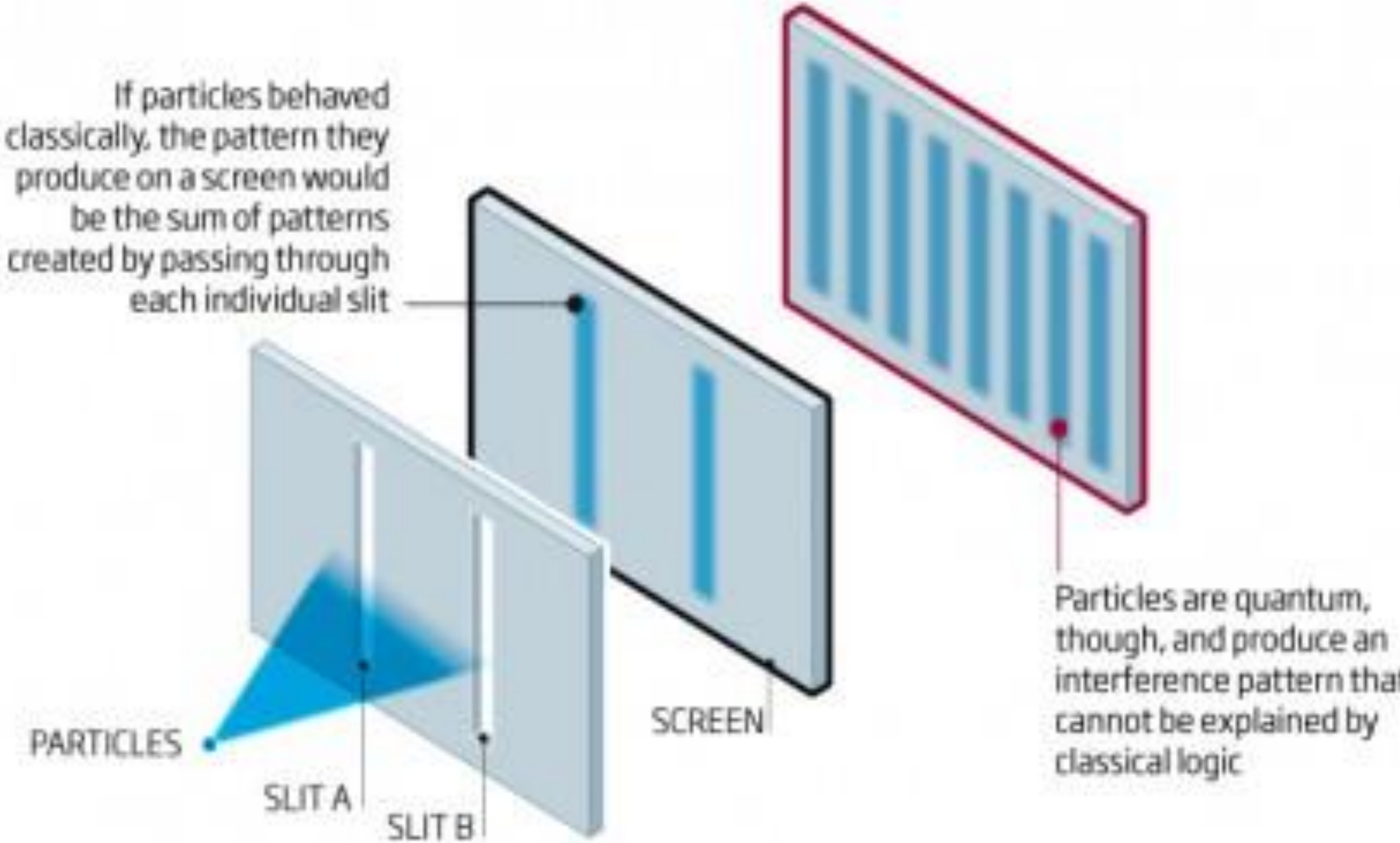


Then

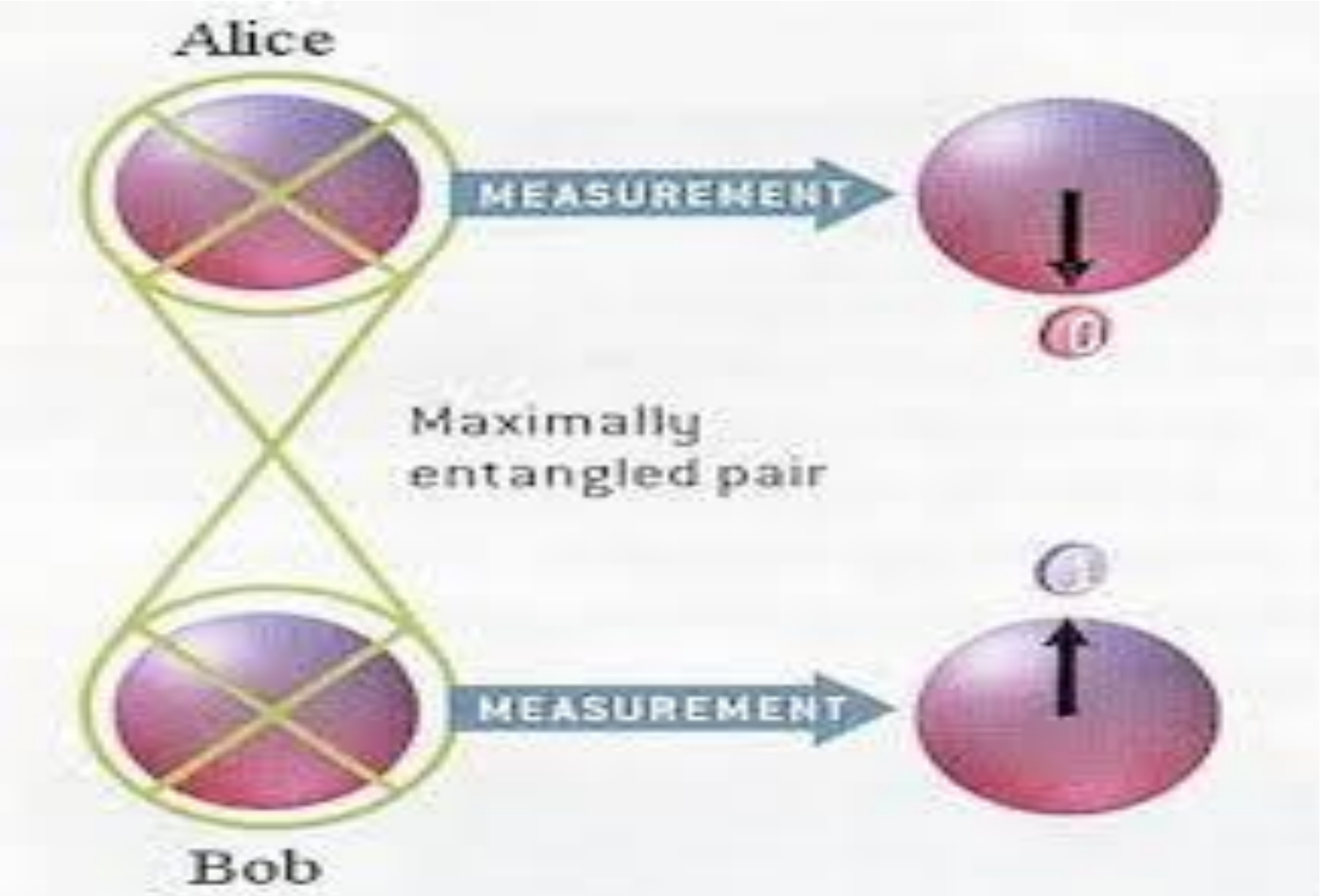
$$|\langle \mathrm{a}|\mathrm{b}\rangle|^2 = 1 - \frac{(\mathrm{m_a} - \mathrm{m_b})^2}{\mathrm{m_a^2 m_b^2}} = \frac{1}{2}\left(1 + \hat{\mathrm{a}} \cdot \hat{\mathrm{b}}\right).$$

The famous double slit experiment

This experiment illustrates the difference between quantum and classical mathematics



Quantum Entanglement



Expected Value: $\langle S \rangle := \langle \alpha | S | \alpha \rangle = s_0 + \mathbf{s} \cdot \hat{\mathbf{a}},$

Standard deviation:

$$\sigma_S^2 := \langle \alpha | \left(S - \langle S \rangle \right)^2 | \alpha \rangle = ||^2 \langle \alpha | \left(\hat{s} - \hat{s} \cdot \hat{a} \right)^2 | \alpha \rangle = \left(s \times \hat{a} \right)^2$$

Uncertainty Principle for observables S and T:

$$(s \times \hat{a})^2 (t \times \hat{a})^2 = |(s \times \hat{a}) \cdot (t \times \hat{a})|^2 + |(s \times t) \cdot \hat{a}|^2,$$

or

$$\sigma_S^2 \sigma_T^2 \geq \langle \mathbf{s} \times \mathbf{t} \rangle^2 .$$

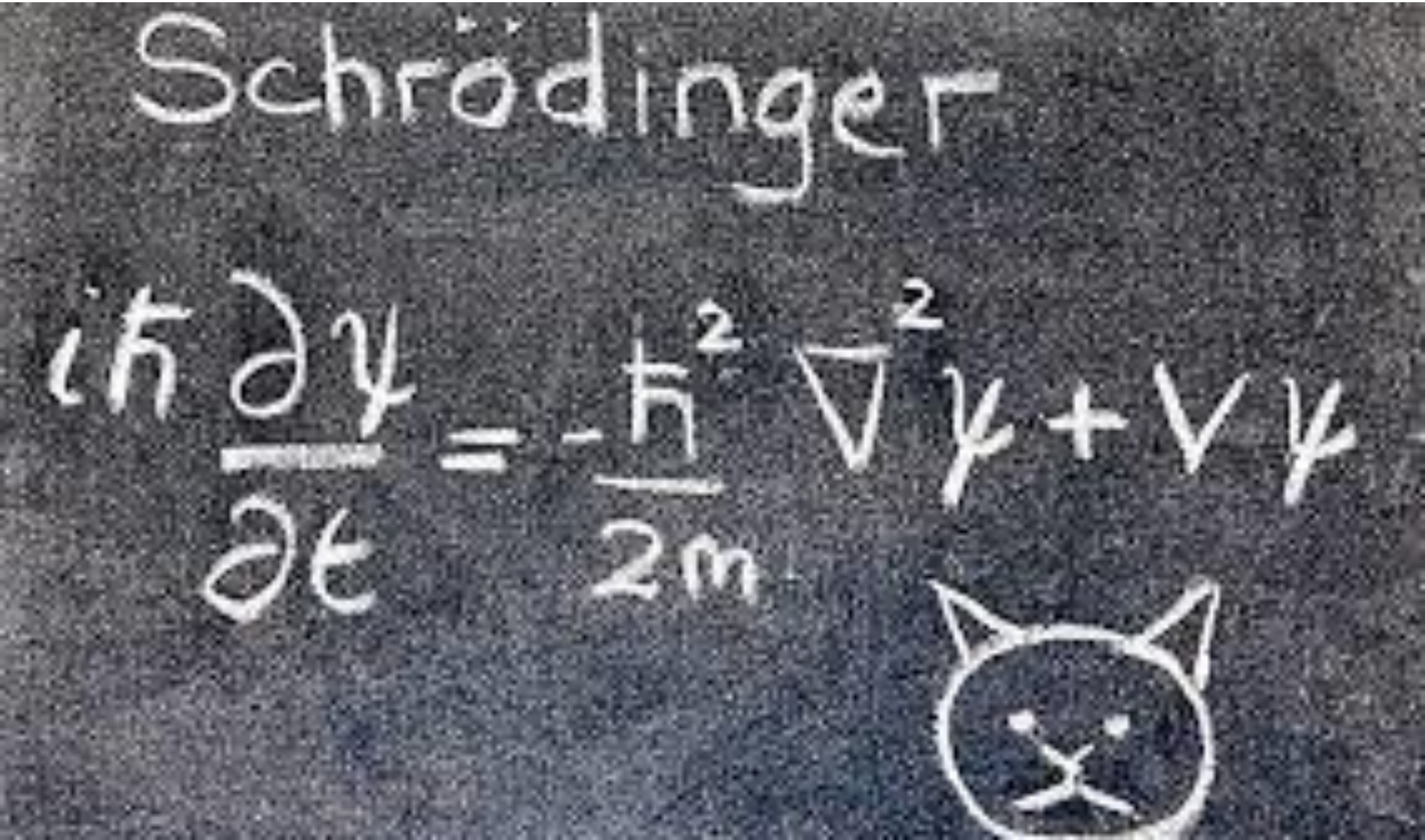
Schrodinger Equation

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)|\alpha\rangle = H|\alpha\rangle$$

If the Hamiltonian is time independent, then H = S = s_0+**s;** with solution:

$$|\alpha\rangle = \sqrt{2}e^{-\frac{iS}{\hbar}t}u_+ = \sqrt{2}e^{-i\frac{s_0}{\hbar}t}\Big(\cos\frac{|\mathbf{s}|t}{\hbar} + i\,\hat{\mathbf{s}}\sin\frac{|\mathbf{s}|t}{\hbar}\Big)u_+.$$

Schrodinger’s Equation



Garret Sobczyk

New Foundations in Mathematics

The Geometric Concept of Number

Birkhäuser

G. Sobczyk, *Hyperbolic Number Plane*, The College Mathematics

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G. Sobczyk, *The Generalized Spectral Decomposition of a Linear Operator*, The College Mathematics Journal, 28:1 (1997) 27-38.

G. Sobczyk, *Spectral integral domains in the classroom*,

APORTACIONES MATEMATICAS, Serie Comunicaciones Vol. 20,

(1997) 169-188.

G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number, Springer-Birkhauser* 2012.

Note: Copies of many of my papers can be found on my website:

http://www.garretstar.com/

Complex Pauli Algebra

$$\mathbf{G_3}(\mathbf{C}) = gen\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}_{\mathbf{C}}$$

Complex spectral basis:

$$\begin{pmatrix} 1 \\ \mathbf{e_1} \end{pmatrix} E_3^+ \begin{pmatrix} 1 & \mathbf{e_1} \end{pmatrix} = \begin{pmatrix} E_3^+ & \mathbf{e_1} E_3^- \\ \mathbf{e}_1 E_3^+ & E_3^- \end{pmatrix}$$

where

$$E_3^{\pm} := \frac{1}{2}(1 \pm J\mathbf{e_3})$$

for

$$J := -i\mathbf{e_{123}} = -iI.$$

Complex Pauli Matrices

$$[1]_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\mathbf{e_1}]_\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$[\mathbf{e_2}]_\mathbf{\Omega} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [\mathbf{e_3}]_\mathbf{\Omega} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

For real $\Omega \in \mathbf{G_3}(\mathbf{C}),$

the matrix $[\Omega]$ of Ω is

$$[\Omega] := \begin{pmatrix} \Omega_0 & \overline{\Omega_1} \\ \Omega_1 & \overline{\Omega_0} \end{pmatrix},$$

where $\Omega_0 := \varphi_1 + J\varphi_3, \Omega_1 := \varphi_4 + J\varphi_2$

for the Dirac spinor $\Psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}.$

Spacetime Algebra $G_{1,3}$

We start with

$$\mathbf{G_3} = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}$$

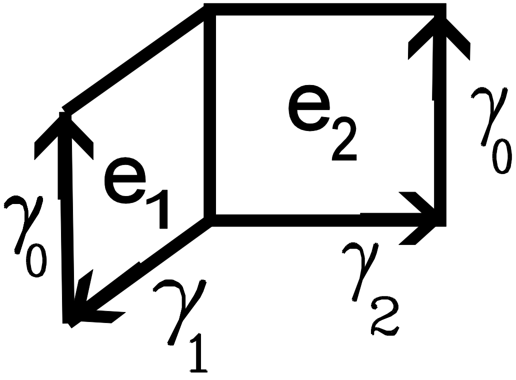
We factor e_1, e_2, e_3 into Dirac bivectors,

$$\mathbf{e}_1 = \gamma_1\gamma_0, \mathbf{e}_2 = \gamma_2\gamma_0, \mathbf{e}_3 = \gamma_3\gamma_0,$$

where

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \gamma_0^2 = 1$$

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \text{ for } \mu \neq \nu.$$



$$\mathbf{G}_{1,3} = span\{1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\omega}, \gamma_{0123}\}$$

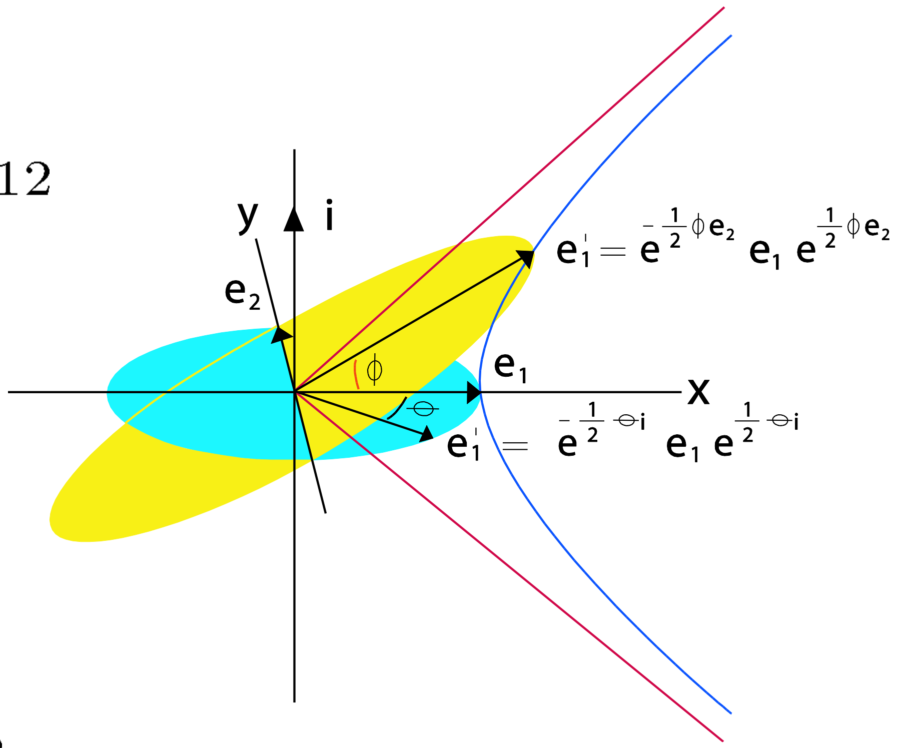
Splitting Space and Time

The ordinary rotation

$$R(\mathbf{x}) = e^{-\frac{1}{2}\theta\mathbf{e}_{12}}\mathbf{x}e^{\frac{1}{2}\theta\mathbf{e}_{12}}$$

is in the blue plane of the bivector $i=e_{12}$. The blue plane is boosted into the yellow plane by

$$B(\mathbf{x}) = e^{-\frac{1}{2}\phi\mathbf{e}_2}\mathbf{x}e^{\frac{1}{2}\phi\mathbf{e}_2}$$



with the velocity $v/c = \tanh\phi$. The light cone is shown in red.

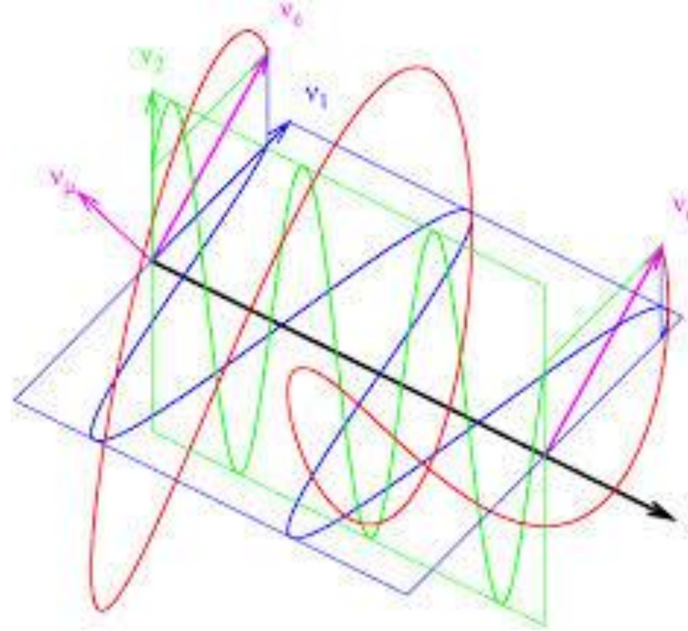
Neutrino Oscillation (two states)

Electron neutrino state: $|\upsilon_\epsilon\rangle := |0\rangle := \sqrt{2}u_+$

Muon neutrino state: $|\upsilon_\mu\rangle := |i\,\hat{\mathbf{s}}\,\mathbf{e_3}\rangle$

Oscillation:

$$\left(\cos\frac{|\mathbf{s}|t}{\hbar} + i\,\hat{\mathbf{s}}\sin\frac{|\mathbf{s}|t}{\hbar}\right)u_+ = \left(\mathbf{e_3}\cos\frac{|\mathbf{s}|t}{\hbar} + i\,\hat{\mathbf{s}}\mathbf{e_3}\sin\frac{|\mathbf{s}|t}{\hbar}\right)u_+,$$



The neutrino in the evolving state $|\alpha(t)\rangle$
will be observed in the state $|\beta\rangle$

with the probability

$$|\langle\alpha|\beta\rangle|^2 = 2\langle\beta|\hat{\mathbf{a}_+}|\beta\rangle_{0+3} = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}).$$