# Geometry of Bivectors

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**Abstract**

The general Lie algebra $gl_n(\mathbb{C})$ is formulated as a Lie algebra of bivectors which is a subalgebra of the geometric algebra $\mathbb{G}_{n,n+1}$. The so-called spinor algebra of $\mathbb{C}_2$, the language of the ubiquitous quantum mechanics, is formulated in terms of the idempotents and nilpotents of geometric algebra $\mathbb{G}_3$. We start by studying the Lie algebra $gl_n(\mathbb{C})$ and extend these ideas to apply to higher dimensional spin algebras.

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## 1 The general Lie algebra $gl_n(\mathbb{C})$

By the *standard matrix basis* $\Omega^2_{n,n}$ of bivectors, we mean

$$\Omega^2_{n,n} = (\mathbf{a})^T_{(n)} \wedge (\mathbf{b})_{(n)} = \begin{pmatrix} \mathbf{a}_1\wedge\mathbf{b}_1 & \mathbf{a}_1\wedge\mathbf{b}_2 & \dots & \mathbf{a}_1\wedge\mathbf{b}_n \\ \mathbf{a}_2\wedge\mathbf{b}_1 & \mathbf{a}_2\wedge\mathbf{b}_2 & \dots & \mathbf{a}_2\wedge\mathbf{b}_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathbf{a}_n\wedge\mathbf{b}_1 & \mathbf{a}_n\wedge\mathbf{b}_2 & \dots & \mathbf{a}_n\wedge\mathbf{b}_n \end{pmatrix}, \tag{1}$$

where the basis vectors $(\mathbf{a})_{(n)}$ and $(\mathbf{b})_{(n)}$ form a *Witt basis*, satisfying

$$(\mathbf{a})^T_{(n)} \cdot (\mathbf{b})_{(n)} = [1]_n, \tag{2}$$

for the identity $n\times n$ matrix $[1]_n$.

The matrix $\Omega^2_{n,n}$ spans a $n^2$-dimensional linear space of bivectors, defined by the same symbol,

$$\Omega^2_{n,n} = \mathrm{span}\{\mathbf{a}\wedge\mathbf{b}|\ \ \mathbf{a}\in\mathscr{A}^n, \mathbf{b}\in\mathscr{B}^n\} = \{\mathbf{F}|\ \mathbf{F} = (\mathbf{a})_{(n)}[\mathbf{F}]\wedge(\mathbf{b})^T_{(n)}\}$$

where $[\mathbf{F}]$ is a real (or complex) $n \times n$ matrix, called the *matrix* of the bivector $\mathbf{F}$. The *algebra* $\Omega_{n,n}$ is generated by taking all sums of geometric products of the elements of $\Omega^2_{n,n}$ and is an even subalgebra of the geometric algebra $\mathbb{G}_{n,n}$. We also use the notation $\Omega_{n,n+1}$ when studying the complex Lie algebra where the imaginary unit $i$ is interpreted to be the pseudoscalar element of the geometric algebra $\mathbb{G}_{n,n+1}$.

Let $\mathbf{K} = (\mathbf{a})_{(n)} \wedge (\mathbf{b})^T_{(n)}$, then we find that $[\mathbf{K}] = [1]_n$ is the identity matrix, and $\mathbf{F} \cdot \mathbf{K} = \mathrm{tr}([\mathbf{F}])$ is the trace of the matrix $[\mathbf{F}]$ of $\mathbf{F}$. By the *rank* of a bivector $\mathbf{F} \in \mathscr{F}$, we mean the highest positive integer $k$ such that $\wedge^k \mathbf{F} \neq 0$. A bivector $\mathbf{F}$ is said to be *nonsingular* if $\wedge^n \mathbf{F} = \Big(\det[\mathbf{F}]\Big) \wedge^n \mathbf{K} \neq 0$.

For two bivectors $\mathbf{F}, \mathbf{G} \in \Omega_{n,n}$, we find that

$$\mathbf{F} \cdot \mathbf{G} = \Big((\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})^T_{(n)}\Big) \cdot \Big((\mathbf{a})_{(n)}[\mathbf{G}] \wedge (\mathbf{b})^T_{(n)}\Big)$$

$$= \Big(\big((\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})^T_{(n)}\big) \cdot (\mathbf{a})_{(n)}\Big) \cdot \Big([\mathbf{G}] \wedge (\mathbf{b})^T_{(n)}\Big) = tr\big([\mathbf{F}][\mathbf{G}]\big), \tag{3}$$

$$\mathbf{F} \otimes \mathbf{G} = \Big((\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})^T_{(n)}\Big) \otimes \Big((\mathbf{a})_{(n)}[\mathbf{G}] \wedge (\mathbf{b})^T_{(n)}\Big)$$

$$= \Big(\big((\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})^T_{(n)}\big) \cdot (\mathbf{a})_{(n)}\Big) \wedge \Big([\mathbf{G}](\mathbf{b})^T_{(n)}\Big)$$

$$+(\mathbf{a})_{(n)}[\mathbf{G}] \wedge \Big(\big((\mathbf{a})_{(n)} \wedge [\mathbf{F}](\mathbf{b})^T_{(n)}\big) \cdot (\mathbf{b})^T_{(n)}\Big) = (\mathbf{a})_{(n)} \wedge \big[[\mathbf{F}],[\mathbf{G}]\big](\mathbf{b})^T_{(n)}, \tag{4}$$

where $\big[[\mathbf{F}],[\mathbf{G}]\big] = [\mathbf{F}][\mathbf{G}] - [\mathbf{G}][\mathbf{F}]$ is the usual anticommutative *bracket operation* for the matrices $[\mathbf{F}]$ and $[\mathbf{G}]$. It follows that $[\mathbf{F} \otimes \mathbf{G}] = \big[[\mathbf{F}],[\mathbf{G}]\big]$. For $\mathbf{F} \wedge \mathbf{G}$, we find that

$$\mathbf{F} \wedge \mathbf{G} = \sum_{i<j} \left[\det\begin{pmatrix} f_{ii} & f_{ij} \\ g_{ji} & g_{jj} \end{pmatrix} + \det\begin{pmatrix} g_{ii} & g_{ij} \\ f_{ji} & f_{jj} \end{pmatrix}\right] \mathbf{a}_i \wedge \mathbf{b}_i \wedge \mathbf{a}_j \wedge \mathbf{b}_j. \tag{5}$$

Other interesting formulas:

$$\mathbf{K} \cdot (\mathbf{F} \wedge \mathbf{F}) = 2\Big((\mathbf{F} \cdot \mathbf{K})\mathbf{F} - (\mathbf{a})_{(n)} \wedge [\mathbf{F}]^2 (\mathbf{b})^T_{(n)}\Big),$$

$$(\mathbf{K} \wedge \mathbf{K}) \cdot (\mathbf{F} \wedge \mathbf{F}) = 2\Big((\mathbf{F} \cdot \mathbf{K})^2 - \mathbf{F} \cdot \mathbf{F}\Big),$$

$$\mathbf{F} \cdot (\mathbf{F} \wedge \mathbf{F}) = 2\Big((\mathbf{F} \cdot \mathbf{F})\mathbf{F} - (\mathbf{a})_{(n)} \wedge [\mathbf{F}]^3 (\mathbf{b})^T_{(n)}\Big),$$

and

$$(\mathbf{K} \wedge \mathbf{F}) \cdot (\mathbf{F} \wedge \mathbf{F}) = 2\Big((\mathbf{F} \cdot \mathbf{K})(\mathbf{F} \cdot \mathbf{F}) - tr([\mathbf{F}]^3)\Big).$$

The most general identity of this kind is

$$\mathbf{F} \cdot (\mathbf{G} \wedge \mathbf{H}) = (\mathbf{F} \cdot \mathbf{G})\mathbf{H} + (\mathbf{F} \cdot \mathbf{H})\mathbf{G} - (\mathbf{a})_{(n)} \wedge \Big([\mathbf{H}][\mathbf{F}][\mathbf{G}] + [\mathbf{G}][\mathbf{F}][\mathbf{H}]\Big)(\mathbf{b})^T_{(n)}. \tag{6}$$

We include a higher order identity,

$$\mathbf{K} \cdot (\mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F}) = 3(\mathbf{K} \cdot \mathbf{F})\mathbf{F} \wedge \mathbf{F} - 6\Big((\mathbf{a})_{(n)} \wedge [\mathbf{F}]^2 (\mathbf{b})^T_{(n)}\Big) \wedge \mathbf{F},$$

from which an identity for $(\mathbf{K}\wedge\mathbf{K})\cdot(\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F})$ can be easily established.

The method of proof of these identities is amply illustrated in the proof of (6) given below. Noting that $\mathbf{F}\cdot(\mathbf{a})_{(n)}=(\mathbf{a})_{(n)}[\mathbf{F}]$, we find that

$$\mathbf{F}\cdot(\mathbf{G}\wedge\mathbf{H})=\big[\big(\mathbf{F}\cdot(\mathbf{a})_{(n)}\big)\wedge(\mathbf{b})^T_{(n)}\big]\cdot\big[(\mathbf{a})_{(n)}\wedge[\mathbf{G}](\mathbf{b})^T_{(n)}\wedge(\mathbf{a})_{(n)}\wedge[\mathbf{H}](\mathbf{b})^T_{(n)}\big]$$

$$=\big(\mathbf{F}\cdot(\mathbf{a})_{(n)}\big)\cdot\big[[\mathbf{G}](\mathbf{b})^T_{(n)}\wedge\mathbf{H}+\mathbf{G}\wedge[\mathbf{H}](\mathbf{b})^T_{(n)}\big]$$

$$=(\mathbf{F}\cdot\mathbf{G})\mathbf{H}+\big[\big(\mathbf{F}\cdot(\mathbf{a})_{(n)}\big)\cdot\mathbf{H}\big]\wedge[\mathbf{G}](\mathbf{b})^T_{(n)}+\big[\big(\mathbf{F}\cdot(\mathbf{a})_{(n)}\big)\cdot\mathbf{G}\big]\wedge[\mathbf{H}](\mathbf{b})^T_{(n)}+(\mathbf{F}\cdot\mathbf{H})\mathbf{G}$$

$$=(\mathbf{F}\cdot\mathbf{G})\mathbf{H}+(\mathbf{F}\cdot\mathbf{H})\mathbf{G}-(\mathbf{a})_{(n)}\big([\mathbf{H}][\mathbf{F}][\mathbf{G}]+[\mathbf{G}][\mathbf{F}][\mathbf{H}]\big)(\mathbf{b})^T_{(n)}.$$

By using the above formulas, we find in general that

$$\mathbf{F}=(\mathbf{F}\cdot\mathbf{K})\mathbf{K}-\frac{1}{2}\mathbf{F}\cdot(\mathbf{K}\wedge\mathbf{K}).$$

The bivector that represents the product of two matrices $[\mathbf{F}]$ and $[\mathbf{G}]$ is given by

$$(\mathbf{a})_{(n)}[\mathbf{F}][\mathbf{G}]\wedge(\mathbf{b})^T_{(n)}=\frac{1}{2}\Big(\mathbf{F}\otimes\mathbf{G}+(\mathbf{F}\cdot\mathbf{K})\mathbf{G}+(\mathbf{G}\cdot\mathbf{K})\mathbf{F}-\mathbf{K}\cdot(\mathbf{F}\wedge\mathbf{G})\Big),$$

which can be used to calculate higher order matrix products.

Given a bivector $\mathbf{F}\in\Omega^2_{n,n}$, we now calculate its inverse $\mathbf{G}=\mathbf{F}^{-1}$. Since $\mathbf{F}\mathbf{G}=1$, it follows that

$$\mathbf{F}\cdot\mathbf{G}=1,\quad \mathbf{F}\otimes\mathbf{G}=0,\quad\text{and}\quad \mathbf{F}\wedge\mathbf{G}=0.$$

Since $\mathbf{F}\wedge\mathbf{G}=0$, it follows using identity (6) that

$$0=\mathbf{F}\cdot(\mathbf{F}\wedge\mathbf{G})=(\mathbf{F}\cdot\mathbf{F})\mathbf{G}+\mathbf{F}-2(\mathbf{a})_{(n)}\wedge[\mathbf{F}]^2[\mathbf{G}](\mathbf{b})^T_{(n)},$$

or

$$\mathbf{F}=2(\mathbf{a})_{(n)}\wedge[\mathbf{F}]^2[\mathbf{G}](\mathbf{b})^T_{(n)}-(\mathbf{F}\cdot\mathbf{F})\mathbf{G}.$$

Solving this last equation for the matrix $[\mathbf{G}]=[\mathbf{F}^{-1}]$, we find that

$$[\mathbf{G}]=\Big(2[\mathbf{F}]^2-(\mathbf{F}\cdot\mathbf{F})[\mathbf{K}]\Big)^{-1}[\mathbf{F}] \tag{7}$$

where, of course, $[\mathbf{K}]$ is the identity $(n\times n)$-matrix.

The exponential function is key to relating Lie algebras to their corresponding Lie groups. Here we establish a key identity. Let $\mathbf{F},\mathbf{B}\in\Omega^2_{n,n}$. We have

$$e^{\frac{t}{2}\mathbf{F}}\mathbf{B}e^{-\frac{t}{2}\mathbf{F}}=(\mathbf{a})_{(n)}\wedge e^{t[\mathbf{F}]}[\mathbf{B}]e^{-t[\mathbf{F}]}(\mathbf{b})^T_{(n)}. \tag{8}$$

This important identity follows from the closely related identities

$$e^{\frac{t}{2}\mathbf{F}}(\mathbf{a})_{(n)}e^{-\frac{t}{2}\mathbf{F}}=(\mathbf{a})_{(n)}e^{t[\mathbf{F}]},\quad\text{and}\quad e^{\frac{t}{2}\mathbf{F}}(\mathbf{b})^T_{(n)}e^{-\frac{t}{2}\mathbf{F}}=e^{-t[\mathbf{F}]}(\mathbf{b})^T_{(n)}, \tag{9}$$

which are established by noting that the Taylor series expansion of both sides around $t=0$ agree for all powers of the expansion. Successively differentiating the identites in (9), gives the useful identities

$$\mathbf{F}^{(k)}\otimes(\mathbf{a})_{(n)}=(\mathbf{a})_{(n)}[\mathbf{F}]^k,\quad\text{and}\quad \mathbf{F}^{(k)}\otimes(\mathbf{b})^T_{(n)}=(-1)^k[\mathbf{F}]^k(\mathbf{b})^T_{(n)}.$$

## 2 Linear algebra of bivectors

Classification schemes for a linear operator are usually built around the characteristic and minimal polinomials of that operator. A modification of that scheme can used to classify a bivector

$$\mathbf{F} = (\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})^T_{(n)} \in \Omega^2_{n,n},$$

defined by the matrix $[\mathbf{F}]$.

First, consider the bivector $\mathbf{K} = \sum_{i=1}^n \mathbf{a}_i \wedge \mathbf{b}_i$ of the identity matrix $[1]_n$. We have the following

**Identity Form 2.1** *For even $n = 2k$, the bivector $\mathbf{K}$ satisfies the minimal polynomial*

$$\mathbf{K}\prod_{r=1}^{k}(\mathbf{K}-2r)(\mathbf{K}+2r), \tag{10}$$

*and for odd $n = 2k+1$, the bivector $\mathbf{K}$ satisfies the minimal polynomial*

$$\prod_{r=0}^{k}(\mathbf{K}-2r-1)(\mathbf{K}+2r+1). \tag{11}$$

**Proof:**

The proof proceeds by induction. Assuming that the theorem is true for $n = k$, we prove that it is true for $n = k+1$ by showing that when the polynomial for $n = k+1$ is divided by the two appropriate lower order polynomials, the remainder is zero.

□

The minimal polynomials of $\mathbf{K}$ for even and odd $n$ are examples of *interlacing polynomials*. Each higher even or odd order polynomials contain the zeros of the lower even or odd order polynomials, respectively.

We now characterize index of nilpotency for nilpotens $\mathbf{N}_k$ of the form, for $k < n$,

$$\mathbf{N}_k = \sum_{r=1}^{k} \mathbf{a}_r \wedge \mathbf{b}_{r+1} = (\mathbf{a})_{(n)} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & \dots & & \dots \\ \dots & & \dots & & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \wedge (\mathbf{b})^T_{(n)},$$

where there are $k$-one's off the main diagonal. The matrix $[\mathbf{N}]$ has *nilpotency* of order $k+1$ in-so-far as that $[\mathbf{N}]^{k+1} = 0$. However, the nilpotency $nil(\mathbf{N}_k)$ of the bivector $\mathbf{N}_k$ is considerably more complicated as the following theorem shows.

**Nilpotent Form 2.2** *The nilpotency of $\mathbf{N}_k$ is given by the recursive formula for $k \ge 1$,*

$$nil(\mathbf{N}_0) := nil(0) = 1, \quad \text{and} \quad nil(\mathbf{N}_k) = nil(\mathbf{N}_{k-1}) + \left[\frac{k+1}{2}\right]. \tag{12}$$

**Proof:** Assume by induction that the theorem is true for $k$. Then for $k+1$, write $\mathbf{N}_{k+1} = \mathbf{N}_k + \mathbf{a}_{k+1}\mathbf{b}_{k+2}$ and calculate

$$\mathbf{N}^2_{k+1} = \mathbf{N}^2_k + \mathbf{a}_{k+1}\mathbf{b}_{k+2}\mathbf{N}_k + \mathbf{N}_k\mathbf{a}_{k+1}\mathbf{b}_{k+2}$$

$$= \mathbf{N}_k^2 + \mathbf{a}_{k+1}\mathbf{b}_{k+2}\big(\mathbf{N}_{k-1} + \mathbf{a}_k\mathbf{b}_{k+1}\big) + \big(\mathbf{N}_{k-1} + \mathbf{a}_k\mathbf{b}_{k+1}\big)\mathbf{a}_{k+1}\mathbf{b}_{k+2}$$

$$= \mathbf{N}_k^2 + 2\Big(\mathbf{N}_{k-1}\mathbf{a}_{k+1} + \mathbf{b}_{k+1} \wedge \mathbf{a}_{k+1}\mathbf{a}_k\Big)\mathbf{b}_{k+2}.$$

At this point the proof is incomplete... Values for which the nilpotency of $\mathbf{N}_k$ are known are given in the Table below. The last two values for $nil(\mathbf{N}_8) = 21$ and $nil(\mathbf{N}_9 = 26)$ were calculated with Mathematica, but by eliminating some of the higher order terms in the expansions. Thus, they can be in error. The general proof is still an open problem, so the general theorem is at this point only a conjecture.

□

| | $\mathbf{N}_0 = 0$ | $\mathbf{N}_1$ | $\mathbf{N}_2$ | $\mathbf{N}_3$ | $\mathbf{N}_4$ | $\mathbf{N}_5$ | $\mathbf{N}_6$ | $\mathbf{N}_7$ | $\mathbf{N}_8$ | $\mathbf{N}_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $nil(\mathbf{N}_k)$ | 1 | 2 | 3 | 5 | 7 | 10 | 13 | 17 | 21 | 26 |

Given a bivector $\mathbf{F} = (\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})_{(n)}^T \in \Omega_{n,n}^2$, and an invertible matrix $A$, we see from

$$\mathbf{F} = (\mathbf{a})_{(n)}[\mathbf{F}] \wedge (\mathbf{b})_{(n)}^T = (\mathbf{a})_{(n)}AA^{-1}[\mathbf{F}]A \wedge A^{-1}(\mathbf{b})_{(n)}^T,$$

how the bivector is re-expressed in the new basis of $\mathbb{R}^{n,n}$ given by $(\mathbf{a}')_{(n)} = (\mathbf{a})_{(n)}A$ and $(\mathbf{b}')_{(n)}^T = A^{-1}(\mathbf{b})_{(n)}^T$. Of course, the new basis vectors $(\mathbf{a}')_{(n)}^T$ and $(\mathbf{b}')_{(n)}$ make up a new Witt basis satisfying (2). From this it follows that we can apply all the usual rules for changing the basis for a linear operator and still maintain the properties of a Witt basis for the corresponding bivector basis of the algebra $\Omega_{n,n}$. In particular, we can always find a prefered basis in which the matrix $[\mathbf{F}]$ of the bivector $\mathbf{F}$ consists of Jordan blocks. Armed with the **Theorems 2.1**, and **2.2**, above, we can efficiently solve the corresponding *eigenbivector problem* for the bivector $\mathbf{F}$ by using the *spectral decomposition* of the matrix $[\mathbf{F}] = \sum_{k=1}^r [\mathbf{J}_k]$ into the Jordan blocks $[\mathbf{J}_k]$.

Thus, suppose that the bivector $\mathbf{F} = \sum_{k=1}^r \mathbf{J}_k$, where the bivectors $\mathbf{J}_k$ correspond to the Jordan blocks $[\mathbf{J}_k]$ will be commuting. We now study the structure of the bivector $\mathbf{J} = \mathbf{J}_k$ of a single block $[\mathbf{J}_k]$, from which the totality of the structure of $\mathbf{F}$ follows. We first note that each Jordan block bivector $\mathbf{J}$ can itself be broken down in a commuting sum of a nilpotent bivector $\mathbf{N}$ and a sub-identity block bivector $\mathbf{K}$, $\mathbf{J} = \lambda\mathbf{K} + \mathbf{N}$. Whereas the minimal polynomial of the matrix $[\mathbf{F}]$ has the form

$$\varphi(x) := \prod_{k=1}^{r}(x - \lambda_k)^{m_k},$$

the corresponding minimal polynomial of the bivector $\mathbf{F}$ is more complicated, but can still be constructed by using the knowledge of the minimal polynomial $\varphi(x)$ of $[\mathbf{F}]$, and the Jordan block structure of each $[\mathbf{J}] = \lambda[\mathbf{K}] + [\mathbf{N}]$ for the bivector $\mathbf{J} = \mathbf{J}_k$.

| Bivector | Minimal Polynomial | Constraint |
|---|---|---|
| $\mathbf{J} = \mathbf{K}_2 + \mathbf{N}_1$ | $\mathbf{J}^2(\mathbf{J}^2 - 4)$ | $\mathbf{N}_1\mathbf{K}_2$ |
| $\mathbf{K}_2$ | $\mathbf{K}_2(\mathbf{K}_2^2 - 4)$ | |
| $\mathbf{J} = \mathbf{K}_3 + \mathbf{N}_2$ | $(\mathbf{J}^2 - 1)^3(\mathbf{J}^2 - 9)$ | $\mathbf{N}_2(\mathbf{K}_3^2 - 1)$ |
| $\mathbf{K}_3$ | $(\mathbf{K}_3^2 - 1)(\mathbf{K}_3^2 - 9)$ | |
| $\mathbf{J} = \mathbf{K}_4 + \mathbf{N}_1$ | $\mathbf{J}^2(\mathbf{J}^2 - 4)^2(\mathbf{J}^2 - 16)$ | $\mathbf{N}_1\mathbf{K}_4(\mathbf{K}_4^2 - 4)$ |
| $\mathbf{K}_4$ | $\mathbf{K}_4(\mathbf{K}_4^2 - 4)(\mathbf{K}_4^2 - 16)$ | |
| $\mathbf{J} = \mathbf{K}_4 + \mathbf{N}_2$ | $\mathbf{J}^3(\mathbf{J}^2 - 4)^3(\mathbf{J}^2 - 16)$ | $\mathbf{N}_2\mathbf{K}_4(\mathbf{K}_4^2 - 4)$ |
| $\mathbf{J} = \mathbf{K}_4 + \mathbf{N}_3$ | $\mathbf{J}^5(\mathbf{J}^2 - 4)^4(\mathbf{J}^2 - 16)$ | $\mathbf{N}_3\mathbf{K}_4(\mathbf{K}_4^2 - 4),\ \mathbf{N}_3^4\mathbf{K}_4$ |
| $\mathbf{J} = \mathbf{K}_5 + \mathbf{N}_2$ | $(\mathbf{J}^2 - 1)^3(\mathbf{J}^2 - 9)^3(\mathbf{J}^2 - 25)$ | $\mathbf{N}_2(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9)$ |
| $\mathbf{K}_5$ | $(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9)(\mathbf{K}_5^2 - 25)$ | |
| $\mathbf{J} = \mathbf{K}_5 + \mathbf{N}_3$ | $(\mathbf{J}^2 - 1)^5(\mathbf{J}^2 - 9)^4(\mathbf{J}^2 - 25)$ | $\mathbf{N}_3(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9)$ |
| $\mathbf{J} = \mathbf{K}_5 + \mathbf{N}_4$ | $(\mathbf{J}^2 - 1)^7(\mathbf{J}^2 - 9)^7(\mathbf{J}^2 - 25)$ | $\mathbf{N}_4(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9)$ |
| $\mathbf{J} = \mathbf{K}_6 + \mathbf{N}_2$ | $\mathbf{J}^3(\mathbf{J}^2 - 4)^3(\mathbf{J}^2 - 16)^3(\mathbf{J}^2 - 36)$ | $\mathbf{N}_2\mathbf{K}_6(\mathbf{K}_6^2 - 4)(\mathbf{K}_6^2 - 16)$ |
| $\mathbf{K}_6$ | $\mathbf{K}_6(\mathbf{K}_5^2 - 4)(\mathbf{K}_6^2 - 16)(\mathbf{K}_6^2 - 36)$ | $\mathbf{N}_5^9\mathbf{K}_6,\ \mathbf{N}_7^{16}\mathbf{K}_8$ |
| $\mathbf{J} = \mathbf{K}_6 + \mathbf{N}_3$ | $\mathbf{J}^5(\mathbf{J}^2 - 4)^5(\mathbf{J}^2 - 16)^4(\mathbf{J}^2 - 36)$ | $\mathbf{N}_3\mathbf{K}_6(\mathbf{K}_6^2 - 4)(\mathbf{K}_6^2 - 16)$ |
| $\mathbf{J} = \mathbf{K}_7 + \mathbf{N}_5$ | $(\mathbf{J}^2 - 1)^{10}(\mathbf{J}^2 - 9)^{10}(\mathbf{J}^2 - 25)^{10}(\mathbf{J}^2 - 49)$ | $\mathbf{N}_5(\mathbf{K}_7^2 - 25)(\mathbf{K}_7^2 - 9)(\mathbf{K}_7^2 - 1)$ |
| $\mathbf{K}_7$ | $(\mathbf{K}_7^2 - 1)(\mathbf{K}_7^2 - 9)(\mathbf{K}_7^2 - 25)(\mathbf{K}_7^2 - 49)$ | |

All the values of the Table can be generalized to apply to $\mathbf{J} = \lambda\mathbf{K}_n + \mathbf{N}_m$ For example, for $\mathbf{J} = \lambda\mathbf{K}_5 + \mathbf{N}_2$, the generalized Minimal Polynomial (middle column) is

$$(\mathbf{J}^2 - \lambda^2)^3(\mathbf{J}^2 - 9\lambda^2)^3(\mathbf{J}^2 - 25\lambda^2).$$

The constraints on the right column of the table are easily established. For example, to prove that $\mathbf{N}_1(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9) = 0$, we note that $\mathbf{N}_1\mathbf{K}_n^k = \mathbf{K}'_{n-2}$ where

$$\mathbf{K}'_{n-2} := \sum_{i=3}^{n} \mathbf{a}_i \wedge \mathbf{b}_{i+1}.$$

It then follows that

$$\mathbf{N}_1(\mathbf{K}_5^2 - 1)(\mathbf{K}_5^2 - 9) = (\mathbf{K}_3'^2 - 1)(\mathbf{K}_3'^2 - 9) = 0.$$

The higher order constraints follow by linearity, since $\mathbf{N}_k = \sum_{i=1}^{k} \mathbf{a}_i\mathbf{b}_{i+1}$, and each term $\mathbf{a}_i \wedge \mathbf{b}_{i+1}$ multiplied by $\mathbf{K}_n$ gives

$$\mathbf{a}_i\mathbf{b}_{i+1}\mathbf{K}_n = \mathbf{K}'_{n-2}.$$

Once the constaints in the right column are established, the minimal polynomials in the middle column follow as a consequence of the binomial expansion of $\mathbf{J}^n = (\mathbf{N} + \mathbf{K})^n$, taking into consideration the extra constraints of the form $\mathbf{N}_{2k+1}^{nil(2k+1)-1}\mathbf{K}_{2k+2} = 0$ for $k \geq 1$.

There are many other identities which greatly simplify calculations with nilpotents of the form

$$\mathbf{N}_k = \sum_{i=1}^{k} \mathbf{a}_i\mathbf{b}_{i+1}.$$

We have

$$(\wedge^{k-1}\mathbf{N}_k)(\wedge^k\mathbf{N}_k) = (\wedge^{k-1}\mathbf{N}_k)\cdot(\wedge^k\mathbf{N}_k) = 0.$$

$$(\wedge^n\mathbf{N}_k)\cdot(\wedge^n\mathbf{N}_k) = 0 \;\text{ for all }\; n \le k, \;\text{ and }\; \wedge^n\mathbf{N}_k = 0 \;\text{ for }\; n > k.$$

$$\mathbf{N}_k \otimes [\mathbf{N}_k\cdot(\mathbf{N}_k\wedge\mathbf{N}_k)] = 0,$$

$$(n-2)\mathbf{N}_k\cdot\Big(\wedge^n\mathbf{N}_k\Big) = n\mathbf{N}_k\wedge\Big[\mathbf{N}_k\cdot\Big(\wedge^{n-1}\mathbf{N}_k\Big)\Big].$$

Special cases of the last identity are

$$\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\big) = 3\mathbf{N}_k\wedge\big[\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\big)\big],$$

$$\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\big) = 2\mathbf{N}_k\wedge\big[\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\big)\big],$$

and

$$\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\big) = \frac{5}{3}\mathbf{N}_k\wedge\big[\mathbf{N}_k\cdot\big(\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\wedge\mathbf{N}_k\big)\big].$$

We also have the very important identities

$$\mathbf{N}_3^4 = 4\mathbf{N}_3\wedge\big[\mathbf{N}_3\cdot(\mathbf{N}_3\wedge\mathbf{N}_3)\big] = \frac{4}{3}\mathbf{N}_3\big(\mathbf{N}_3\wedge\mathbf{N}_3\wedge\mathbf{N}_3\big),$$

from which it follows from the previous Table that

$$\mathbf{N}_3^5 = \frac{4}{3}\mathbf{N}_3^2\big(\mathbf{N}_3\wedge\mathbf{N}_3\wedge\mathbf{N}_3\big) = 0.$$

This last result generalizes nicely to

$$\mathbf{N}_4^7 = \mathbf{N}_4^3\big(\mathbf{N}_4\wedge\mathbf{N}_4\wedge\mathbf{N}_4\wedge\mathbf{N}_4\big) = 0,$$

and even more generally to

$$\mathbf{N}_n^{nil(\mathbf{N}_n)} = \mathbf{N}_n^{nil(\mathbf{N}_n)-n}\Big(\wedge^n\mathbf{N}_n\Big) = 0,$$

which gives us the interesting recursive relationship

$$nil(\mathbf{N}_n) = nil(\mathbf{N}_{n-2}) + n,$$

where $n \ge 2$, for the index of nilpotency $nil(\mathbf{N}_n)$. Using these recursive relationships, we find that for all integers $r \ge 0$

$$nil(\mathbf{N}_{2r}) = r^2 + r + 1, \quad \text{and} \quad nil(\mathbf{N}_{2r-1}) = r^2 + 1,$$

which gives $nil(\mathbf{N}_0) \equiv 1 \equiv nil(\mathbf{N}_{-1})$ when $r = 0$.

Let us carry out the calculations for $\mathbf{N}_k^n$ for a number of steps to see what is going on. We find that

$$\mathbf{N}_k^n = \mathbf{N}^{n-2}\mathbf{N}_k\wedge\mathbf{N}_k = \mathbf{N}_k^{n-3}\Big(\mathbf{N}_k\cdot(\mathbf{N}_k\wedge\mathbf{N}_k) + \wedge^3\mathbf{N}_k\Big)$$

$$= \mathbf{N}_k^{n-4}\Big(\frac{4}{3}\mathbf{N}_k\cdot(\wedge^3\mathbf{N}_k) + \wedge^4\mathbf{N}_k\Big) = \mathbf{N}_k^{n-5}\Big(\frac{4}{3}(\wedge^2\mathbf{N}_k)\cdot(\wedge^3\mathbf{N}_k) + \frac{5}{3}\mathbf{N}_k\cdot(\wedge^4\mathbf{N}_k) + \wedge^5\mathbf{N}_k\Big)$$

$$= \mathbf{N}_k^{n-6}\Big[\frac{4}{3}\Big((\wedge^2\mathbf{N}_k)\cdot(\wedge^3\mathbf{N}_k)\Big)\wedge\mathbf{N}_k + \frac{5}{3}(\wedge^2\mathbf{N}_k)\cdot(\wedge^4\mathbf{N}_k) + 2\mathbf{N}_k\cdot(\wedge^5\mathbf{N}_k) + \wedge^6\mathbf{N}_k\Big] = \cdots,$$

the steps terminating after $n$-iterations when the first term becomes $\mathbf{N}_k^{n-n} = 1$.

## 3 The bivector algebra $\Omega^2_{3,3}$

As a simple example, let us consider the subgroup $SO(3)$ of rotations of the general linear group $GL(3)$ in $\Omega^2_{3,3}$. The matrices of the generators of rotations in the Lie algebra $so(3)$ are

$$[\mathbf{A}_z] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ [\mathbf{A}_y] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ [\mathbf{A}_x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

corresponding to the bivectors

$$\mathbf{A}_3 = -\mathbf{a}_1\mathbf{b}_2 + \mathbf{a}_2\mathbf{b}_1, \quad \mathbf{A}_2 = \mathbf{a}_1\mathbf{b}_3 - \mathbf{a}_3\mathbf{b}_1, \quad \mathbf{A}_1 = -\mathbf{a}_2\mathbf{b}_3 + \mathbf{a}_3\mathbf{b}_2.$$

Each of these bivectors $\mathbf{A}_k$ are the generators of rotations around the $z$, $y$ and $x$ axes, respectively, and satisfy the minimal polynomial

$$(\mathbf{A}_k^2 + 4)\mathbf{A}_k = (\mathbf{A}_k - 2i)(\mathbf{A}_k + 2i)\mathbf{A}_k = 0.$$

The spectral basis for this minimal polynomial is

$$s_1 = \frac{(\mathbf{A}_k + 2i)\mathbf{A}_k}{-8}, \quad s_2 = \frac{(\mathbf{A}_k - 2i)\mathbf{A}_k}{-8}, \quad s_3 = \frac{\mathbf{A}_k^2 + 4}{4}, \tag{13}$$

and the spectral equation for $\mathbf{A}_k$ is

$$\mathbf{A}_k = 2is_1 - 2is_2 + 0s_3 = 2i(s_1 - s_2) + 0s_3. \tag{14}$$

For the vector $\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$, a counterclockwise rotation, through the angle $\theta$, around the $k$-axis is specified by

$$\mathbf{x}' = e^{\frac{1}{2}\theta\mathbf{A}_k}\mathbf{x}e^{-\frac{1}{2}\theta\mathbf{A}_k}.$$

Using (14), and the properties of the spectral basis (13), we find that

$$e^{\frac{1}{2}\theta\mathbf{A}_k} = \frac{1}{4}\Big[(1-\cos\theta)\mathbf{A}_k^2 + 2\sin\theta\mathbf{A}_k + 4\Big].$$

For a rotation around the $x$-axis, we find that

$$\mathbf{x}' = \frac{1}{16}\Big[(1-\cos\theta)\mathbf{A}_1^2 + 2\sin\theta\mathbf{A}_1 + 4\Big]\mathbf{x}\Big[(1-\cos\theta)\mathbf{A}_1^2 - 2\sin\theta\mathbf{A}_1 + 4\Big]$$
$$= (x_1\cos\theta - x_2\sin\theta)\mathbf{a}_1 + (x_1\sin\theta + x_2\cos\theta)\mathbf{a}_2 + x_3\mathbf{a}_3.$$

Let us consider another example of a subgroup of $GL(3)$. In this case we take the generators of the corresponding Lie algebra to be

$$[\mathbf{B}_z] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ [\mathbf{B}_y] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ [\mathbf{B}_x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

corresponding to the bivectors

$$\mathbf{B}_3 = \mathbf{a}_1\mathbf{b}_2 + \mathbf{a}_2\mathbf{b}_1, \quad \mathbf{B}_2 = \mathbf{a}_1\mathbf{b}_3 + \mathbf{a}_3\mathbf{b}_1, \quad \mathbf{B}_1 = \mathbf{a}_2\mathbf{b}_3 + \mathbf{a}_3\mathbf{b}_2.$$

Each of these bivectors $\mathbf{B}_k$ are the generators of hyperbolic rotations around the $z$, $y$ and $x$ axes, respectively, and satisfy the minimal polynomial

$$(\mathbf{B}_k^2 - 4)\mathbf{B}_k = (\mathbf{B}_k - 2)(\mathbf{B}_k + 2)\mathbf{B}_k = 0.$$

The spectral basis for this minimal polynomial is

$$s_1 = \frac{(\mathbf{B}_k + 2)\mathbf{B}_k}{8}, \quad s_2 = \frac{(\mathbf{B}_k - 2)\mathbf{A}_k}{8}, \quad s_3 = \frac{\mathbf{B}_k^2 - 4}{-4}, \tag{15}$$

and the spectral equation for $\mathbf{B}_k$ is

$$\mathbf{B}_k = 2s_1 - 2s_2 + 0s_3 = 2(s_1 - s_2) + 0s_3. \tag{16}$$

Using (16), and the properties of the spectral basis (15), we find that

$$e^{\frac{1}{2}\phi\mathbf{B}_k} = \frac{1}{4}\Big[(\cosh\phi - 1)\mathbf{B}_k^2 + 2\sinh\phi\mathbf{B}_k + 4\Big].$$

For a hyperbolic rotation around the $x$-axis, we find that

$$\mathbf{x}' = \frac{1}{16}\Big[(\cosh\phi - 1)\mathbf{B}_1^2 + 2\sinh\phi\mathbf{B}_1 + 4\Big]\mathbf{x}\Big[(\cosh\phi - 1)\mathbf{B}_1^2 - 2\sinh\phi\mathbf{B}_1 + 4\Big]$$

$$= (x_1\cosh\phi + x_2\sinh\phi)\mathbf{a}_1 + (x_1\sinh\phi + x_2\cosh\phi)\mathbf{a}_2 + x_3\mathbf{a}_3.$$

# Acknowledgement

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