Geometric Algebra, Spectral Basis and Interpolation

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February 24, 2006

El Álgebra Geométrica de Clifford es una extención del concepto de los números reales para incluir el concepto general de dirección. El Bases Espectral de polinomios es una concepto fundamental para hacer Lagrange, Hermite, Padé, y otros tipos de interpolación. Damos aplicaciones.

PART I: Geometric Extension of Number.

PART II: Spectral Basis and Rational Interpolation.

PART III: Álgebra Geométrica Conforme in $\mathbb{G}_{4,1}$

Geometric Algebra, Spectral Basis and Interpolation

PART I: Geometric Extension of Number.

1. The real number system $\mathbb{R}$.

2. $i := \sqrt{-1} \notin \mathbb{R}$ leads to *complex numbers* $\mathbb{C}$.

   a) $z = x + iy \in \mathbb{C}$ in the *standard basis* $\{1, i\}$.

   b) $i^2 = -1$ and $\mathbb{C}$ enjoys all the rules of $\mathbb{R}$.

   c) $\mathbb{C}$ is a *field*.

   d) $\mathbb{C}$ is algebraically closed.

   e) Euler formula: $z = x + iy = r \exp i\theta$.

   f) Euclidean distance: $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

   g) Equation of circle with radius r:

   $z\overline{z} = x^2 + y^2 = r^2$.

3. Let $u := \sqrt{1} \notin \mathbb{R}$ leads to *hyperbolic numbers* $w = x + uy \in \mathbb{H}$.

   a) $w = x + uy \in \mathbb{H}$ in the *standard basis* $\{1, u\}$.

   b) $u^2 = 1$ and $\mathbb{H}$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{H}$ has divisors of zero.

   c) $w = w_+ u_+ + w_- u_-$ in the *spectral basis* $\{u_+, u_-\}$ where $w_+ = x + y$ and $w_- = x - y$, and

   $u_+ = \frac{1}{2}(1 + u)$ and $u_- = \frac{1}{2}(1 - u)$. Note that

   $u_+ + u_- = 1$, $u_+^2 = u_+$, $u_-^2 = u_-$, $u_+ u_- = 0$.

d) $I\!H$ is a *ring.*

e) $I\!H$ is NOT algebraically closed.

f) Euler formula: $w = x + uy = \pm\rho \exp u\phi$,

$w = x + uy = \pm\rho u \exp u\phi$.

g) Hyperbolic distance: $|w_1 - w_2| = \sqrt{|(x_1 - x_2)^2 - (y_1 - y_2)^2|}$.

h) Equation of hyperbola (4 branches):

$|ww^-| = |x^2 - y^2| = \rho^2$.

4. *Complex Hyperbolic Numbers*

$I\!H_{\mathbb{C}} = span_{I\!R}\{1, u, i, ui = iu\}$, $w = z_1 + z_2 u$.

a) $w = z_1 + z_2 u \in I\!H_{\mathbb{C}}$ in the *standard basis over* $\mathbb{C}$ $\{1, u\}$.

b) $I\!H_{\mathbb{C}}$ enjoys all the rules of $I\!R$, except $I\!H$ has zero divisors.

c) $w = w_+ u_+ + w_- u_-$ in the *spectral basis*$_{\mathbb{C}}$ $= \{u_+, u_-\}$ where $w_+ = z_1 + z_2$ and $w_- = z_1 - z_2$,

$u_+ = \frac{1}{2}(1 + u)$ and $u_- = \frac{1}{2}(1 - u)$. Note that

$u_+ + u_- = 1$, $u_+^2 = u_+$, $u_-^2 = u_-$, $u_+ u_- = 0$.

d) $I\!H_{\mathbb{C}}$ is a *ring.*

e) $\mathbb{H}_{\mathbb{C}}$ is algebraically closed:

The zeros of any polynomial

$f(w) = f(w_+)u_+ + f(w_-)u_-$, are just the complex zeros of $f(w_+) = 0$ and $f(w_-) = 0$.

f) Euler formula:

$w = z_1 + z_2 u = Z(\frac{z_1}{Z} + \frac{z_1}{Z}u) = Z \exp \Omega u,$

$Z = \sqrt{z_1^2 - z_2^2}$, and Ω is complex hyperbolic angle. (Singular at $z_1 = \pm z_2$.)

g) Complex Hyperbolic distance to origin:

$|w| = \sqrt{Z\overline{Z}}.$

h) Equation of 3-D hypersurface:

$Z\overline{Z} = \rho^2.$

5. Geometric Numbers of the Plane $\mathbb{G}_2$.

$\mathbb{G}_2 = span_{\mathbb{R}}\{1, e_1, e_2, e_1 e_2\},$

where $e_1^2 = e_2^2 = 1$ and $e_{12} := e_1 e_2 = -e_2 e_1$.

$i = e_{12}$ has the geometric interpretation of a *unit bivector* in the plane of e_1 and e_2, and $i^2 = -1$.

a) $g = (\alpha_1 + \alpha_2 e_{12}) + (v_1 e_1 + v_2 e_2)$

in the *standard basis* $\{1, e_1, e_2, e_1 e_2\}$.

b) $\mathbb{G}_2$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_2$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_2$ is

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u_+ & e_1 u_- \\ e_1 u_+ & u_- \end{pmatrix},$$

where $u_\pm = \frac{1}{2}(1 \pm e_2)$.

d) Let $g = (\alpha_1 + \alpha_2 i) + (v_1 e_1 + v_2 e_2)$. Then

$$g = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} \alpha_1 + v_2 & v_1 - \alpha_2 \\ v_1 + \alpha_2 & \alpha_1 - v_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.$$

$$[g] = u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_-$$

$$= \begin{pmatrix} \alpha_1 + v_2 & v_1 - \alpha_2 \\ v_1 + \alpha_2 & \alpha_1 - v_2 \end{pmatrix}$$

is called the (real) *matrix* of g.

e) $\mathbb{G}_2$ is algebraically closed:

Using d) above, the zeros of any polynomial $f(g)$ can be defined in terms of the zeros of the matrix $f([g])$.

6. Geometric Numbers of the 3-Space $\mathbb{G}_3$.

$\mathbb{G}_3 = span_{\mathbb{R}}\{1, e_1, e_2, e_3, e_2 e_3, e_3 e_1, e_1 e_2, i = e_{123}\}$, where $e_1^2 = e_2^2 = e_3^2 = 1$ and $e_{ij} := e_i e_j = -e_j e_i$ for $i \neq j$. $i = e_{123}$ has the geometric interpretation of a *unit trivector* in the 3-space of e_1, e_2, e_3, and $i^2 = -1$.

a) $g = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

in the *standard basis* $\{1, e_1, e_2, e_3\}_{\mathbb{C}}$.

b) $\mathbb{G}_3$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_3$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_3$ is

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u_+ & e_1 u_- \\ e_1 u_+ & u_- \end{pmatrix},$$

where $u_\pm = \frac{1}{2}(1 \pm e_2)$.

d) For $g = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$,

$$g = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} \alpha_0 + \alpha_2 & \alpha_1 + i\alpha_3 \\ \alpha_1 - i\alpha_3 & \alpha_0 - \alpha_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.$$

$$[g] = u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_-$$

$$= \begin{pmatrix} \alpha_0 + \alpha_2 & \alpha_1 + i\alpha_3 \\ \alpha_1 - i\alpha_3 & \alpha_0 - \alpha_2 \end{pmatrix}.$$

is called the (complex) *matrix* of g.

e) $\mathbb{G}_3$ is algebraically closed:

Using d) above, the zeros of any polynomial $f(g)$ can be difined in terms of the zeros of the matrix $f([g])$.

7. Geometric Numbers of the (4,1)-Space $\mathbb{G}_{4,1}$.

$\mathbb{G}_{4,1} = span_{\mathbb{R}}\{\mathbb{G}_3, e_4, e_5\}$, where $e_4^2 = 1 = -e_5^2$, $i := e_{12345}$ and has the geometric interpretation of a *unit 5-vector* in the 5-space of e_1, e_2, e_3, e_4, e_5, and $i^2 = -1$.

a) $g = (h_0 + ih_2) + (h_1 + ih_3)e_4$

in the *standard basis*

$\{h_0 + ih_2, (h_1 + ih_3)e_4 | \; h_0, h_1, h_2, h_3 \in \mathbb{G}_3\}_{\mathbb{G}_3}$.

b) $\mathbb{G}_{4,1}$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_{4,1}$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_{4,1}$ over $\mathbb{G}_3$ is

$$\begin{pmatrix} 1 \\ e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 & e_4 \end{pmatrix} = \begin{pmatrix} v_+ & e_4 v_- \\ e_4 v_+ & v_- \end{pmatrix},$$

where $v_\pm = \frac{1}{2}(1 \pm e_{45})$.

d) For $g = (h_0 + ih_2) + (h_1 + ih_3)e_4$,

$$g = \begin{pmatrix} 1 & e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix} g \begin{pmatrix} 1 & e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_4 \end{pmatrix} v_+ \begin{pmatrix} g & ge_4 \\ e_4 g & e_4 g e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_4 \end{pmatrix} v_+ \begin{pmatrix} h_0 + ih_2 & h_1 + ih_3 \\ \tilde{h}_1 - i\tilde{h}_3 & \tilde{h}_0 - i\tilde{h}_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_4 \end{pmatrix}.$$

$$[g] = v_+ \begin{pmatrix} g & ge_4 \\ e_4 g & e_4 g e_4 \end{pmatrix} v_+ + v_- \begin{pmatrix} e_4 g e_4 & e_4 g \\ ge_4 & g \end{pmatrix} v_-$$

$$= \begin{pmatrix} h_0 + ih_2 & h_1 + ih_3 \\ \tilde{h}_1 - i\tilde{h}_3 & \tilde{h}_0 - i\tilde{h}_2 \end{pmatrix}.$$

is called the (complex $\mathbb{G}_3$) *matrix* of g, where

for $h \in \mathbb{G}_3$, $\tilde{h} := e_4 h e_4$.

e) We can also represent g as a complex scalar 4×4 matrix, by employing the Kronecker product of matrices:

$$\begin{pmatrix} 1 & e_1 \end{pmatrix} \begin{pmatrix} 1 & e_4 \end{pmatrix} u_+ v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & e_1 & e_4 & e_{14} \end{pmatrix} u_+ v_+ \begin{pmatrix} 1 \\ e_1 \\ e_4 \\ e_{41} \end{pmatrix} = 1.$$

f) Similarly, the geometric algebras $\mathbb{G}_{n,n+1} \cong Mat_{\mathbb{C}}(2^n \times 2^n)$.

f) The geometric algebras $\mathbb{G}_{n,n+1}$ are all algebraically closed.

Using d) above, the zeros of any polynomial $f(g)$ can be defined in terms of the zeros of the matrix $f([g])$.

PART II: Spectral Basis and Rational Interpolation.

1. Let g be a geometric number and $[g]$ be its corresponding real or complex matrix.

DEFINITION: The *minimal polynomial* $h(x)$ of g is the smallest degree monic polynomial

$$h(x) := \Pi_{i=1}^{r}(x - x_i)^{m_i}$$

for distinct $x_i \in \mathbb{C}$, such that $h(g) = 0$. We shall be primarily interested in the case where $x_i \in \mathbb{R}$.

2. The algebra $\mathbb{C}[x]_{h(x)}$.

a) The *standard basis* of $\mathbb{C}[x]_{h(x)}$ is

$$\mathcal{B} = \{1, x, x^2, \ldots, x^{m-1}\}$$

where $m = \Sigma_{i=1}^{r} m_i = deg(h(x))$.

b) **Euclidean Algorithm:** Given $f(x) \in \mathbb{C}[x]$, we say that $f(x) = r(x)$ *modulo* $h(x)$, or

$$f(x) \overset{h}{=} r(x) \text{ if } f(x) = q(x)h(x) + r(x)$$

where $deg(r(x)) < deg(h(x))$ or $r(x) = 0$.

c) Operations $+$ and $\times$ are defined in $\mathbb{C}[x]_{h(x)}$ modulo $h(x)$.

d) $\mathbb{C}[x]_{h(x)}$ is a *ring*.

e) $\mathbb{C}[x]_{h(x)}$ is algebraically closed.

f) The *Spectral Basis* of $\mathbb{C}[x]_{h(x)}$ consists of *idempotents* $s_i := s_i(x)$ and *nilpotents* $q_i := q_i(x)$,

$$\mathcal{S} = \{s_1, q_1, \ldots, q_1^{m_1-1}, \ldots, s_r, q_r, \ldots, q_r^{m_1-1}\}$$

which have the characterizing properties

Property 1. $s_1 + s_2 + \cdots + s_r = 1$, and

$s_i s_j \overset{\text{h}}{=} \delta_{ij} s_i$ for $i, j = 1, \ldots, r$, where

$\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

Property 2. $q_i s_i \overset{\text{h}}{=} q_i$, and $q_i^{m_i-1} \neq 0 \; mod(h)$, but

$q_i^{m_i} \overset{\text{h}}{=} 0$, for $i = 1, \ldots, r$.

Property 3. For each $f(x) \in \mathbb{R}[x]$,

$f(x)s_i \overset{\text{h}}{=} (f(x) mod[(x - x_i)^{m_i}])s_i$

for $i = 1, \ldots, r$.

To solve for the spectral basis elements:

Define $h_i = h_i(x) = h(x)/(x - x_i)^{m_i}$.

$h_i s_i \overset{\text{h}}{=} h_i$ (Properties 1 and 3)

Since $h_i(x_i) \neq 0$,

$s_i(x) = (h_i^{-1} mod \; (x - x_i)^{m_i}) h_i(x)$.

The nilpotents q_i are given by

$q_i^k : \overset{\text{h}}{=} (x - x_i)^k s_i$

$= (x - x_i)^k (h_i^{-1} mod \; (x - x_i)^{m_i}) h_i(x)$

$\stackrel{h}{=} (h_i^{-1} mod\ (x - x_i)^{m_i - k}) h_i(x).$

for $k = 0, 1, \ldots, m_i - 1$. Note that

$$q_i^0 \stackrel{h}{=} s_i.$$

Representing x in the spectral basis $\mathcal{S}$, we get

$$x = x(\Sigma_{i=1}^r s_i) = \Sigma_{i=1}^r (x - x_i + x_i) s_i$$

$$\stackrel{h}{=} \Sigma_{i=1}^r (x_i + q_i) s_i, \ (***)$$

called the *spectral form* of $x \in \mathbb{C}[x]_h$.

It follows that

$$x^k \stackrel{h}{=} \Sigma_{i=1}^r (x_i + q_i)^k s_i.$$

3. The spectral form of a geometric number g (or matrix $[g]$.)

Since g has the minimal polynomial $h(g) = 0$,

$$g \stackrel{h}{=} \Sigma_{i=1}^r (x_i + q_i(g)) s_i(g).$$

or, for the matrix $[g]$ of g,

$$[g] \stackrel{h}{=} \Sigma_{i=1}^r (x_i + q_i([g])) s_i([g]).$$

The *spectral form* of a matrix is more fundamental than the Jordan form of a matrix.

4. Rational Interpolation.

Let $h(x) := \Pi_{i=1}^r (x - x_i)^{m_i}$

for distinct $x_i \in \mathbb{R}$.

a) Let $f(x) \in \mathbb{R}$ be continuous and have derivatives to the orders $\{m_1 - 1, \ldots, m_r - 1\}$ at $\{x_1, \ldots, x_r\}$.

b) Substituting

$x \stackrel{h}{=} \Sigma_{i=1}^{r}(x_i + q_i)s_i$, we get

$g(x) := f(\Sigma_{i=1}^{r}(x_i + q_i)s_i) \stackrel{h}{=} \Sigma_{i=1}^{r} f(x_i + q_i)s_i$

Expanding $f(x_i + q_i)$ in a Taylor series about $x = x_i$, gives

$g(x) \stackrel{h}{=} \Sigma_{i=1}^{r} \Sigma_{k=0}^{m_i-1} \frac{1}{k!} f^{(k)}(x_i)q_i^k s_i$.

c) $g(x)$ is called the *Birkhoff* or *osculating* interpolation polynomial of $f(x)$ with respect to $h(x)$.

d) If $m_1 = \cdots = m_r = 1$,

$g(x)$ is called the *Lagrange* interpolation polynomial of $f(x)$.

e) If $m_1 = \ldots = m_r = 2$,

$g(x)$ is called the *Hermite* interpolation polynomial of $f(x)$.

f) If $r = 1$, $g(x)$ is the first $m_1 - 1$ terms of the Taylor series of $f(x)$ about $x = x_1$.

g) More generally, we say that

$g(x) = \frac{a(x)}{b(x)}$

is a *rational* interpolate of $f(x)$ at

$\{x_1, \ldots, x_r\}$ with multiplicities $\{m_1, \ldots, m_r\}$

if

$f(x)b(x) - a(x) \stackrel{h}{=} 0$.

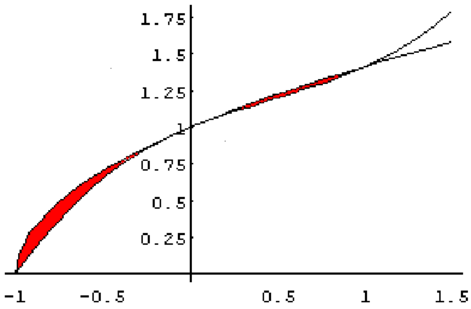

Figure 1: Polynomial Interpolation in $\mathbb{R}^2$.

h) Chebyshev and other kinds of rational interpolation are obtained by replacing the powers of x^k of x in $a(x)$ and $b(x)$ by the corresponding Chebyshev or other sets of orthogonal polynomials of the same order.

5. Cubic spline interpolation.

a) The spectral basis for $h = h(t) = t^2(t-1)^2$

is

$$\mathcal{S}_{2,2} = \{s_1 = (2t+1)(t-1)^2,$$
$$q_1 = t(t-1)^2, s_2 = (3-2t)t^2, q_2 = t^2(t-1)\}.$$

b) The piecewise *natural cubic spline*

$$\{g_1(t_1), g_2(t_2), \ldots, g_{k-1}(t_{k-1})\},$$

for $0 \le t_i < 1$ and $k \ge 3$, connecting the successive points $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ in $\mathbb{R}^n$,

is defined by

$$g_i(t_i) = \mathbf{x}_i s_1(t_i) + \mathbf{v}_i q_1(t_i) + \mathbf{x}_{i+1} s_2(t_i) + \mathbf{v}_{i+1} q_2(t_i),$$

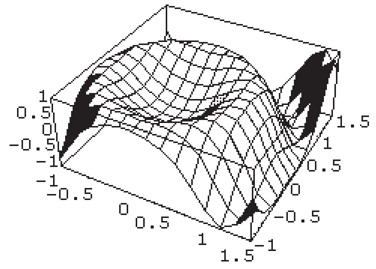

Figure 2: Polynomial Surface Interpolation in $I\!R^3$.

with the requirements that

$$g_1''(0) = 0 = g_{k-1}''(1) \quad \text{and} \quad g_i''(1) = g_{i+1}''(0)$$

for $i = 1, \ldots, k-2$.

Taking the second derivatives of $g_i(t_i)$, and evaluating at $t_i = 0, 1$ gives

$$g_i''(0) = 6(\mathbf{x}_{i+1} - \mathbf{x}_i) - 4\mathbf{v}_i - 2\mathbf{v}_{i+1},$$

and

$$g_i''(1) = -6(\mathbf{x}_{i+1} - \mathbf{x}_i) + 2\mathbf{v}_i + 4\mathbf{v}_{i+1}.$$

The resulting k linear vector equations are uniquely solved for the k-unknown tangent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

c) Similarly, a *bounded cubic spline* can be defined.

6. Rational cubic spline interpolation.

a) The rational spectral basis

$$\mathcal{R}_{2,2} = \{s_{r1}, q_{r1}, s_{r2}, q_{r2}\}$$

is defined by

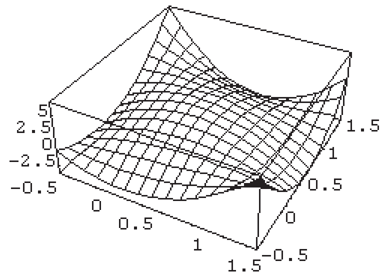

Figure 3: Polynomial Interpolating Surface in $I\!R^3$.

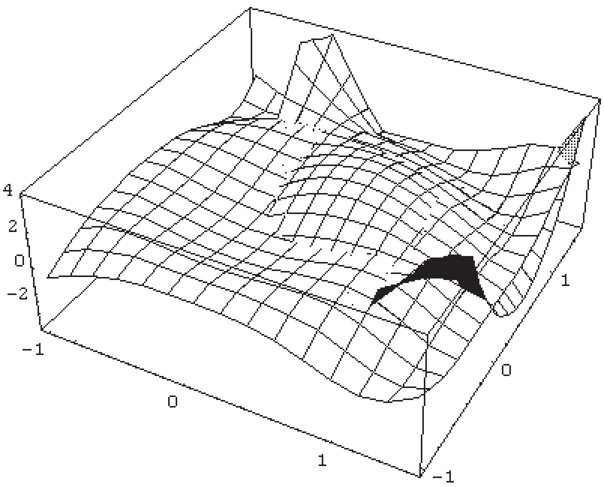

Figure 4: Polynomial Interpolation Patch $I\!R^3$.

$$s_{r1}(t) = \frac{b(t)s_1(t) \ mod(h(t))}{b(t)},$$

$$q_{r1}(t) = \frac{b(t)q_1(t) \ mod(h(t))}{b(t)}$$

and

$$s_{r2}(t) = \frac{b(t)s_2(t) \ mod(h(t))}{b(t)},$$

$$q_{r2}(t) = \frac{b(t)q_2(t) \ mod(h(t))}{b(t)},$$

where $b = b(t) = 1 + b_1t + b_2t^2 + b_3t^3$ and $b(1) \neq 0$.

b) The rational spectral basis $\mathcal{R}_{2,2}(b)$ reduces to the ordinary spectral basis $\mathcal{S}_{2,2}$, for

$b = b(t) = 1.$

c) When using the rational spectral basis, the second derivatives $g_i''(0)$ and $g_i''(1)$, must be recalculated.

7. Examples: **Circles**.

a) The rational spectral basis

$\mathcal{S}_h = \{s_{r1}, s_{r2}, s_{r3}\}$ for $b = 1 + b_1t + b_2t^2$

and $h(t) = (t+1)t(t-1)$,

is defined by

$$s_{r1} = \frac{\frac{1}{2}(1 - b_1 + b_2)t(t-1)}{1 + b_1 t + b_2 t^2},$$

$$s_{r2} = -\frac{(t-1)(t+1)}{1 + b_1 t + b_2 t^2},$$

and

$$s_{r3} = \frac{\frac{1}{2}(1 + b_1 + b_2)t(t+1)}{1 + b_1 t + b_2 t^2}.$$

Optimizing b_1 and b_2 (least squares) on semi-circle, gives

$g(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$.

b) Nurbs on quarter circle: Using the rational spectral basis

$\mathcal{R}_{2,2} = \{s_{r1}, q_{r1}, s_{r2}, q_{r2}\}$,

$g(t) = (1,0)s_{r1} + (0,v)q_{r1} + (0,1)s_{r2} + (-v,0)q_{r2}$,

Letting

$b_1 = -2 + \sqrt{2} = -b_2$, $v = \sqrt{2}$

eliminates the t^3 term in the numerator,

$$g(t) = (1,0)s_{r1} + (0,\sqrt{2})q_{r1} + (0,1)s_{r2} + (-\sqrt{2},0)q_{r2}$$

$$= (\frac{1 + (-2+\sqrt{2})t + (1-\sqrt{2})t^2}{1 + (-2+\sqrt{2})t + (2-\sqrt{2})t^2},$$

$$\frac{\sqrt{2}t + (1-\sqrt{2})t^2}{1 + (-2+\sqrt{2})t + (2-\sqrt{2}t^2)}).$$

c) For $h(t) = t^2(t-1)$,

$$\mathcal{S}_{2,1} = \{s_1 = -(t+1)(t-1), q_1 = -(-1+t)t, s_2 = t^2\}$$

Letting

$$g(t) = \frac{(1,0)s_1 + (b_1(1,0) + (0,\frac{\pi}{2}))q_1 + (0,1)(1+b_1+b_2)s_2}{1+b_1 t + b_2 t^2}$$

and

optimizing b_1 and b_2, gives the perfect circle

$$g(t) = (\frac{8+4(\pi-4)t - 4(\pi-2)t^2}{8+4(\pi-4)t+(\pi^2-4\pi+8)t^2},$$

$$\frac{4\pi t + (\pi-4)\pi t^2}{8+4(\pi-4)t+(\pi^2-4\pi+8)t^2}).$$

b) Using the rational spectral basis

$$\mathcal{R}_{2,2} = \{s_{r1}, q_{r1}, s_{r2}, q_{r2}\},$$

$$g(t) = (1,0)s_{r1} + (0,v)q_{r1} + (0,1)s_{r2} + (-v,0)q_{r2},$$

Letting

$b = 1 - t + t^2$, and $v = 3$ gives

$$g(t) = (\frac{1-t-3t^2+2t^3}{1-t+t^2}, \frac{-3(t-1)t}{1-t+t^2}) \cong (\cos \pi t, \sin \pi t).$$

with a least square error less than .000071.

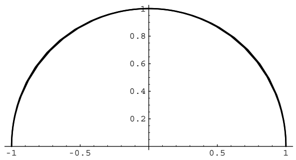

Figure 5: The unit semicircle is shown together with its approximation.

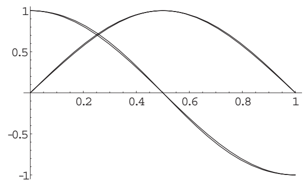

Figure 6: Both sine and cosine curves are shown together with their approximations.

The series expansions for $\cos\pi t$ and $\sin\pi t$ are

$$\cos\pi t \cong 1-4t^2-2t^3+\sum_{k=1}^{\infty}(-1)^{k+1}[2t^{3k+1}+4t^{3k+2}+2t^{3k+3}]$$

and

$$\sin\pi t \cong 3t+3\sum_{k=1}^{\infty}(-1)^{k}[t^{3k}+t^{3k+1}],$$

which are interesting in their own right.

PART III: Álgebra Geométrica Conforme in $\mathbb{G}_{4,1}$

1. The matrix geometric algebra $G_{4,1}$.

The basis elements of $\mathbb{G}_{4,1} = Mat_{\mathbb{G}_3}(2 \times 2)$ are:

$$[e_k] = \begin{pmatrix} e_k & 0 \\ 0 & -e_k \end{pmatrix}, \; k = 1, 2, 3.$$

$$[\sigma] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$[\gamma] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$[u] = [\sigma\gamma] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[u_+] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$[u_-] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Other elements are calcuated by taking sums and products of the matrix representations of the vector basis elements.

For example, for $e = \frac{1}{2}(\sigma + \gamma)$ and $\overline{e} = \sigma - \gamma$,

$$[e] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; [\overline{e}] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

A (real) vector $x \in R^{4,1}$ can be written

$$x = \mathbf{x} + \alpha e + \frac{1}{2}\beta \overline{e},$$

where $\mathbf{x} \in R^3$ and $\alpha, \beta \in R$.

Definition: By a *complex vector* $[x] \in M_{2\times 2}$ we mean any element of the form $[x] = \begin{pmatrix} \mathbf{x} & \beta \\ \alpha & -\mathbf{x} \end{pmatrix}$ for $\mathbf{x} \in G_3^{1+2}$ and $\alpha, \beta \in G_3^{0,3}$.

A general *complex vector* $x \in G_{4,1}^{1+4}$ has the form

$$x = \mathbf{x} + iu\mathbf{y} + (\alpha_1 + iu\alpha_2)e + \frac{1}{2}(\beta_1 + iu\beta_2)\overline{e}$$

The matrix representation of the real x is

$$[x] = \begin{pmatrix} \mathbf{x} & \beta \\ \alpha & -\mathbf{x} \end{pmatrix},$$

and for the complex x,

$$[x] = \begin{pmatrix} \mathbf{x} + i\mathbf{y} & \beta_1 + i\beta_2 \\ \alpha_1 + i\alpha_2 & -\mathbf{x} - i\mathbf{y} \end{pmatrix}.$$

The determinant of $[x]$ for both the real and complex x is

$$\det{[x]} = \mathbf{x}^4 + 2\alpha\beta\mathbf{x}^2 + \alpha^2\beta^2 = (\mathbf{x}^2 + \alpha\beta)^2.$$

The *pseudodeterminant* of $[x]$ is

$$pdet[x] := -(x^2 + \alpha\beta),$$

and x is invertible iff $pdet[x] \neq 0$.

2. The group $G^*_{4,1}$ of all invertible elements of $G_{4,1}$ is isomorphic to the general linear group $M_{4\times 4}(C)$.

3. The *Lipschitz subgroup* $\Gamma_{4,1}$ of $G^*_{4,1}$ consists of those elements in $G^*_{4,1}$ for which $gx\overline{g} \in R^{4,1}$ for all $x \in R^{4,1}$ and is generated by the product of invertible vectors $x \in R^{4,1}$.

4. The *complex Lipschitz subgroup*

$\Gamma^c_{4,1}$ of $G^*_{4,1}$

consists of those elements of $g \in G^*_{4,1}$ for which

$gxg^\dagger \in G^{1+4}_{4,1}$ for all $x \in G^{1+4}_{4,1}$.

Letting $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$[g][x][g^\dagger] = \begin{pmatrix} a\mathbf{x}\overline{d} + \alpha b\overline{d} + \beta a\overline{c} - b\mathbf{x}\overline{c} & a\mathbf{x}\overline{b} + \alpha b\overline{b} + \beta a\overline{a} - b\mathbf{x}\overline{a} \\ c\mathbf{x}\overline{d} + \alpha d\overline{d} + \beta c\overline{c} - d\mathbf{x}\overline{c} & c\mathbf{x}\overline{b} + \alpha d\overline{b} + \beta c\overline{a} - d\mathbf{x}\overline{a} \end{pmatrix}.$$

Examining the complex products,

$$< a\mathbf{x}\overline{d} - b\mathbf{x}\overline{c} >_{0+3} = \mathbf{x}\circ[b_0\mathbf{c} - a_0\mathbf{d} + d_0\mathbf{a} - c_0\mathbf{b} + \mathbf{a}\otimes\mathbf{d} - \mathbf{b}\otimes\mathbf{c}] = 0$$

for all $\mathbf{x}$, or equivalently, $< \overline{a}d - \overline{b}c >_{1+2} = 0$.

Also

$$< \alpha b\overline{d} + \beta a\overline{c} >_{0+3} = 0$$

for all $\alpha, \beta \in G^{0+3}_3$, or equivalently, $b\overline{d} = -d\overline{b}$ and $a\overline{c} = -c\overline{a}$.

5. The pseudodeterminant function is related to the ordinary determinant function for elements $g \in \Gamma^c_{4,1}$.

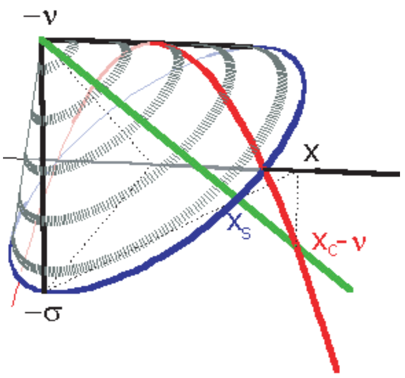

Figure 7: The Horosphere in $I\!R^3$.

$$\det[g] = \det\begin{pmatrix} a & b \\ b & c \end{pmatrix} = (\overline{d}a\overline{a} - \overline{b}a\overline{c})\frac{1}{a\overline{a}}(a\overline{a}d - c\overline{a}b)$$

$$= (\overline{d}a\overline{a} + \overline{b}c\overline{a})\frac{1}{a\overline{a}}(a\overline{a}d + a\overline{c}b)$$

$$= (\overline{d}a + \overline{b}c)(\overline{a}d + \overline{c}b)$$

$$= (\overline{a}d + \overline{c}b)^2 = (\mathrm{pdet}[g])^2,$$

since $\overline{a}d + \overline{c}b = <\overline{a}d + \overline{c}b>_{0+3}$.

6. The 3-Affine space $\mathcal{A}_e(R^3)$.

DEFINITION: The *affine space* $\mathcal{A}_e(R^3) := \{x_h = \mathbf{x} + e |\ \mathbf{x} \in R^3\}$.

Note that $x_h \cdot \overline{e} = 1$ for all $x_h \in \mathcal{A}_e(R^3)$.

7. The 3-Dimensional Horosphere.

DEFINITION: The *horosphere* $H(R^3)$, is defined by the condition that $x_c := x_h + \beta\overline{e}$ is a null vector for all $x_h = \mathbf{x} + e \in \mathcal{A}_e(R^3)$.

Calculating

$$x_c^2 = x_h^2 + 2\beta\overline{e} \cdot x_h = \mathbf{x}^2 + 2\beta = 0$$

or $\beta = -\frac{1}{2}\mathbf{x}^2$. Thus,

$$H(I\!R^3) := \{x_c = \mathbf{x} - \frac{\mathbf{x}^2}{2}\overline{e} + e | \ \ \mathbf{x} \in R^3\}.$$

The horosphere consists of homogeneous points, since

$x_c = \frac{\alpha x_c}{\overline{e} \cdot (\alpha x_c)}$ for all $x_c \in H(R^3)$

and $\alpha \in R^*$.

Calculating $[x_c]$,

$$[x_c] = [x_h] - [\frac{\mathbf{x}^2}{2}\overline{e}] = \begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & -\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{x} \end{pmatrix}.$$

By the *column h-twistor* $[x_c]_t$, we mean

$$[x_c]_t = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}.$$

By the space of *column h-twistor*

$\mathcal{T}_{G_3}$ of G_3
we mean

$$\mathcal{T}_{G_3} := \{[w]_t = \begin{pmatrix} a \\ b \end{pmatrix} | \ \ a, b \in G_3\}.$$

For the column h-twistor

$[w]_t = \begin{pmatrix} a \\ b \end{pmatrix},$

we define a *conjugate row h-twistor* by

$[w]_t^\dagger = (\overline{b} \quad \overline{a})$.

The *h-twistor inner product* is

$$< [w_1]_t, [w_2] >_t := [w_1]_t^\dagger [w_2]_t = \overline{b_1} a_2 + \overline{a_1} b_2 \in G_3^{1+3},$$

where $[w_1]_t = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $[w_2]_t = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ are h-twistors.

8. We can now express any point x_c on the horosphere by

$$[x_c] = [x_c]_t [x_c]_t^\dagger.$$

Actually, we can do much more.

DEFINITION: (*Equivalence of h-twistors*)

$[w_1]_t \equiv [w_2]_t$ iff $[w_1]_t [w_1]_t^\dagger = [w_2]_t [w_2]_t^\dagger$,

and that they are *projectively equivalent* iff

$[w_1]_t [w_1]_t^\dagger = \alpha [w_2]_t [w_2]_t^\dagger$ for $\alpha \in R^*$.

It follows that $[x_c]_t$ and $[x_c h]_t$ are projectively equivalent for all $h \in G_3$ such that $h\overline{h} \in R^*$.

Thus points on the horosphere need only be defined up to a invertible multivector $h \in G_3$.

The concept of an h-twistor cuts calculations on the horosphere in half. For example, for any $g \in G_{4,1}$ with $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$[g x_c g^\dagger] = [g][x_c][g]^\dagger = [g][x_c]_t ([g][x_c]_t)^\dagger.$$

Reflections on the horosphere have the form

$$S_{\mathbf{a}}(x_c) = \mathbf{a}ux_c(\mathbf{a}u)^\dagger = -\mathbf{a}ux_c\mathbf{a}u.$$

In terms of the h-twistor representation, we have

$$[S_{\mathbf{a}}(x_c)] = ([\mathbf{a}u][x_c]_t)([\mathbf{a}u][x_c]_t)^\dagger$$

$$= \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} \mathbf{ax} \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} -\mathbf{a} & \mathbf{xa} \end{pmatrix} = \begin{pmatrix} -\mathbf{axa} & \mathbf{a}^2\mathbf{x}^2 \\ -\mathbf{a}^2 & \mathbf{axa} \end{pmatrix}.$$

Rotations are the composition of two reflections. We find that

$$S_{\mathbf{b}}S_{\mathbf{a}}(x_c) = \mathbf{ba}x_c(\mathbf{ba})^\dagger = \mathbf{ba}x_c\mathbf{ab}.$$

In terms of the h-twistor construction, we find

$$[S_{\mathbf{c}}S_{\mathbf{a}}(x_c)] = ([\mathbf{ba}][x_c]_t)([\mathbf{ba}][x_c]_t)^\dagger$$

$$= \begin{pmatrix} \mathbf{ba} & 0 \\ 0 & \mathbf{ba} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{ba} & 0 \\ 0 & \mathbf{ba} \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} \mathbf{bax} \\ \mathbf{ba} \end{pmatrix} \begin{pmatrix} \mathbf{ab} & -\mathbf{xab} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{baxab} & -\mathbf{a}^2\mathbf{b}^2\mathbf{x}^2 \\ \mathbf{a}^2\mathbf{b}^2 & -\mathbf{baxab} \end{pmatrix}.$$

We can also represent *translations* in the horosphere. For $\mathbf{a} \in R^3$,

$$T_{\mathbf{a}}(x_c) := (1 + \frac{\mathbf{a}\overline{e}}{2})x_c(1 - \frac{\mathbf{a}\overline{e}}{2}).$$

The expression for translations on the horosphere is simpler than translations in affine space, since the last term in the expression for translation in affine space is no longer necessary.

In terms of the h-twistor construction,

$$[T_{\mathbf{a}}(x_c)] = [1 + \frac{\mathbf{a}\overline{e}}{2}][x_c]_t([1 + \frac{\mathbf{a}\overline{e}}{2}][x_c]_t)^\dagger$$

$$= \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}^\dagger =$$

$$\begin{pmatrix} \mathbf{x} + \mathbf{a} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{x} - \mathbf{a} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{x} + \mathbf{a} & -(\mathbf{x} + \mathbf{a})^2 \\ 1 & -\mathbf{x} - \mathbf{a} \end{pmatrix}.$$

For a general element $g \in G^*_{4,1}$, with $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the h-twistor transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} a\mathbf{x} + b \\ c\mathbf{x} + d \end{pmatrix},$$

leads to the general linear fraction *Möbius transformation* or conformal transformation

$$f(\mathbf{x}) = (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1},$$

because of the projective equivalence of the h-twistors

$$\begin{pmatrix} a\mathbf{x}+b \\ c\mathbf{x}+d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1} \\ 1 \end{pmatrix}$$

at all points for which $(c\mathbf{x}+d)^{-1}$ is defined.

The linear fractional transformation $f(\mathbf{x})$ defines a conformal transformation:

Letting $f = f(\mathbf{x})$ we calculate the differential of

$f(\mathbf{x})(c\beta x + d) = (a\mathbf{x}+b)$, getting

$df(c\mathbf{x}+d) + fcd\mathbf{x} = ad\mathbf{x}$ or

$df = (a - fc)d\mathbf{x}(c\mathbf{x}+d)^{-1}$. Continuing the calculation,

$$df = \frac{[a(c\mathbf{x}+b)(\overline{d}-\mathbf{x}\overline{c}) - (a\mathbf{x}+b)(\overline{d}-\mathbf{x}\overline{c})c]d\mathbf{x}(\overline{d}-\mathbf{x}\overline{c})}{(c\mathbf{x}+b)^2(\overline{d}-\mathbf{x}\overline{c})^2}.$$

Simplifying the first part of the numerator,

$$a(c\mathbf{x}+b)(\overline{d}-\mathbf{x}\overline{c}) - (a\mathbf{x}+b)(\overline{d}-\mathbf{x}\overline{c})c$$

$$= ac\mathbf{x}\overline{d} - ad\mathbf{x}\overline{c} + b\mathbf{x}\overline{c}c - a\mathbf{x}\overline{d}c + ad\overline{d} - ac\overline{c}\mathbf{x}^2 - b\overline{d}c + a\mathbf{x}^2c\overline{c}$$

$$= ac\mathbf{x}\overline{d} - ad\mathbf{x}\overline{c} + b\mathbf{x}\overline{c}c - a\mathbf{x}\overline{d}c + d(\overline{d}a + \overline{b}c)$$

$$= ac\mathbf{x}\overline{d} + (ad\mathbf{x} + b\mathbf{x}c)\overline{c} - a\mathbf{x}\overline{d}c + d\ \mathrm{pdet}(g)$$

$$= \cdots = -a\overline{c}d\mathbf{x} + b\overline{c}c\mathbf{x} + d\,\mathrm{pdet}(g) = \mathrm{pdet}(g)(c\mathbf{x} + d).$$

Finally, we get

$$df = \frac{\mathrm{pdet}(g)(c\mathbf{x}+d)d\mathbf{x}(\overline{d}-\mathbf{x}\overline{c})}{(c\mathbf{x}+d)^2(\overline{d}-\mathbf{x}\overline{c})^2}$$

$$= \mathrm{pdet}(g)(\overline{d}-\mathbf{x}\overline{c})^{-1}d\mathbf{x}(c\mathbf{x}+d)^{-1}.$$

Squaring both sides gives

$$(df)^2 = \frac{\det g}{(c\mathbf{x}+d)^2(\mathbf{x}\overline{c}-\overline{d})^2}(d\mathbf{x})^2,$$

showing that $\beta y = f(\mathbf{x})$ is conformal at all points at which it is defined.