Part II: Spacetime Algebra of Dirac Spinors

Garret Sobczyk
Universidad de las Américas-Puebla
Departamento de Físico-Matemáticas
72820 Puebla, Pue., México
http://www.garretstar.com

July 19, 2015

Abstract

In Part I: *Vector Analysis of Spinors*, the author studied the geometry of two component spinors as points on the Riemann sphere in the geometric algebra $\mathbb{G}_3$ of three dimensional Euclidean space. Here, these ideas are generalized to apply to four component Dirac spinors on the complex Riemann sphere in the complexified geometric algebra $\mathbb{G}_3(\mathbb{C})$ of spacetime, which includes Lorentz transformations. The development of generalized Pauli matrices eliminate the need for the traditional Dirac gamma matrices. We give the discrete probability distribution of measuring a spin $1/2$ particle in an arbitrary spin state, assuming that it was prepared in a given state immediately prior to the measurement, independent of the inertial system in which measurements are made. The Fierz identities between the physical observables of a Dirac spinor are discussed.

AMS Subject Classication: 15A66, 81P16

Keywords: bra-ket formalism, geometric algebra, spacetime algebra, Dirac equation, Dirac-Hestenes equation, Riemann sphere, complex Riemann sphere, spinor, spinor operator, Fierz identities.

0 Introduction

Since the birth of quantum mechanics a Century ago, scientists have been both puzzled and amazed about the seemingly inescapable occurrence of the imaginary number $i = \sqrt{-1}$, first in the Pauli-Schrödinger equation for spin $\frac{1}{2}$ particles in space, and later in the more profound Dirac equation of spacetime. Exactly what role complex numbers play in quantum mechanics is even today hotly debated. In a previous paper, "Vector Analysis of Spinors", I show that the i occurring in the Schrödinger-Pauli equation for the electron should be interpreted as the unit pseudoscalar, or directed volume element, of the geometric algebra $\mathbb{G}_3$. This follows directly from the assumption that the famous Pauli matrices are nothing more than the components of the orthonormal space vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ with respect to the *spectral basis* of the geometric algebra $\mathbb{G}_3$,

[1]. Another basic assumption made is that the geometric algebra $\mathbb{G}_3$ of space is naturally identified as the even sub-algebra $\mathbb{G}_{1,3}^+$ of the spacetime algebra $\mathbb{G}_{1,3}$, also known as the algebra of *Dirac matrices*, [2], [3].

This line of research began when I started looking at the foundations of quantum mechanics. In particular, I wanted to understand in exactly what sense the *Dirac-Hestenes* equation for the electron is equivalent to the standard Dirac equation. What I discovered was that the equations are equivalent only so long as the issues of parity and complex conjugation are not taken into consideration, [4]. In the present work, I show that $i = \sqrt{-1}$ in the Dirac equation must have a different interpretation, than the i that occurs in simpler Schrödinger-Pauli theory. In order to turn both the Schrödinger-Pauli theory, and the relativistic Dirac theory, into strictly equivalent geometric theories, we replace the study of 2 and 4-component spinors with corresponding 2 and 4-component geometric spinors, defined by the minimal left ideals in the appropriate geometric algebras. As pointed out by the late Perrti Lounesto, [5, p.327], "Juvet 1930 and Sauter 1930 replaced column spinors by square matrices in which only the first column was non-zero - thus spinor spaces became minimial left ideals in a matrix algebra". In order to gives the resulting matrices a unique geometric interpretation, it is then only necessary to interpret these matrices as the components of geometric numbers with respect to the *spectral basis* of the appropriate geometric algebra [6, p.205].

The important role played by an idempotent, and its interpretation as a point on the Riemann sphere in the case of Pauli spinors, and as a point on the complex Riemann sphere in the case of Dirac spinors, make up the heart of our new geometric theory. Just as the spin state of an electron can be identified with a point on the Riemann sphere, and a corresponding unique point in the plane by stereographic projection from the South Pole, we find that the spin state of a relativistic electron can be identified by a point on the complex Riemann sphere, and its corresponding point in the complex 2-plane by a *complex stereographic projection* from the South Pole. In developing this theory, we find that the study of geometric Dirac spinors can be carried out by introducing a generalized set of 2×2 *Pauli E-matrices* over a 4-dimensional commuative ring with the basis $\{1, i, I, iI\}$, where $i = \sqrt{-1}$ and $I = \mathbf{e}_{123}$ is the unit pseudo-scalar of the geometric algebra $\mathbb{G}_3$. The setting for the study of quantum mechanics thereby becomes the complex geometric algebra $\mathbb{G}_3(\mathbb{C})$. In order to study quantum mechanics in a real geometric algebra, eliminating the need for any artificial $i = \sqrt{-1}$, we would have to consider at least one of higher dimensional geometric algebras $\mathbb{G}_{2,3}, \mathbb{G}_{4,1}, \mathbb{G}_{0,5}$ of the respective pseudoeuclidean spaces $\mathbb{R}^{2,3}, \mathbb{R}^{4,1}, \mathbb{R}^{0,5}$, [5, p.217], [7, p.326].

1 Geometric algebra of spacetime

The geometric algebra $\mathbb{G}_3$ of an orthonormal rest frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathbb{R}^3$ can be factored into an orthonormal frame $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ in the geometric algebra $\mathbb{G}_{1,3}$ of the pseudo-Euclidean space $\mathbb{R}^{1,3}$ of *Minkowski spacetime*, by writing

$$\mathbf{e}_k := \gamma_k \gamma_0 = -\gamma_0 \gamma_k \quad \text{for} \quad k = 1, 2, 3. \tag{1}$$

In doing so, the geometric algebra $\mathbb{G}_3$ is identified with the elements of the even subalgebra $\mathbb{G}_{1,3}^+ \subset \mathbb{G}_{1,3}$. A consequence of this identification is that space vectors $\mathbf{x} =$

$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{G}_3^1$ become *spacetime bivectors* in $\mathbb{G}_{1,3}^2 \subset \mathbb{G}_{1,3}^+$. In summary, the geometric algebra $\mathbb{G}_{1,3}$, also known as *spacetime algebra* [2], has $2^4 = 16$ basis elements generated by geometric multiplication of the γ_μ for $\mu = 0,1,2,3$. Thus,

$$\mathbb{G}_{1,3} := gen\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$$

obeying the rules

$$\gamma_0^2 = 1,\ \gamma_k^2 = -1,\ \gamma_{\mu\nu} := \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu = \gamma_{\nu\mu}$$

for $\mu \neq \nu$, $\mu, \nu = 0,1,2,3$, and $k = 1,2,3$. Note also that the *pseudo-scalar*

$$\gamma_{0123} := \gamma_{10}\gamma_{20}\gamma_{30} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{123} =: I$$

of $\mathbb{G}_{1,3}$ is the same as the pseudo-scalar of the rest frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{G}_3$, and it anti-commutes with each of the spacetime vectors γ_μ for $\mu = 0,1,2,3$.

In the above, we have carefully distinguished the rest frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the geometric algebra $\mathbb{G}_3 := \mathbb{G}_{1,3}^+$. Any other rest frame $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ can be obtained by an ordinary space rotation of the rest frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ followed by a *Lorentz boost*. In the spacetime algebra $\mathbb{G}_{1,3}$, this is equivalent to defining a new frame of spacetime vectors $\{\gamma'_\mu |\ 0 \leq \mu \leq 3\} \subset \mathbb{G}_{1,3}$, and the corresponding rest frame $\{\mathbf{e}'_k = \gamma'_k \gamma'_0 |\ k = 1,2,3\}$ of a Euclidean space $\mathbb{R}^{3'}$ moving with respect to the Euclidean space $\mathbb{R}^3$ defined by the rest frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Of course, the primed rest-frame $\{\mathbf{e}'_k\}$, itself, generates a corresponding geometric algebra $\mathbb{G}'_3 := \mathbb{G}_{1,3}^+$. A much more detailed treatment of $\mathbb{G}_3$ is given in [6, Chp.3], and in [8] I explore the close relationship that exists between geometric algebras and their matrix counterparts. The way we introduced the geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_{1,3}$ may appear novel, but they perfectly reflect all the common relativistic concepts [6, Chp.11].

The well-known *Dirac matrices* can be obtained as a real sub-algebra of the 4×4 matrix algebra $Mat_{\mathbb{C}}(4)$ over the complex numbers where $i = \sqrt{-1}$. We first define the idempotent

$$u_{++} := \frac{1}{4}(1+\gamma_0)(1+i\gamma_{12}) = \frac{1}{4}(1+i\gamma_{12})(1+\gamma_0), \tag{2}$$

where the unit imaginary $i = \sqrt{-1}$ is assumed to commute with all elements of $\mathbb{G}_{1,3}$. Whereas it would be nice to identify this unit imaginary i with the pseudo-scalar element $\gamma_{0123} = \mathbf{e}_{123}$ as we did in $\mathbb{G}_3$, this is no longer possible since γ_{0123} anti-commutes with the spacetime vectors γ_μ as previously mentioned.

Noting that

$$\gamma_{12} = \gamma_1\gamma_0\gamma_0\gamma_2 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_{21},$$

and similarly $\gamma_{31} = \mathbf{e}_{13}$, it follows that

$$\mathbf{e}_{13}u_{++} = u_{+-}\mathbf{e}_{13},\ \ \mathbf{e}_3 u_{++} = u_{-+}\mathbf{e}_3,\ \ \mathbf{e}_1 u_{++} = u_{--}\mathbf{e}_1, \tag{3}$$

where

$$u_{+-} := \frac{1}{4}(1+\gamma_0)(1-i\gamma_{12}),\ u_{-+} := \frac{1}{4}(1-\gamma_0)(1+i\gamma_{12}),\ u_{--} := \frac{1}{4}(1-\gamma_0)(1-i\gamma_{12}).$$

The idempotents $u_{++}, u_{+-}, u_{-+}, u_{--}$ are *mutually annihilating* in the sense that the product of any two of them is zero, and *partition unity*

$$u_{++} + u_{+-} + u_{-+} + u_{--} = 1. \tag{4}$$

By the *spectral basis* of the Dirac algebra $\mathbb{G}_{1,3}$, we mean the elements of the matrix

$$\begin{pmatrix} 1 \\ \mathbf{e}_{13} \\ \mathbf{e}_3 \\ \mathbf{e}_1 \end{pmatrix} u_{++} \begin{pmatrix} 1 & -\mathbf{e}_{13} & \mathbf{e}_3 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_{++} & -\mathbf{e}_{13}u_{+-} & \mathbf{e}_3 u_{-+} & \mathbf{e}_1 u_{--} \\ \mathbf{e}_{13}u_{++} & u_{+-} & \mathbf{e}_1 u_{-+} & -\mathbf{e}_3 u_{--} \\ \mathbf{e}_3 u_{++} & \mathbf{e}_1 u_{+-} & u_{-+} & -\mathbf{e}_{13}u_{--} \\ \mathbf{e}_1 u_{++} & -\mathbf{e}_3 u_{+-} & \mathbf{e}_{13}u_{-+} & u_{--} \end{pmatrix}. \tag{5}$$

Any geometric number $g \in \mathbb{G}_{1,3}$ can be written in the form

$$g = \begin{pmatrix} 1 & \mathbf{e}_{13} & \mathbf{e}_3 & \mathbf{e}_1 \end{pmatrix} u_{++} [g] \begin{pmatrix} 1 \\ -\mathbf{e}_{13} \\ \mathbf{e}_3 \\ \mathbf{e}_1 \end{pmatrix} \tag{6}$$

where $[g]$ is the *complex Dirac matrix* corresponding to the geometric number g. In particular,

$$[\gamma_0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, [\gamma_1] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

and

$$[\gamma_2] = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, [\gamma_3] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is interesting to see what the representation is of the basis vectors of $\mathbb{G}_3$. We find that for $k = 1,2,3$,

$$[\mathbf{e}_k]_4 = [\gamma_k][\gamma_0] = \begin{pmatrix} [0]_2 & [\mathbf{e}_k]_2 \\ [\mathbf{e}_k]_2 & [0]_2 \end{pmatrix} \quad \text{and} \quad [\mathbf{e}_{123}]_4 = i \begin{pmatrix} [0]_2 & [1]_2 \\ [1]_2 & [0]_2 \end{pmatrix},$$

where the outer subscripts denote the order of the matrices and, in particular, $[0]_2$, $[1]_2$ are the 2×2 zero and unit matrices, respectively. The last relationship shows that the $I := \mathbf{e}_{123}$ occurring in the Pauli matrix representation, which represents the oriented unit of volume, is different than the $i = \sqrt{-1}$ which occurs in the complex matrix representation (7) of the of Dirac algebra. In particular, $[\mathbf{e}_2]_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which is **not** the Pauli matrix for $\mathbf{e}_2 \in \mathbb{G}_3$ since $i \neq I$. We will have more to say about this important matter later.

A *Dirac spinor* is a 4-component column matrix $[\varphi]_4$,

$$[\varphi]_4 := \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \quad \text{for} \quad \varphi_k = x_k + iy_k \in \mathbb{C}. \tag{8}$$

Just as in [4], from the Dirac spinor $[\varphi]_4$, using (6), we construct its equivalent $S \in \mathbb{G}_{1,3}(\mathbb{C})$ as an element of the *minimal left ideal* generated by u_{++},

$$[\varphi]_4 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} \varphi_1 & 0 & 0 & 0 \\ \varphi_2 & 0 & 0 & 0 \\ \varphi_3 & 0 & 0 & 0 \\ \varphi_4 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow S := \big(\varphi_1 + \varphi_2 \mathbf{e}_{13} + \varphi_3 \mathbf{e}_3 + \varphi_4 \mathbf{e}_1\big) u_{++}. \quad (9)$$

Because of its close relationship to a Dirac spinor, we shall refer to S as a *geometric Dirac spinor* or a *Dirac g-spinor*.

Noting that

$$u_{++}\gamma_{21} = \frac{1}{4}(1+\gamma_0)(\gamma_{21} + i\gamma_{12}\gamma_{21}) = \frac{1}{4}(1+\gamma_0)(i-\gamma_{12}) = iu_{++} = \gamma_{21}u_{++},$$

it follows that $\varphi_k u_{++} = (x_k + \gamma_{21} y_k) u_{++} = u_{++}\big(x_k + \gamma_{21} y_k\big)$ and hence

$$S = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)u_{++} = \big(\alpha_1 + \alpha_2^\dagger \mathbf{e}_{13} + \mathbf{e}_3\alpha_3 + \alpha_4^\dagger \mathbf{e}_1\big)u_{++}, \qquad (10)$$

where each of the elements α_k in S is defined by $\alpha_k = \varphi_k|_{i\to\gamma_{21}}$, and $\alpha_k^\dagger := \varphi_k|_{i\to-\gamma_{21}}$.

Expanding out the terms in (10),

$$S = \Big((x_1 + x_4\mathbf{e}_1 + y_4\mathbf{e}_2 + x_3\mathbf{e}_3) + I(y_3 + y_2\mathbf{e}_1 - x_2\mathbf{e}_2 + y_1\mathbf{e}_3)\Big)u_{++}. \qquad (11)$$

This suggest the substitution

$$\varphi_1 \to x_0 + iy_3, \;\; \varphi_2 \to -y_2 + iy_1, \;\; \varphi_3 \to x_3 + iy_0, \;\; \varphi_4 \to x_1 + ix_2, \qquad (12)$$

in which the geometric Dirac spinor S takes the more perspicuous forms

$$S = \big(X + IY\big)u_{++} = \big(X + IY\big)\gamma_0 u_{++} = (x + Iy)u_{++}, \qquad (13)$$

for $X := x_0 + \mathbf{x}$, $Y := y_0 + \mathbf{y} \in \mathbb{G}_3$ where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3$, and $x := \sum_{\mu=0}^3 x_\mu\gamma_\mu, y := \sum_{\mu=0}^3 y_\mu\gamma_\mu \in \mathbb{G}_{1,3}$.

We now calculate

$$\overline{S} = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)u_{+-},$$

where $\overline{S}$ is the *complex conjugate* of S, defined by $i \to -i$,

$$S^\# = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)u_{-+},$$

where $S^\#$ is the parity transformation defined by $\gamma_\mu \to -\gamma_\mu$, and

$$S^\star := (\overline{S})^\# = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)u_{--}.$$

Using (4), we then define the *even spinor operator*

$$\psi := S + \overline{S} + S^\# + S^\star = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\big(u_{++} + u_{+-} + u_{-+} + u_{--}\big)$$

$$= \alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4 = X + IY \in \mathbb{G}_{1,3}^+, \tag{14}$$

and the *odd spinor operator*

$$\Phi := S + \overline{S} - S^\# - S^\star = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\big(u_{++} + u_{+-} - u_{-+} - u_{--}\big)$$
$$= \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\gamma_0 = x + Iy \in \mathbb{G}_{1,3}^-. \tag{15}$$

In addition to the two real even and odd spinor operators in $\mathbb{G}_{1,3}$, we have two *complex spinor operators* in $\mathbb{G}_{1,3}(\mathbb{C})$, given by

$$Z_+ := S - \overline{S} + S^\# - S^\star = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\big(u_{++} - u_{+-} + u_{-+} - u_{--}\big)$$
$$= \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)E_3 = (X + IY)E_3 \in \mathbb{G}_{1,3}^+(\mathbb{C}), \tag{16}$$

where $E_3 := -iI\mathbf{e}_3$, and

$$Z_- := S - \overline{S} - S^\# + S^\star = \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\big(u_{++} - u_{+-} - u_{-+} + u_{--}\big)$$
$$= \big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\big)\gamma_0 E_3 = (x + Iy)E_3 \in \mathbb{G}_{1,3}^-(\mathbb{C}). \tag{17}$$

Using (5), the matrix $[\psi]$ of the even spinor operator ψ is found to be

$$[\psi] = \begin{pmatrix} \varphi_1 & -\overline{\varphi}_2 & \varphi_3 & \overline{\varphi}_4 \\ \varphi_2 & \overline{\varphi}_1 & \varphi_4 & -\overline{\varphi}_3 \\ \varphi_3 & \overline{\varphi}_4 & \varphi_1 & -\overline{\varphi}_2 \\ \varphi_4 & -\overline{\varphi}_3 & \varphi_2 & \overline{\varphi}_1 \end{pmatrix}, \tag{18}$$

[4], [5, p.143], and using (7) and (18), the matrix $[\Phi]$ of the odd spinor operator Φ is found to be

$$[\Phi] = \begin{pmatrix} \varphi_1 & -\overline{\varphi}_2 & -\varphi_3 & -\overline{\varphi}_4 \\ \varphi_2 & \overline{\varphi}_1 & -\varphi_4 & +\overline{\varphi}_3 \\ \varphi_3 & \overline{\varphi}_4 & -\varphi_1 & +\overline{\varphi}_2 \\ \varphi_4 & -\overline{\varphi}_3 & -\varphi_2 & -\overline{\varphi}_1 \end{pmatrix}. \tag{19}$$

Unlike the Dirac spinor $[\varphi]_4$, the even spinor operator $[\psi]$ is invertible iff $\det[\psi] \neq 0$. We find that

$$\det[\psi] = r^2 + 4a^2 \geq 0, \tag{20}$$

where

$$r = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 \quad \text{and} \quad a = im\big(\overline{\varphi}_1\varphi_3 + \overline{\varphi}_2\varphi_4\big).$$

Whereas the even spinor operator $[\psi]$ obviously contains the same information as the Dirac spinor $[\varphi]_4$, it acquires in (14) the geometric interpretation of an even multivector in $\mathbb{G}_{1,3}^+$. With the substitution (12), the expansion of the determinant (20) takes the interesting form

$$\det[\psi] = \big((x_0^2 - x_1^2 - x_2^2 - x_3^2) - (y_0^2 - y_1^2 - y_2^2 - y_3^2)\big)^2 + 4(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)^2$$
$$= (x^2 - y^2)^2 + 4(x \cdot y)^2 \quad \text{for} \quad x, y \in \mathbb{G}_{1,3}^1. \tag{21}$$

Since the matrix $[\Phi]$ of the odd spinor operator is the same as the matrix $[\psi]$ of the even spinor operator, except for the change of sign in the last two columns, $\det[\Phi] =$

$\det[\psi]$. Indeed, similar arguments apply to the matrices $[Z_+]$ and $[Z_-]$, and so $\det[\psi] = \det[Z_+] = \det[Z_-]$.

As an even geometric number in $\mathbb{G}^+_{1,3}$, ψ generates Lorentz boosts in addition to ordinary rotations in the Minkowski space $\mathbb{R}^{1,3}$. Whereas we started by formally introducing the complex number $i = \sqrt{-1}$, in order to represent the Dirac gamma matrices, we have ended up with the real spinor operators $\psi, \Phi \in \mathbb{G}^+_{1,3}$, in which the role of $i = \sqrt{-1}$ is taken over by the $\gamma_{21} = \mathbf{e}_{12} \in \mathbb{G}_3$. This is the key idea in the Hestenes representation of the *Dirac equation* [9]. However, in the process, the crucial role played by the mutually annihilating idempotents $u_{\pm\pm}$ has been obscured, and the role played by $i = \sqrt{-1}$ in the definition of the Dirac matrices has been buried. Idempotents are slippery objects which can change the identities of everything they touch. As such, they should always be treated gingerly with care. Idempotents naturally arise in the study of number systems that have zero divisors, [10].

2 Geometric Dirac Spinors to Geometric E-Spinors

Recall from equation (14) that the real, even, Dirac spinor operator is

$$\psi = \alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4 = X + IY \in \mathbb{G}^+_{1,3},$$

for the geometric Dirac spinor,

$$S = \psi u_{++} = \Big(\varphi_1 + \varphi_2\mathbf{e}_{13} + \varphi_3\mathbf{e}_3 + \varphi_4\mathbf{e}_1\Big)u_{++} = \Big(\alpha_1 + \mathbf{e}_{13}\alpha_2 + \mathbf{e}_3\alpha_3 + \mathbf{e}_1\alpha_4\Big)u_{++}$$

as an element in the minimal left ideal $\{\mathbb{G}^+_{1,3}u_{++}\}$.

Defining $J := -iI$, we can express

$$u_{++} = \gamma_0^+ E_3^+, \ \text{ where } \ \gamma_0^\pm = \frac{1}{2}(1 \pm \gamma_0), \ \ E_3^\pm = \frac{1}{2}(1 \pm J\mathbf{e}_3).$$

Noting that $J\mathbf{e}_3 u_{++} = u_{++}$,

$$S = \Big(\varphi_1 + \varphi_2\mathbf{e}_{13} + \varphi_3\mathbf{e}_3 + \varphi_4\mathbf{e}_1\Big)u_{++} = \Big(\varphi_1 + \varphi_2\mathbf{e}_1 J + \varphi_3 J + \varphi_4\mathbf{e}_1\Big)u_{++}$$

$$= \Big((\varphi_1 + \varphi_3 J) + (\varphi_4 + \varphi_2 J)\mathbf{e}_1\Big)u_{++} = \Omega u_{++}, \tag{22}$$

where $\Omega := \Omega_0 + \Omega_1\mathbf{e}_1$ for $\Omega_0 := (\varphi_1 + \varphi_3 J)$ and $\Omega_1 := (\varphi_4 + \varphi_2 J)$. Alternatively, Ω_0 and Ω_1 can be defined in the highly useful, but equivalent, way

$$\Omega_0 = z_1 + Jz_3, \quad \text{and} \quad \Omega_1 = z_4 + z_2 J, \tag{23}$$

for $z_1 := x_1 + y_3 I$, $z_3 := x_3 + y_1 I$, $z_4 := x_4 + y_2 I$, $z_2 := x_2 + y_4 I$ are all in $\mathbb{G}^{0+4}_{1,3}$. With the substitution variables (12), the z_k's become

$$z_1 = x_0 + y_0 I, \ \ z_2 = -y_2 + x_2 I, \ \ z_3 = x_3 + y_3 I, \ \ z_4 = x_1 + y_1 I. \tag{24}$$

Once again, we calculate

$$\overline{S} = \Big((\overline{\varphi}_1 - \overline{\varphi}_3 J) + (\overline{\varphi}_4 - \overline{\varphi}_2 J)\mathbf{e}_1\Big)u_{+-} = \overline{\Omega}u_{+-},$$

where $\overline{S}$ is the *complex conjugate* of S defined by $i \to -i$,

$$S^{\#} = \Big((\varphi_1 + \varphi_3 J) + (\varphi_4 + \varphi_2 J)\mathbf{e}_1\Big)u_{-+} = \Omega u_{-+},$$

where $S^{\#}$ is the parity transformation defined by $\gamma_\mu \to -\gamma_\mu$, and

$$S^{\star} := (\overline{S})^{\#} = \Big((\overline{\varphi}_1 - \overline{\varphi}_3 J) + (\overline{\varphi}_4 - \overline{\varphi}_2 J)\mathbf{e}_1\Big)u_{--} = \overline{\Omega}u_{--}.$$

The even spinor operator (14), satisfies

$$\psi = S + \overline{S} + S^{\#} + S^{\star} = \Omega E_3^+ + \overline{\Omega}E_3^- = \frac{1}{2}(\Omega + \overline{\Omega}) + \frac{1}{2}(\Omega - \overline{\Omega})E_3, \qquad (25)$$

where as before $E_3 := J\mathbf{e}_3 = E_3^+ - E_3^-$. Equations (14) and (25) express the even spinor operator ψ in two very different, but equivalent ways, the first as an element of $\mathbb{G}_{1,3}^+$, and the second by multiplication in the complex geometric algebra $\mathbb{G}_{1,3}^+(\mathbb{C})$. The second approach is closely related to the *twistor theory* of Roger Penrose [11, p.974].

It follows from (22) and (25), that any even spinor operator $\psi \in \mathbb{G}_{1,3}^+$ can be written in the matrix form

$$\psi = (1 \quad \mathbf{e}_1)E_3^+ \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \iff \psi^\dagger = (1 \quad \mathbf{e}_1)E_3^+ \begin{pmatrix} \overline{\Omega}_0^\dagger & \overline{\Omega}_1^\dagger \\ \Omega_1^\dagger & \Omega_0^\dagger \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \qquad (26)$$

where the matrix $[\psi]_\Omega := \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix}$, and the matrix $[\psi^\dagger]_\Omega := \begin{pmatrix} \overline{\Omega}_0^\dagger & \overline{\Omega}_1^\dagger \\ \Omega_1^\dagger & \Omega_0^\dagger \end{pmatrix}$. Analogous to the Pauli matrices, we have

$$[1]_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; [\mathbf{e}_1]_\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; [\mathbf{e}_2]_\Omega = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; [\mathbf{e}_3]_\Omega = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \qquad (27)$$

which can be obtained from the Pauli matrix representation by multiplying the Pauli matrices for $\mathbf{e}_2$ and $\mathbf{e}_3$ by $J = -iI$. Alternatively, the Pauli matrices can be obtained from the above Dirac-like representation, simply by replacing i by I, in which case $J \to 1$. Indeed, if we let $i \to I$ in (26), and using the substitution variables z_k defined in (24), we get

$$\psi = (1 \quad \mathbf{e}_1)u_+ \begin{pmatrix} z_1 + z_3 & z_4 - z_2 \\ z_4 + z_2 & z_1 - z_3 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}, \qquad (28)$$

where $u_+ = \frac{1}{2}(1 + \mathbf{e}_3)$, which is exactly the Pauli algebra representation of the geometric number $\psi \in \mathbb{G}_3$. Recently, I was surprised to discover that I am not the first to consider such a representation of the Pauli matrices [12].

Calculating the determinant of the matrix $[\Omega] := [\psi]_\Omega$ of ψ,

$$\det[\Omega] = \det\begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 + 2im(\overline{\varphi}_1\varphi_3 + \overline{\varphi}_2\varphi_4)I, \quad (29)$$

or in the more elegant alternative form in the spacetime algebra $\mathbb{G}_{1,3}$,

$$\det[\Omega] = x^2 - y^2 + 2I(x\cdot y) = (x + Iy)(x - Iy) = (X + IY)(\widetilde{X} + I\widetilde{Y}), \quad (30)$$

where $x = X\gamma_0$ and $y = Y\gamma_0$, and $\widetilde{A}$ denotes the operation of reverse of the element $A \in \mathbb{G}_{1,3}$. Note that the determinant $\det[\psi]$, found in (20), is related to (29) or (30), by $\det[\psi] = |\det[\psi]_\Omega|^2$ for the matrix of the even spinor operator ψ defined in (18). The spinor operator ψ will have an inverse only when $\det[\psi] \neq 0$.

We have already noted that the geometric algebra $\mathbb{G}_3$ can be algebraically identified with the even sub-algebra $\mathbb{G}_{1,3}^+ \subset \mathbb{G}_{1,3}$. Each timelike Dirac vector γ_0 determines a different rest frame (1) of spacelike bivectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. For $\psi \in \mathbb{G}_{1,3}^+$, the matrix $[\psi]_\Omega$ is *Hermitian* with respect to the rest-frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of γ_0 if

$$\overline{[\psi^\dagger]}_\Omega^T = [\psi]_\Omega,$$

or equivalently, $\psi^\dagger = \psi$, where $\dagger$ is the conjugation of reverse in the Pauli algebra $\mathbb{G}_3 \widetilde{=} \mathbb{G}_{1,3}^+$. Using the more transparent variables (12), for a Hermitian $[\psi]_\Omega$,

$$[\psi]_\Omega = \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} = \begin{pmatrix} x_0 + Jx_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - Jx_3 \end{pmatrix} = \overline{[\psi^\dagger]}_\Omega^T$$

with $\det[\psi]_\Omega = x_0^2 - x_1^2 - x_2^2 - x_3^2$, and for anti-Hermitian $[\psi]_\Omega$,

$$[\psi]_\Omega = \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} = I\begin{pmatrix} y_0 + Jy_3 & y_1 - iy_2 \\ y_1 + iy_2 & y_0 - Jy_3 \end{pmatrix} = -\overline{[\psi^\dagger]}_\Omega^T$$

with $\det[\psi]_\Omega = -y_0^2 + y_1^2 + y_2^2 + y_3^2$.

Thus, a *Hermitian Dirac g-spinor*, corresponding to $X = x_0 + \mathbf{x} \in \mathbb{G}^3$, has the form

$$S := (\Omega_0 + \Omega_1\mathbf{e}_1)u_{++} = \Big((x_0 + x_3J) + (x_1 + ix_2)\mathbf{e}_1\Big)u_{++} = X\,u_{++},$$

for $\Omega_0 = x_0 + x_3J$ and $\Omega_1 = x_1 + ix_2$, and an anti-Hermitian Dirac g-spinor, corresponding to $IY = I(y_0 + \mathbf{y}) \in \mathbb{G}_3$, has the form

$$Q := (\Omega_0 + \Omega_1\mathbf{e}_1)u_{++} = I\Big((y_0 + y_3J) + (y_1 + iy_2)\mathbf{e}_1\Big)u_{++} = IY\,u_{++},$$

for $\Omega_0 = I(y_0 + y_3J)$ and $\Omega_1 = I(y_1 + iy_2)$.

In the bra-ket notation, a Dirac g-spinor (9), and its conjugate Dirac g-spinor, in the spacetime algebra $\mathbb{G}_{1,3}^+(\mathbb{C})$, takes the form

$$|\Omega\rangle := 2S = 2(\Omega_0 + \Omega_1\mathbf{e}_1)u_{++} = 2(x + Iy)u_{++}, \quad (31)$$

and

$$\langle\Omega| := \widetilde{\overline{|\Omega\rangle}} = 2u_{++}(\overline{\Omega}_0 - \overline{\Omega}_1\mathbf{e}_1) = 2u_{++}(x - Iy), \tag{32}$$

respectively. Taking the product of the conjugate of a g-spinor $|\Phi\rangle = 2(r+Is)u_{++}$, with a g-spinor $|\Omega\rangle$, gives

$$\langle\Phi||\Omega\rangle = 4u_{++}\big(\overline{\Phi}_0 - \mathbf{e}_1\overline{\Phi}_1\big)\big(\Omega_0 + \mathbf{e}_1\Omega_1\big)u_{++} = 4u_{++}\big(\overline{\phi}_1\varphi_1 + \overline{\phi}_2\varphi_2 - \overline{\phi}_3\varphi_3 - \overline{\phi}_4\varphi_4\big)$$

$$= 4u_{++}\Big(r\cdot x - s\cdot y + \gamma_{21}\big(\gamma_{12}\cdot(r\wedge x - s\wedge y) + \gamma_{30}\cdot(r\wedge y - s\wedge x)\big)\Big). \tag{33}$$

The complex *sesquilinear inner product* in $\mathbb{G}_3^{0+3}$ of the two g-spinors $|\Phi\rangle$ and $|\Omega\rangle$ is then defined by

$$\langle\Phi|\Omega\rangle := \Big\langle\langle\Phi||\Omega\rangle\Big\rangle_{\mathbb{C}} = \Big\langle\overline{\phi}_1\varphi_1 + \overline{\phi}_2\varphi_2 - \overline{\phi}_3\varphi_3 - \overline{\phi}_4\varphi_4\Big\rangle_{\mathbb{C}} = a + ib \in \mathbb{C}. \tag{34}$$

Defining $|\Omega\rangle = 2(\Omega_0 + \Omega_1\mathbf{e}_1)u_{++}$ and $|\Phi\rangle = 2(\Phi_0 + \Phi_1\mathbf{e}_1)u_{++}$ for the substituted variables (12), with $|\Phi\rangle$ being defined in the coresponding variable r and s, the inner product (34) takes the form

$$\langle\Phi|\Omega\rangle = (r_0x_0 - r_1x_1 - r_2x_2 - r_3x_3) - (s_0y_0 - s_1y_1 - s_2y_2 - s_3y_3)$$

$$+ i(r_0y_3 - r_3y_0 + s_0x_3 - s_3x_0 + r_2x_1 - r_1x_2 + s_1y_2 - s_2y_1). \tag{35}$$

Alternatively, writing $|\Omega\rangle = 2(x+Iy)u_{++}$ and $\langle\Phi| = 2u_{++}(r - Is)$, we have

$$\langle\Phi|\Omega\rangle = (r\cdot x - s\cdot y) + i\big(\gamma_{12}\cdot(r\wedge x - s\wedge y) + \gamma_{30}\cdot(r\wedge y - s\wedge x)\big) \in \mathbb{C}.$$

Consider now the parity invarient part S_E of the geometric Dirac spinor S,

$$S_E := S + S^{\#} = (\Omega_0 + \Omega_1\mathbf{e}_1)E_3^+ = \Omega E_3^+ = (X + IY)E_3^+.$$

Writing

$$S_E = \Omega_0(1 + \Omega_0^{-1}\Omega_1\mathbf{e}_1)E_3^+ = \Omega_0 T_E^+, \tag{36}$$

for $T_E^+ := (1 + \Lambda\mathbf{e}_1)E_3^+$ and $\Lambda := \Omega_0^{-1}\Omega_1$, we then find that T_E^+ is an idempotent. We now study idempotents $P \in \mathbb{G}_3(\mathbb{C})$ of the form

$$P = \frac{1}{2}(1 + J\mathbf{M} + I\mathbf{N}), \tag{37}$$

for $\mathbf{N}, \mathbf{M} \in \mathbb{G}_{1,3}^2$ and $(J\mathbf{M} + I\mathbf{N})^2 = 1$.

Defining the *symmetric product* $\mathbf{M}\circ\mathbf{N} := \frac{1}{2}(\mathbf{MN} + \mathbf{NM})$, the condition

$$(J\mathbf{M} + I\mathbf{N})^2 = 1 \quad \leftrightarrow \quad \mathbf{M}^2 - \mathbf{N}^2 = 1 \quad \text{and} \quad \mathbf{M}\circ\mathbf{N} = 0,$$

implies that

$$P = \frac{1}{2}(1 + J\mathbf{M} + I\mathbf{N}) = J\mathbf{M}\frac{1}{2}\Big(1 + \frac{J}{\mathbf{M}^2}\big(\mathbf{M} + I\mathbf{MN}\big)\Big) = J\mathbf{M}\frac{1}{2}(1 + J\hat{\mathbf{b}}),$$

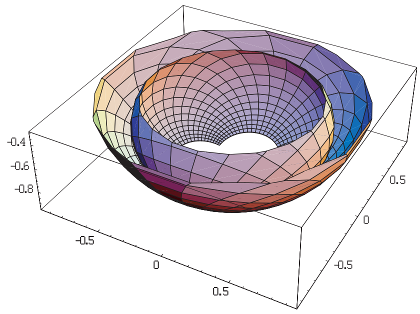

Figure 1: The larger dish shows unit direction velocities on the Riemann sphere defined by $\hat{\mathbf{M}}$. The inner dish shows the velocity vectors $\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2 \tanh 2\phi$, measured from the origin. The dishes coincide at the south pole, when $2\phi \to \infty$ and the velocity $v \to c$.

for $\hat{\mathbf{b}} := \frac{1}{\mathbf{M}^2}\left(\mathbf{M} + I\mathbf{MN}\right) \in \mathbb{G}^2_{1,3}$. For the idempotent T_E^+, defined in (36), we have

$$T_E^+ = (1 + \Lambda \mathbf{e}_1)E_3^+ = \frac{1}{2}(1 + J\mathbf{M} + I\mathbf{N}) = J\mathbf{M}E_3^+ = \mathbf{M}\mathbf{e}_3 E_3^+, \tag{38}$$

since in this case $\hat{\mathbf{b}} = \mathbf{e}_3$. Because T_E^+ is an idempotent, it further follows that

$$T_E^+ = J\mathbf{M}E_3^+ = \mathbf{M}^2(\hat{\mathbf{M}}E_3^+\hat{\mathbf{M}})E_3^+ = \mathbf{M}^2 \hat{A}_+ E_3^+, \tag{39}$$

where $\hat{A}_+ := (\hat{\mathbf{M}}E_3^+\hat{\mathbf{M}})$, and $\hat{\mathbf{M}} := \frac{\mathbf{M}}{\sqrt{\mathbf{M}^2}}$. We also identify $J\hat{\mathbf{a}}$ in $\hat{A}_+$, by $\hat{\mathbf{a}} = \hat{\mathbf{M}}\mathbf{e}_3\hat{\mathbf{M}} \in \mathbb{G}^2_{1,3}$.

Now decompose the complex unit vector $\hat{\mathbf{M}} = \mathbf{m}_1 + I\mathbf{m}_2 \in \mathbb{G}_3^{1+2}$, by writing

$$\hat{\mathbf{M}} = \hat{\mathbf{m}}_1 \cosh\phi + I\hat{\mathbf{m}}_2 \sinh\phi = e^{\phi \hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2}\hat{\mathbf{m}}_1, \tag{40}$$

where $\cosh\phi = |\mathbf{m}_1|$ and $\sinh\phi = |\mathbf{m}_2|$, and note that since $\hat{\mathbf{M}}^2 = 1$,

$$\hat{\mathbf{m}}_1\hat{\mathbf{m}}_2 = -\hat{\mathbf{m}}_2\hat{\mathbf{m}}_1 = I(\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2),$$

so $I = \mathbf{e}_{123} = \hat{\mathbf{m}}_1\hat{\mathbf{m}}_2(\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2)$. It then follows that the expression for $\hat{\mathbf{a}}$, given after equation (39), becomes

$$\hat{\mathbf{a}} = \hat{\mathbf{M}}\mathbf{e}_3\hat{\mathbf{M}} = e^{\phi \hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2}\hat{\mathbf{m}}_1\mathbf{e}_3\hat{\mathbf{m}}_1 e^{-\phi \hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2}, \tag{41}$$

which expresses that the North Pole $\mathbf{e}_3$ of the Riemann sphere is rotated π radians in the plane of $I\hat{\mathbf{m}}_1$ into the point $\hat{\mathbf{m}}_1\mathbf{e}_3\hat{\mathbf{m}}_1$, and then undergoes the *Lorentz Boost* defined by the unit vector $\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2$, with velocity $\tanh 2\phi = v/c$, to give the unit vector $\hat{\mathbf{a}}$. We can think of the Riemann sphere, itself, as undergoing a Lorentz boost with velocity $\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2 \tanh 2\phi$. See Figure 1.

We can now decompose the parity invariant part S_E of the geometric Dirac Spinor S, into

$$S_E = \Omega_0 T_E^+ = J\Omega_0\mathbf{M}E_3^+ = \Omega_0\mathbf{M}^2\hat{A}_3^+E_3^+, \tag{42}$$

where $\Omega_0=\varphi_1+J\varphi_3$, and

$$T_E^+=(1+\Lambda\mathbf{e}_1)E_3^+=\frac{1}{2}\big(1+J\mathbf{M}+I\mathbf{N}\big),$$

for $\Lambda=\Omega_0^{-1}\Omega_1$. We then have

$$\Lambda\mathbf{e}_1-JI\Lambda\mathbf{e}_2+J\mathbf{e}_3=J\mathbf{M}+I\mathbf{N}, \text{ and } \overline{\Lambda}\mathbf{e}_1+JI\overline{\Lambda}\mathbf{e}_2-J\mathbf{e}_3=-J\mathbf{M}+I\mathbf{N}.$$

Solving these equations for $\mathbf{M}$, gives

$$\mathbf{M}=\frac{\Lambda-\overline{\Lambda}}{2}J\mathbf{e}_1-\frac{\Lambda+\overline{\Lambda}}{2}I\mathbf{e}_2+\mathbf{e}_3, \quad \mathbf{e}_3\circ\mathbf{M}=1, \quad \text{and} \quad \mathbf{M}^2=1-\Lambda\overline{\Lambda}. \tag{43}$$

Writing $\mathbf{M}=\mathbf{x}+\mathbf{e}_3=\sum_{k=1}^3\alpha_k\mathbf{e}_k$, for $\alpha_k\in\mathbb{G}_3^{0+3}$, where $\mathbf{x}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$ and $\alpha_3=1$, the Ω-matrix (27) of $\mathbf{M}$ takes the form

$$[\mathbf{M}]_\Omega=\begin{pmatrix} J & \alpha_1-i\alpha_2\\ \alpha_1+i\alpha_2 & -J\end{pmatrix}_\Omega.$$

If for the complex idempotent P, given in (37), we make the substitution $\mathbf{M}=\mathbf{m}$ and $\mathbf{N}=\mathbf{n}$ for the vectors $\mathbf{m},\mathbf{n}\in\mathbb{G}_3$, then the idempotent $P=\frac{1}{2}(1+J\mathbf{m}+I\mathbf{n})$. Letting $i\to I$, so that $J\to 1$, the idempotent P becomes $s=\frac{1}{2}(1+\mathbf{m}+I\mathbf{n})$, which is the idempotent that was studied in the case of the representation of the Pauli spinors on the Riemann sphere, [1, (23)].

The complex unit vector $\hat{\mathbf{a}}$, defined in (41), can be directly expressed in terms of Λ. Using (43), we find that

$$\hat{\mathbf{a}}=\hat{\mathbf{M}}\mathbf{e}_3\hat{\mathbf{M}}=(-\mathbf{e}_3\hat{\mathbf{M}}+2\mathbf{e}_3\circ\hat{\mathbf{M}})\hat{\mathbf{M}}=-\mathbf{e}_3+2(\mathbf{e}_3\circ\hat{\mathbf{M}})\hat{\mathbf{M}}$$
$$=\frac{2}{1-\Lambda\overline{\Lambda}}\mathbf{M}-\mathbf{e}_3,$$

which in turn gives the projective relation

$$\mathbf{M}=\mathbf{x}+\mathbf{e}_3=\frac{1-\Lambda\overline{\Lambda}}{2}(\hat{\mathbf{a}}+\mathbf{e}_3) \quad\iff\quad \mathbf{M}=\frac{2}{\hat{\mathbf{a}}+\mathbf{e}_3} \tag{44}$$

showing that the projection $\mathbf{x}$ of $\mathbf{M}$ onto the complex hyperplane defined by $\mathbf{e}_1,\mathbf{e}_2$, is on the complex ray extending from the south pole $-\mathbf{e}_3$. This is equivalent to saying that $\mathbf{M}$ is a complex multiple of the complex vector $\hat{\mathbf{a}}+\mathbf{e}_3$.

Using (42) and (43), we find that

$$S_E^2=\Omega_0^2T_E^+=\Omega_0^2J\mathbf{M}E_3^+$$

or

$$S_E=J\Omega_0\mathbf{M}E_3^+=J\Omega_0\sqrt{\mathbf{M}^2}\hat{\mathbf{M}}E_3^+=J\frac{\Omega_0}{\sqrt{\Omega_0\overline{\Omega}_0}}\sqrt{\Omega_0\overline{\Omega}_0-\Omega_1\overline{\Omega}_1}\,\hat{\mathbf{M}}E_3^+,$$

so that

$$S_E=Je^{\omega}\hat{\mathbf{M}}E_3^+=\hat{\mathbf{M}}\mathbf{e}_3e^{\omega_1+\omega_2\mathbf{e}_3}E_3^+=e^{\omega_1+\omega_2\hat{\mathbf{a}}}\hat{\mathbf{a}}\hat{\mathbf{M}}E_3^+=Je^{\omega}\hat{A}_3^+\hat{\mathbf{M}}, \tag{45}$$

for $\omega = \omega_1 + \omega_2 J$ where $\omega_k = \phi_k + \theta_k I \in \mathbb{G}_3^{0+3}$ for $k = 1, 2$, and

$$\hat{\mathbf{a}} := \hat{\mathbf{M}}\mathbf{e}_3\hat{\mathbf{M}}, \quad A_3^+ := \hat{\mathbf{M}}E_3^+\hat{\mathbf{M}}.$$

The $\phi_k, \theta_k \in \mathbb{R}$ are defined in such a way that $e^{\omega_1} = \sqrt{\Omega_0\overline{\Omega}_0 - \Omega_1\overline{\Omega}_1} = \sqrt{\det[\Omega]}$, and $e^{J\omega_2} := \frac{\Omega_0}{\sqrt{\Omega_0\overline{\Omega}_0}}$. Note that $\hat{\mathbf{M}}$ can be further decomposed using (40).

There are two additional canonical forms, derived from (45), that are interesting. We have

$$S_E = e^{\omega_1} e^{I\hat{\mathbf{b}}_e z_e} e^{\omega_2 \mathbf{e}_3} E_3^+ = e^{\omega_1} e^{\omega_2 \hat{\mathbf{a}}} e^{I\hat{\mathbf{b}}_a z_a} E_3^+, \tag{46}$$

where

$$\hat{\mathbf{M}}\mathbf{e}_3 = \hat{\mathbf{M}} \circ \mathbf{e}_3 + \hat{\mathbf{M}} \otimes \mathbf{e}_3 = \cos z_e + I\hat{\mathbf{b}}_e \sin z_e = e^{I\hat{\mathbf{b}}_e z_e}$$

for $z_e \in \mathbb{G}_{1,3}^{0+4}$, $\hat{\mathbf{b}}_e \in \mathbb{G}_{1,3}^2$, and $\hat{\mathbf{M}} \otimes \mathbf{e}_3 := \frac{1}{2}(\hat{\mathbf{M}}\mathbf{e}_3 - \mathbf{e}_3\hat{\mathbf{M}})$ is the *anti-symmetric product*. Similarly,

$$\hat{\mathbf{a}}\hat{\mathbf{M}} = \hat{\mathbf{a}} \circ \hat{\mathbf{M}} + \hat{\mathbf{a}} \otimes \hat{\mathbf{M}} = \cos z_a + I\hat{\mathbf{b}}_a \sin z_a = e^{I\hat{\mathbf{b}}_a z_a}$$

for $z_a \in \mathbb{G}_{1,3}^{0+4}$ and $\hat{\mathbf{b}}_a \in \mathbb{G}_{1,3}^2$. Note that $\hat{\mathbf{b}}_e$ and $\mathbf{e}_3$ anti-commute, as do $\hat{\mathbf{b}}_a$ and $\hat{\mathbf{a}}$. In our decomposition, it is the unit bivector $\hat{\mathbf{M}}$ that defines the Lorentz transformation (41) associated with a Dirac spinor.

Recalling (26), (42), and (45), we define a Pauli E-spinor by

$$|\Omega\rangle_E := \sqrt{2}(\Omega_0 + \Omega_1\mathbf{e}_1)E_3^+ = \sqrt{2}\Omega_0 T_E^+ = \sqrt{2}Je^{\omega}\hat{\mathbf{M}}E_3^+, \tag{47}$$

Given the E-spinor $|\Phi\rangle_E = \sqrt{2}Je^{\omega'}\hat{\mathbf{M}}'E_3^+$, its conjugate is specified by

$$\langle\Phi|_E = \widetilde{\overline{|\Phi\rangle}}_E = \sqrt{2}Je^{\overline{\omega}'}E_3^+\hat{\mathbf{M}}',$$

which we use to calculate

$$\langle\Phi|_E|\Omega\rangle_E = 2e^{\overline{\omega}'+\omega}E_3^+\hat{\mathbf{M}}'\hat{\mathbf{M}}E_3^+ = 2e^{\overline{\omega}'+\omega}E_3^+\Big(\hat{\mathbf{M}}' \circ \hat{\mathbf{M}} + J(\hat{\mathbf{M}}' \otimes \hat{\mathbf{M}}) \circ \mathbf{e}_3\Big), \tag{48}$$

which can be expressed in the alternative form,

$$\langle\Phi|_E|\Omega\rangle_E = 2e^{\overline{\omega}'+\omega}\hat{\mathbf{M}}'\hat{B}_3^+\hat{A}_3^+\hat{\mathbf{M}} \quad \iff \quad \langle\Omega|_E|\Phi\rangle_E = 2e^{\overline{\omega}+\omega'}\hat{\mathbf{M}}\hat{A}_3^+\hat{B}_3^+\hat{\mathbf{M}}', \tag{49}$$

and

$$|\Omega\rangle_E\langle\Omega|_E = 2e^{2\omega_1}\hat{\mathbf{M}}E_3^+\hat{\mathbf{M}} = 2e^{2\omega_1}\hat{A}_3^+. \tag{50}$$

The result (49) is in agreement with (29) when $\Phi = \Omega$, and identical in terms of the complex components ϕ_k and φ_k, for which

$$\langle\Phi|_E|\Omega\rangle_E = 2E_3^+\Big(\overline{\phi}_1\varphi_1 + \overline{\phi}_2\varphi_2 - \overline{\phi}_3\varphi_3 - \overline{\phi}_4\varphi_4 + 2im(\overline{\phi}_1\varphi_3 + \overline{\phi}_2\varphi_4)I\Big),$$

so the Dirac inner product (34) can also be expressed by

$$\langle\Phi|\Omega\rangle = \Big\langle\langle\Phi|_E|\Omega\rangle_E\Big\rangle_{\mathbb{C}} = \frac{1}{4}\Big(\langle\Phi|_E|\Omega\rangle_E + \langle\Phi|_E|\Omega\rangle_E^- + \langle\Phi|_E|\Omega\rangle_E^\dagger + \langle\Phi|_E|\Omega\rangle_E^*\Big), \tag{51}$$

Of course, the $\langle\,.\,\rangle^-$, $\langle\,.\,\rangle^\dagger$ and $\langle\,.\,\rangle^*$-conjugation operators all depend upon the decomposition of $I=\mathbf{e}_{123}$ in the rest-frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}\in\mathbb{G}^+_{1,3}$. The importance of this result is that we are able to directly define the Dirac inner product in terms of conjugations of the Pauli E-inner product given in (49). It is also interesting to compare this result with (33).

By a *gauge transformation* of an E-spinor $|\Omega\rangle_E$, we mean the spinor $e^\alpha|\Omega\rangle_E$, where $\alpha=I\theta+J\phi$ for $\theta,\phi\in\mathbb{R}$. Now note that

$$\det[e^\alpha\Omega]=\det\begin{pmatrix} e^\alpha\Omega_0 & e^{\overline{\alpha}}\,\overline{\Omega}_1\\ e^\alpha\Omega_1 & e^{\overline{\alpha}}\,\overline{\Omega}_0\end{pmatrix}=e^{2I\theta}\det[\Omega]=e^{2I\theta}\Omega_0\overline{\Omega}_0\mathbf{M}^2,$$

since $\det[\Omega]=\Omega_0\overline{\Omega}_0\mathbf{M}^2$. We say that $|\Omega'\rangle_E=e^\alpha|\Omega\rangle_E$ is *gauge normalized* if

$$\det[\Omega']\in\mathbb{R},\;\; e^{i\phi}:=\frac{\Omega_0'}{\sqrt{\Omega_0'\overline{\Omega}_0'}}\in\mathbb{C}. \tag{52}$$

The definition of an E-spinor is closely related to the Dirac g-spinor definition (31),

$$|\Omega\rangle=\sqrt{2}\,|\Omega\rangle_E\,\gamma_0^+=2Je^\omega\hat{\mathbf{M}}E_3^+\gamma_0^+. \tag{53}$$

Given the g-spinor $|\Phi\rangle=2Je^{\omega'}\hat{\mathbf{M}}'E_3^+\gamma_0^+$, its conjugate is

$$\langle\Phi|=\sqrt{2}\gamma_0^+\langle\Phi|_E=2\gamma_0^+Je^{\overline{\omega}'}E_3^+\hat{\mathbf{M}}'.$$

Using (49) and (51), we calculate

$$\langle\Phi||\Omega\rangle=2\gamma_0^+\langle\Phi|_E|\Omega\rangle_E\gamma_0^+=4\gamma_0^+e^{\overline{\omega}'+\omega}E_3^+\hat{\mathbf{M}}'\hat{\mathbf{M}}E_3^+\gamma_0^+. \tag{54}$$

The Dirac inner product, expressed in (51), shows that

$$\langle\Phi|\Omega\rangle=\big\langle\langle\Phi||\Omega\rangle\big\rangle_{\mathbb{C}}=2\big\langle\gamma_0^+\langle\Phi|_E|\Omega\rangle_E\gamma_0^+\big\rangle_{\mathbb{C}}. \tag{55}$$

Using (29), (45), and (49) for $|\Omega\rangle_E=\sqrt{2}Je^\omega\hat{\mathbf{M}}E_3^+$,

$$\langle\Omega|_E|\Omega\rangle_E=2e^{2\omega_1}E_3^+\hat{\mathbf{M}}\hat{\mathbf{M}}E_3^+=2\det[\Omega]E_3^+, \tag{56}$$

so

$$\langle\Omega|\Omega\rangle_E:=\rho_1^2e^{2I\theta_1}=\det[\Omega],$$

where $\rho_1:=e^{\phi_1}$. When $|\Omega\rangle_E$ is gauge normalized to

$$|\Omega'\rangle_E:=e^{-I\theta_1-J\phi_2}|\Omega\rangle_E,$$

and similarly, $|\Phi\rangle_E$ is gauge normalized to $|\Phi'\rangle_E$, then the inner products (55) and (56) become more simply related. For gauge normalized E-spinors $|\Omega\rangle_E$ and $|\Phi\rangle_E$, the relation (54) simplifies to

$$\langle\Phi||\Omega\rangle=2\gamma_0^+\langle\Phi|_E|\Omega\rangle_E\gamma_0^+=4e^{\overline{\omega}'+\omega}\gamma_0^+E_3^+\hat{\mathbf{M}}'\hat{\mathbf{M}}E_3^+\gamma_0^+. \tag{57}$$

A geometric Dirac spinor $|\Omega\rangle$ represents the *physical state* of a spin $\frac{1}{2}$-particle if $\det[\Omega]=1$.

Before our next calculation, for $\hat{A}_+=\frac{1}{2}(1+J\hat{\mathbf{a}})$ and $\hat{B}=\frac{1}{2}(1+J\hat{\mathbf{b}})$ for $\hat{\mathbf{a}},\hat{\mathbf{b}}\in\mathbb{G}^2_{1,3}$, we find that

$$\hat{A}_+\hat{B}_+\hat{A}_+=\frac{1}{4}\hat{A}_+(1+\hat{B}_E)(1+\hat{A}_E)=\frac{1}{4}\hat{A}_+(1+\hat{B}_E+\hat{A}_E+\hat{B}_E\hat{A}_E)$$

$$=\frac{1}{4}\hat{A}_+(1+\hat{B}_E+\hat{A}_E-\hat{A}_E\hat{B}_E+2\hat{A}_E\circ B_E)=\frac{1}{2}\hat{A}_+(1+\hat{\mathbf{a}}\circ\hat{\mathbf{b}}). \tag{58}$$

Using this result, (49) and (51), we calculate

$$\langle\Omega|_E|\Phi\rangle_E\langle\Phi|_E|\Omega\rangle_E=4e^{2(\omega_1+\omega_1')}\hat{\mathbf{M}}_\Omega\hat{A}^+_\Omega\hat{A}^+_\Phi\hat{A}^+_\Omega\hat{\mathbf{M}}_\Omega$$

$$=2e^{2(\omega_1+\omega_1')}(1+\hat{\mathbf{a}}_\Phi\circ\hat{\mathbf{a}}_\Omega)E_3^+=2\det[\Omega]\det[\Phi](1+\hat{\mathbf{a}}_\Phi\circ\hat{\mathbf{a}}_\Omega)E_3^+. \tag{59}$$

Using (51), we would now like to calculate $\langle\Omega|\Phi\rangle\langle\Phi|\Omega\rangle$ for gauge normalized physical states $|\Omega\rangle_E$ and $|\Phi\rangle_E$. Here, we consider a special case where the formula (51) simplifies to

$$\langle\Phi|\Omega\rangle=\frac{1}{2}\Big(\langle\Phi|_E|\Omega\rangle_E+\langle\Phi|_E|\Omega\rangle_E^\dagger\Big). \tag{60}$$

Referring to (48), this will occur when

$$\hat{\mathbf{M}}_\Phi\circ\hat{\mathbf{M}}_\Omega=(\hat{\mathbf{M}}_\Phi\circ\hat{\mathbf{M}}_\Omega)^\dagger \quad\text{and}\quad (\hat{\mathbf{M}}_\Phi\otimes\hat{\mathbf{M}}_\Omega)\circ\mathbf{e}_3=-\Big((\hat{\mathbf{M}}_\Phi\otimes\hat{\mathbf{M}}_\Omega)\circ\mathbf{e}_3\Big)^\dagger. \tag{61}$$

Indeed, if $|\Omega\rangle_E$ and $|\Phi\rangle_E$ are gauge normalized spinor states, then using (51) and (59), we have

$$\langle\Omega|\Phi\rangle\langle\Phi|\Omega\rangle=\frac{1}{4}\Big(\langle\Omega|_E|\Phi\rangle_E+\langle\Omega|_E|\Phi\rangle_E^\dagger\Big)\Big(\langle\Phi|_E|\Omega\rangle_E+\langle\Phi|_E|\Omega\rangle_E^\dagger\Big)$$

$$=\frac{1}{2}\det[\Omega]\det[\Phi]\big(1+\hat{\mathbf{a}}_\Phi\circ\hat{\mathbf{a}}_\Omega\big). \tag{62}$$

If $\det[\Omega]=\det[\Phi]=1$, then the probability of finding the spin $\frac{1}{2}$-particle in the state $|\Phi\rangle$, having prepared the particle in the state $|\Omega\rangle$, is given by $\langle\Omega|\Phi\rangle\langle\Phi|\Omega\rangle$. There is a particularly simple formula for evaluating (62), directly in terms of $\mathbf{M}_\Omega$ and $\mathbf{M}_\Phi$, which follows from (44). We find that

$$\langle\Omega|\Phi\rangle\langle\Phi|\Omega\rangle=\frac{1}{2}\big(1+\hat{\mathbf{a}}_\Phi\circ\hat{\mathbf{a}}_\Omega\big)=1-\frac{(\mathbf{M}_\Omega-\mathbf{M}_\Phi)^2}{\mathbf{M}_\Omega^2\mathbf{M}_\Phi^2}. \tag{63}$$

Let us consider important examples of gauge normalized states for which (63) applies. Let

$$\mathbf{M}_\Omega=e^{\phi_x\mathbf{e}_3}\mathbf{x}+\mathbf{e}_3=\mathbf{x}\cosh\phi_x+\mathbf{e}_3+I\mathbf{e}_3\times\mathbf{x}\sinh\phi_x, \tag{64}$$

for $\mathbf{x}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$ and $\phi_x\in\mathbb{R}$. This $\mathbf{M}_\Omega$ is defined by $\Lambda=\Omega_0^{-1}\Omega_1$, for

$$\Omega_0=\varphi_1+J\varphi_3=1,\ \text{ and }\ \Omega_1=\varphi_4+J\varphi_2=Je^{-J\phi_x}(x_1+x_2i),$$

or alternatively by

$$\Omega_0 = \varphi_1 + J\varphi_3 = e^{\phi_x}, \text{ and } \Omega_1 = \varphi_4 + J\varphi_2 = J(x_1 + x_2 i).$$

For $\mathbf{M}_\Omega$, we calculate $\mathbf{M}_\Omega^2 = \mathbf{x}^2 + 1 \geq 1$, and

$$\hat{\mathbf{M}}_\Omega = \frac{\mathbf{x}\cosh\phi_x + \mathbf{e}_3}{\sqrt{\mathbf{x}^2+1}} + \frac{I(\mathbf{e}_3 \times \mathbf{x})\sinh\phi_x}{\sqrt{\mathbf{x}^2+1}} = \mathbf{m}_1 + I\mathbf{m}_2, \tag{65}$$

which defines the velocity

$$\tanh\frac{\omega_x}{2} = \frac{v}{c} := \sqrt{\frac{(\mathbf{m}_2)^2}{(\mathbf{m}_1)^2}} = \sqrt{\frac{\mathbf{x}^2\sinh^2\phi_x}{\mathbf{x}^2\cosh^2\phi_x + 1}}$$

in the direction

$$\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2 = \frac{-\hat{\mathbf{x}} + \mathbf{e}_3|\mathbf{x}|\cosh\phi_x}{\sqrt{\mathbf{x}^2\cosh^2\phi_x + 1}}.$$

Given $\mathbf{M}_\Omega$, we can also find the element

$$\mathbf{M}_\Omega^\perp := \frac{1}{\mathbf{M}_\Omega \otimes \mathbf{e}_3}\mathbf{M}_\Omega = -\frac{1}{e^{\phi_x\mathbf{e}_3}\mathbf{e}_3}\mathbf{M}_\Omega$$

$$= \mathbf{e}_3\mathbf{x}^{-1}e^{-\phi_x\mathbf{e}_3}\left(e^{\phi_x\mathbf{e}_3}\mathbf{x} + \mathbf{e}_3\right) = -e^{\phi_x\mathbf{e}_3}\frac{\mathbf{x}}{\mathbf{x}^2} + \mathbf{e}_3, \tag{66}$$

for which $\hat{\mathbf{a}}_x^\perp = -\hat{\mathbf{a}}_x$. Notice that $\mathbf{M}_\Omega^\perp$ and $\mathbf{M}_\Omega$ are in the same inertial system defined by ϕ_x.

For $\mathbf{M}_\Omega$, and a second state $\mathbf{M}_\Phi = e^{\phi_y\mathbf{e}_3}\mathbf{y} + \mathbf{e}_3$, we now calculate

$$\frac{1}{2}(1 + \hat{\mathbf{a}}_x \circ \hat{\mathbf{a}}_y) = 1 - \frac{(\mathbf{M}_\Omega - \mathbf{M}_\Phi)^2}{\mathbf{M}_\Omega^2\mathbf{M}_\Phi^2}$$

$$= 1 - \frac{\mathbf{x}^2 + \mathbf{y}^2 - 2\Big(\cosh(\phi_x - \phi_y)\mathbf{x}\cdot\mathbf{y} + \sinh(\phi_x - \phi_y)\mathbf{e}_3 \wedge \mathbf{x} \wedge \mathbf{y}\Big)}{\Big(1 + \mathbf{x}^2\Big)\Big(1 + \mathbf{y}^2\Big)}. \tag{67}$$

When $\phi_x = \phi_y$, so that $\mathbf{M}_\Omega$ and $\mathbf{M}_\Phi$ are in the same inertial system, the formula (67) reduces to

$$\frac{1}{2}(1 + \hat{\mathbf{a}}_x \circ \hat{\mathbf{a}}_y) = \frac{\mathbf{x}^2\mathbf{y}^2 + 2\mathbf{x}\cdot\mathbf{y} + 1}{(\mathbf{x}^2+1)(\mathbf{y}^2+1)}, \tag{68}$$

which is the discrete Pauli probability of measuring the state $\mathbf{M}_\Omega$, given the state $\mathbf{M}_\Phi$, and it is independent of the inertial system in which it is calculated [1]. Note that the inertial system defined by ϕ_x is different than the inertial system of $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_y$ defined by ω_x. See Figure 2.

Also, when $\mathbf{y} = \rho\mathbf{x}$, the pseudoscalar term will vanish. For example, for $\mathbf{x} = \mathbf{y}$, and $\phi_x = 1,\ \phi_y = 2$, the formula (67) gives

$$\frac{1}{2}(1 + \hat{\mathbf{a}}_x \circ \hat{\mathbf{a}}_y) = 1 + 2\mathbf{x}^2\frac{\big(\cosh(1) - 1\big)}{(1 + \mathbf{x}^2)^2} \geq 1,$$

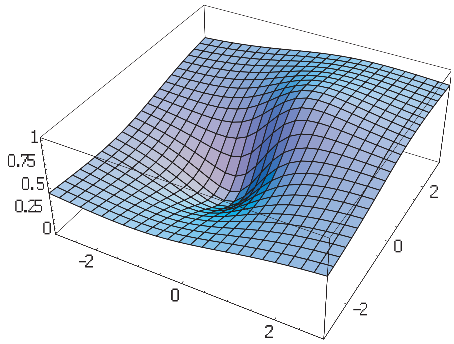

Figure 2: Shown is the discrete probability distribution for observing a particle in a state $\hat{\mathbf{a}}_x$, given that the particle was prepared in the state $\hat{\mathbf{a}}_y$ immediately preceding. For this figure, $\mathbf{y} = (1,1)$, $\phi_x = \phi_y = 0$, $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$.

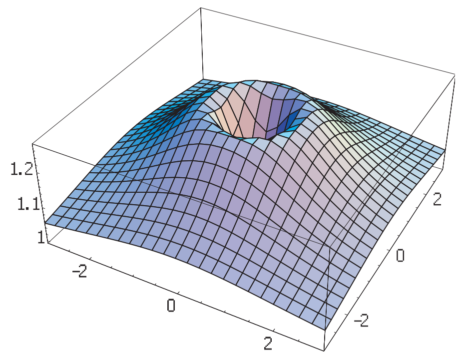

Figure 3: Shown is the discrete distribution for observing a particle in a state $\hat{\mathbf{a}}_x$, given that the particle was prepared in the state $\hat{\mathbf{a}}_y$ immediately preceding. For this figure, $\mathbf{x} = \mathbf{y}$, $\phi_x = 1$, $\phi_y = 2$, $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$. The values are all greater than or equal to one and therefore do not represent a probability.

which cannot represent a probability distribution, but gives the volcano shown in Figure 3. In the general case for $\mathbf{M}_\Omega$ and $\mathbf{M}_\Phi$, the term $\mathbf{M}_\Omega \circ \mathbf{M}_\Phi$ will contain the pseudoscalar element $\mathbf{e}_3 \wedge \mathbf{x} \wedge \mathbf{y}$, which vanishes in the special cases considered above.

There is one other spin state that we will consider. Let

$$\mathbf{M} = \mathbf{x}(1+\mathbf{e}_3)+\mathbf{e}_3 = x\mathbf{e}_1 + y\mathbf{e}_2 + \mathbf{e}_3 + I(y\mathbf{e}_1 - x\mathbf{e}_2) = \hat{\mathbf{M}}.$$

This state is realized for $\varphi_1 = 1, \varphi_3 = 0$ and $\varphi_2 = \varphi_4 = x+iy$, so that $\Lambda = (1+J)(x+iy)$. We then calculate

$$\hat{\mathbf{a}} = \hat{\mathbf{M}}\mathbf{e}_3\hat{\mathbf{M}} = -\mathbf{e}_3 + 2(\mathbf{e}_3 \circ \hat{\mathbf{M}})\hat{\mathbf{M}} = 2\mathbf{x}(1+\mathbf{e}_3)+\mathbf{e}_3.$$

This state is a limiting state of (64), when $\phi_x \to -\infty$, and cannot represent the state of a spin $\frac{1}{2}$ particle.

3 Fierz Identities

The so-called Fierz identities are quadratic relations between the *physical observables* of a Dirac spinor. The identities are most easily calculated in terms of the spinor operator ψ of a Dirac spinor. Whereas spinors are usually classified using irreducible representations of the Lorentz group $SO^+_{1,3}$, Pertti Lounesto has developed a classification scheme based upon the Fierz identities [5, P.152,162]. A Dirac spinor, which describes an electron, the subject of this paper, is characterized by the property that $\det[\psi]_\Omega \neq 0$. Other types of spinors, such as Majorana and Weyl spinors, and even Lounesto's *boomerang spinor*, can all be classifed by bilinear covariants of their spinor operators $\psi \in \mathbb{G}_3 \widetilde{=} \mathbb{G}^+_{1,3}$.

For $g \in \mathbb{G}_3$, let g^- and $g^\dagger$, and g^* denote the conjugations of *inversion*, *reversion*, and their composition $g^* = (g^-)^\dagger$, respectively. Thus for $g = \alpha + \mathbf{x} + I\mathbf{y}$, where $\alpha \in \mathbb{G}^{0+3}$,

$$g^- = \alpha^\dagger - \mathbf{x} + I\mathbf{y}, \quad g^\dagger = \alpha^\dagger + \mathbf{x} - I\mathbf{y}, \quad \text{and} \quad g^* = \alpha - \mathbf{x} - I\mathbf{y}. \tag{69}$$

Any element $\omega \in \mathbb{G}_{1,3}$ can be written $\omega = g_1 + g_2\gamma_0$ for some $g_1, g_2 \in \mathbb{G}_3$. For $\omega \in \mathbb{G}_{1,3}$, as before, let $\widetilde{\omega}$ denote the conjugation of reverse in $\mathbb{G}_{1,3}$. Since $g^\dagger = \gamma_0 g^* \gamma_0$,

$$\widetilde{\omega} = \widetilde{g}_1 + \gamma_0\widetilde{g}_2 = g_1^* + \gamma_0 g_2^* = g_1^* + g_2^\dagger\gamma_0.$$

Each unit timelike vector $\gamma_0 \in \mathbb{G}_{1,3}$ determines a unique Pauli sub-algebra $\mathbb{G}_3 \widetilde{=} \mathbb{G}^+_{1,3}$, and its corresponding *rest-frame* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, satisfying $\mathbf{e}_k\gamma_0 = -\gamma_0\mathbf{e}_k$ for $k = 1,2,3$. The relative conjugations of the Pauli algebra $\mathbb{G}_3$ can be defined directly in terms of the the conjugations of the larger algebra $\mathbb{G}_{1,3}$. For $g \in \mathbb{G}_3$,

$$g^- := \gamma_0 g \gamma_0, \quad g^\dagger := \gamma_0 \widetilde{g} \gamma_0, \quad g^* := (g^-)^\dagger = \widetilde{g}.$$

Using (69), we calculate

$$gg^\dagger = \alpha\alpha^\dagger + \mathbf{x}^2 + \mathbf{y}^2 + (\alpha + \alpha^\dagger)\mathbf{x} - (\alpha - \alpha^\dagger)I\mathbf{y} = r + s\hat{\mathbf{a}} \in \mathbb{G}^{0+1},$$

$$gg^* = (\alpha + \mathbf{x} + I\mathbf{y})(\alpha - \mathbf{x} - I\mathbf{y}) = \alpha^2 - (\mathbf{x} - I\mathbf{y})^2 = \det[g]_\Omega \in \mathbb{G}_3^{0+3},$$

or $gg^* = R_1 + IR_2$ for $R_1, R_2 \in \mathbb{R}$.

Define

$$\mathbf{J} := g\gamma_0 g^* = g\gamma_0 g^* \gamma_0\gamma_0 = gg^\dagger \gamma_0 = \theta\gamma_0 + \phi\hat{a},$$

where $\hat{a}^2 = -1$ and $\gamma_0 \cdot \hat{a} = 0$,

$$\mathbf{S} := g\gamma_{12} g^* = -Ig\mathbf{e}_3 g^*, \quad \text{and} \quad \mathbf{K} := g\gamma_3 g^* = g\mathbf{e}_3 g^\dagger \gamma_0.$$

We then find

$$\mathbf{J}^2 = g\gamma_0 g^* g\gamma_0 g^* = \big(R_1 + IR_2\big)\big(R_1 - IR_2\big) = R_1^2 + R_2^2 = r^2 - s^2 \geq 0,$$

$\mathbf{S}$ is a bivector in $\mathbb{G}_{1,3}$ since $\widetilde{\mathbf{S}} = \mathbf{S}^* = -\mathbf{S}$, and

$$\mathbf{S}^2 = \big(-Ig\mathbf{e}_3 g^*\big)\big(-Ig\mathbf{e}_3 g^*\big) = -g\mathbf{e}_3 R\mathbf{e}_3 g^* = -R^2$$

where $R = R_1 + IR_2$. Similarly, $\widetilde{\mathbf{K}} = \mathbf{K}$ and the *inverse* of $\mathbf{K} \in \mathbb{G}_{1,3}$ is $-\mathbf{K}$, from which it follows that $\mathbf{K} \in \mathbb{G}_{1,3}^1$, and

$$\mathbf{K}^2 = g\gamma_3 g^* g\gamma_3 g^* = g\gamma_3 R\gamma_3 g^* = -RR^\dagger = -\mathbf{J}^2 \leq 0.$$

We also find that

$$\mathbf{KJ} = g\gamma_3 g^* g\gamma_0 g^* = g\gamma_3 R\gamma_0 g^* = g\mathbf{e}_3 g^* R^\dagger = I\mathbf{S}R^\dagger = \mathbf{K} \wedge \mathbf{J},$$

and consequently, $\mathbf{K} \cdot \mathbf{J} = 0$. Note also that

$$\mathbf{JS} = -Ig\gamma_0 g^* g\mathbf{e}_3 g^* = -Ig\gamma_0 R\mathbf{e}_3 g^* = IR^\dagger \mathbf{K},$$

and

$$\mathbf{JSK} = IR^\dagger \mathbf{K}^2 = -I|R|^2 R^\dagger.$$

Writing $g = (x_0 + \mathbf{x}) + I(y_0 + \mathbf{y})$, $g^\dagger = (x_0 + \mathbf{x}) - I(y_0 + \mathbf{y})$, and $g^* = (x_0 - \mathbf{x}) - I(y_0 - \mathbf{y})$, calculate

$$gg^\dagger = (x_0^2 + y_0^2 + \mathbf{x}^2 + \mathbf{y}^2) + 2(x_0\mathbf{x} + y_0\mathbf{y} + \mathbf{x} \times \mathbf{y}),$$

$$g^\dagger g = (x_0^2 + y_0^2 + \mathbf{x}^2 + \mathbf{y}^2) + 2(x_0\mathbf{x} + y_0\mathbf{y} - \mathbf{x} \times \mathbf{y}),$$

and

$$gg^* = g^* g = x_0^2 - y_0^2 + \mathbf{y}^2 - \mathbf{x}^2 + 2I\big(x_0 y_0 - \mathbf{x} \cdot \mathbf{y}\big) = R.$$

We see that

$$\mathbf{J} = gg^\dagger \gamma_0 = \theta\gamma_0 + \phi\hat{a} = (x_0^2 + y_0^2 + \mathbf{x}^2 + \mathbf{y}^2)\gamma_0 + 2(x_0\mathbf{x} + y_0\mathbf{y} + \mathbf{x} \times \mathbf{y})\gamma_0.$$

Acknowledgements

I thank Professor Melina Gomez Bock and her students for many lively discussions about quantum mechanics, and the Universidad de Las Americas-Puebla for many years of support. Professor Leonard Susskind's You Tube Lectures [13], inspired me to look closely at the derivation of the discreet probability distribution of the Pauli spin states of an electron.

References

[1] G. Sobczyk, *Part I: Vector Analysis of Spinors*,

`http://www.garretstar.com/paulispinI-14-05-15.pdf`

[2] D. Hestenes, *Spacetime Algebra*, Gordon and Breach 1966.

[3] G. Sobczyk, *Spacetime Vector Analysis*, Physics Letters A, Vol 84A, p.45-49, 1981.

[4] G. Sobczyk, *Geometry of Spin 1/2 Particles*, Revista Mexicana de Física, **61** (2015) 211-223.

`http://rmf.smf.mx/pdf/rmf/61/3/61_3_211.pdf`

[5] P. Lounesto, *Clifford Algebras and Spinors, 2nd Edition*. Cambridge University Press, Cambridge, 2001.

[6] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, New York 2013.

`http://www.garretstar.com/`

[7] D. Hestenes, *Clifford Algebra and the Interpretation of Quantum Mechanics*, in *Clifford Algebras and Their Applications in Mathematical Physics*, edited by J.S.R. Chisholm and A.K. Common, NATO ASI Series C: Mathematical and Physical Sciences Vol. 183, D. Reidel Publishing Company 1985.

[8] G. Sobczyk, Geometric Matrix Algebra, *Linear Algebra and its Applications*, 429 (2008) 1163-1173.

[9] D. Hestenes, *Zitterbewegung in Quantum Mechanics*, Found Physics (2010) 40:1-54.

`http://geocalc.clas.asu.edu/pdf/ZBWinQM15**.pdf`

[10] G. Sobczyk, Hyperbolic Number Plane, *The College Mathematics Journal*, Vol. 26, No. 4, pp.268-280, September 1995.

[11] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe*, CAPE JONATHAN (RAND) (2004).

[12] W.E. Baylis, J.D. Keselica, *The Complex Algebra of Physical Space: A Framework for Relativity*, *Advances in Applied Clifford Algebras*, Vol. 22, No. 3, pp. 537 - 561, 2012.

[13] L. Susskind, 9 **YouTube Lectures**: *Quantum Entanglements, Part 1*, Stanford University 2008.