Part II: Spacetime Algebra of Dirac Spinors

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Abstract

In Part I: Vector Analysis of Spinors, the author studied the geometry of two component spinors as points on the Riemann sphere in the geometric algebra \( \mathbb{G}_3 \) of three dimensional Euclidean space. Here, these ideas are generalized to apply to four component Dirac spinors on the complex Riemann sphere in the complexified geometric algebra \( \mathbb{G}_3(\mathbb{C}) \). The development of generalized Pauli matrices eliminate the need for the traditional Dirac gamma matrices of spacetime. Based upon our analysis, we make the surprising prediction that a spin \( \frac{1}{2} \) particle prepared in the spin-up state in one inertial system is observed in a calculated spin state in a different inertial system.

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0 Introduction

Since the birth of quantum mechanics a Century ago, scientists have been both puzzled and amazed about the seemingly inescapable occurrence of the imaginary number \( i = \sqrt{-1} \), first in the Pauli-Schrödinger equation for spin \( \frac{1}{2} \) particles in space, and later in the more profound Dirac equation of spacetime. Exactly what role complex numbers play in quantum mechanics is even today hotly debated. In a previous paper, “Vector Analysis of Spinors”, I show that the \( i \) occurring in the Schrödinger-Pauli equation for the electron should be interpreted as the unit pseudoscalar, or directed volume element, of the geometric algebra \( \mathbb{G}_3 \). This follows directly from the assumption that the famous Pauli matrices are nothing more than the components of the orthonormal space vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3 \) with respect to the spectral basis of the geometric algebra \( \mathbb{G}_3 \), [1]. Another basic assumption made is that the geometric algebra \( \mathbb{G}_3 \) of space is naturally identified as the even sub-algebra \( \mathbb{G}^+_3 \) of the spacetime algebra \( \mathbb{G}_{1,3} \), also known as the algebra of Dirac matrices, [2], [3].
This line of research began when I started looking at the foundations of quantum mechanics. In particular, I wanted to understand in exactly what sense the Dirac-Hestenes equation for the electron is equivalent to the standard Dirac equation. What I discovered was that the equations are equivalent only so long as the issues of parity and complex conjugation are not taken into consideration, [4]. In the present work, I show that $i = \sqrt{-1}$ in the Dirac equation must have a different interpretation, than the $i$ that occurs in simpler Schrödinger-Pauli theory. In order to turn both the Schrödinger-Pauli theory, and the relativistic Dirac theory, into strictly equivalent geometric theories, we replace the study of 2 and 4-component spinors with corresponding 2 and 4-component geometric spinors, defined by the minimal left ideals in the appropriate geometric algebras. As pointed out by the late Perni Lounesto, [5, p.327], “Juvet 1930 and Sauter 1930 replaced column spinors by square matrices in which only the first column was non-zero - thus spinor spaces became minimal left ideals in a matrix algebra”. In order to gives the resulting matrices a unique geometric interpretation, it is then only necessary to interpret these matrices as the components of geometric numbers with respect to the spectral basis of the appropriate geometric algebra [6, p.205].

The important role played by an idempotent, and its interpretation as a point on the Riemann sphere in the case of Pauli spinors, and as a point on the complex Riemann sphere in the case of Dirac spinors, make up the heart of our new geometric theory. Just as the spin state of an electron can be identified with a point on the Riemann sphere, and a corresponding unique point in the plane by stereographic projection from the South Pole, we find that the spin state of a relativistic electron can be identified by a point on the complex Riemann sphere, and its corresponding point in the complex 2-plane by a complex stereographic projection from the South Pole. In developing this theory, we find that the study of geometric Dirac spinors can be carried out by introducing a generalized set of $2 \times 2$ Pauli E-matrices over a 4-dimensional commutative ring with the basis $\{1, i, I, iI\}$, where $i = \sqrt{-1}$ and $I = e^{123}$ is the unit pseudo-scalar of the geometric algebra $G_3$. The setting for the study of quantum mechanics thereby becomes the complex geometric algebra $G_3(\mathbb{C})$. In order to study quantum mechanics in a real geometric algebra, eliminating the need for any artificial $i = \sqrt{-1}$, we would have to consider at least one of higher dimensional geometric algebras $G_{2,3}, G_{4,1}, G_{0,5}$ of the respective pseudoeuclidean spaces $\mathbb{R}^{2,3}, \mathbb{R}^{4,1}, \mathbb{R}^{0,5}$, [5, p.217], [7, p.326].

1 Geometric algebra of spacetime

The geometric algebra $G_3$ of an orthonormal rest frame $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ can be factored into an orthonormal frame $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ in the geometric algebra $G_{1,3}$ of the pseudo-Euclidean space $\mathbb{R}^{1,3}$ of Minkowski spacetime, by writing

$$e_k := \gamma_k \gamma_0 = -\gamma_0 \gamma_k \quad \text{for} \quad k = 1, 2, 3. \quad (1)$$

In doing so, the geometric algebra $G_3$ is identified with the elements of the even sub-algebra $G_{1,3}^+ \subset G_{1,3}$. A consequence of this identification is that space vectors $x = x_1 e_1 + x_2 e_2 + x_3 e_3 \in G_3^+ \subset G_{1,3}^+$ become spacetime bivectors in $G_{1,3}^+ \subset G_{1,3}$. In summary, the geometric algebra $G_{1,3}$, also known as spacetime algebra [2], has $2^4 = 16$ basis
elements generated by geometric multiplication of the $\gamma_{\mu}$ for $\mu = 0, 1, 2, 3$. Thus,

$$G_{1,3} := \text{gen}\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$$

obeying the rules

$$\gamma_0^2 = 1, \quad \gamma_0^2 = -1, \quad \gamma_{\mu\nu} := \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu$$

for $\mu \neq \nu, \mu, \nu = 0, 1, 2, 3$, and $k = 1, 2, 3$. Note also that the pseudo-scalar

$$\gamma_{0123} := \gamma_0 \gamma_2 \gamma_3 = e_1 e_2 e_3 = e_{123} = I$$

of $G_{1,3}$ is the same as the pseudo-scalar of the rest frame $\{e_1, e_2, e_3\}$ of $G_3$, and it anti-commutes with each of the spacetime vectors $\gamma_{\mu}$ for $\mu = 0, 1, 2, 3$.

In the above, we have carefully distinguished the rest frame $\{e_1, e_2, e_3\}$ of the geometric algebra $G_3 := \mathbb{G}^{1,3}_\mathbb{R}$. Any other rest frame $\{e'_1, e'_2, e'_3\}$ can be obtained by an ordinary space rotation of the rest frame $\{e_1, e_2, e_3\}$ followed by a Lorentz boost. In the spacetime algebra $G_{1,3}$, this is equivalent to defining a new frame of spacetime vectors $\{\gamma_{\mu}'| 0 \leq \mu \leq 3\} \subset G_{1,3}$, and the corresponding rest frame $\{e'_{\mu} = \gamma_{\mu}' \gamma_\mu\}^k = 1, 2, 3\}$ of a Euclidean space $\mathbb{R}^3$ moving with respect to the Euclidean space $\mathbb{R}^3$ defined by the rest frame $\{e_1, e_2, e_3\}$. Of course, the primed rest-frame $\{e'_1\}$, itself, generates a corresponding geometric algebra $G'_3 := \mathbb{G}^{1,3}_\mathbb{R}$. A much more detailed treatment of $G_3$ is given in [6, Chp.3], and in [8] I explore the close relationship that exists between geometric algebras and their matrix counterparts. The way we introduced the geometric algebras $G_3$ and $G_{1,3}$ may appear novel, but they perfectly reflect all the common relativistic concepts [6, Chp.11].

The well-known Dirac matrices can be obtained as a real sub-algebra of the $4 \times 4$ matrix algebra $\text{Mat}_\mathbb{C}(4)$ over the complex numbers where $i = \sqrt{-1}$. We first define the idempotent

$$u_{++} := \frac{1}{4} (1 + \gamma_0)(1 + i\gamma_2) = \frac{1}{4} (1 + i\gamma_2)(1 + \gamma_0),$$

(2)

where the unit imaginary $i = \sqrt{-1}$ is assumed to commute with all elements of $G_{1,3}$. Whereas it would be nice to identify this unit imaginary $i$ with the pseudo-scalar element $\gamma_{0123} = e_{123}$ as we did in $G_3$, this is no longer possible since $\gamma_{0123}$ anti-commutes with the spacetime vectors $\gamma_{\mu}$ as previously mentioned.

Noting that

$$\gamma_2 = \gamma_0 \gamma_2 \gamma_3 = e_2 e_1 = e_{21},$$

and similarly $\gamma_1 = e_{13}$, it follows that

$$e_{13} u_{++} = u_{+-} e_{13}, \quad e_1 u_{++} = u_{-+} e_1, \quad e_1 u_{++} = u_{-} e_1,$$

(3)

where

$$u_{+-} := \frac{1}{4} (1 + \gamma_0)(1 - i\gamma_2), \quad u_{-+} := \frac{1}{4} (1 - \gamma_0)(1 + i\gamma_2), \quad u_{--} := \frac{1}{4} (1 - \gamma_0)(1 - i\gamma_2).$$

The idempotents $u_{++}, u_{+-}, u_{-+}, u_{--}$ are mutually annihilating in the sense that the product of any two of them is zero, and partition unity

$$u_{++} + u_{+-} + u_{-+} + u_{--} = 1.$$  

(4)
By the spectral basis of the Dirac algebra \( G_{1,3} \), we mean the elements of the matrix

\[
\begin{pmatrix}
1 \\
e_{13} \\
e_3 \\
e_1
\end{pmatrix}
\begin{pmatrix}
1 & -e_{13} & e_3 & e_1
\end{pmatrix}
\begin{pmatrix}
u_{++} \\
u_{+} \\
u_{-} \\
u_{--}
\end{pmatrix}

= 
\begin{pmatrix}
u_{++} & -e_{13}u_{++} & e_3u_{++} & e_1u_{++} \\
e_{13}u_{++} & u_{++} & e_1u_{++} & -e_3u_{++} \\
e_3u_{++} & e_1u_{++} & u_{++} & -e_1u_{++} \\
e_1u_{++} & -e_3u_{++} & e_1u_{++} & u_{--}
\end{pmatrix}.
\]

(5)

Any geometric number \( g \in G_{1,3} \) can be written in the form

\[
g = (1 \ e_{13} \ e_3 \ e_1) u_{++} [g]
\]

where \([g]\) is the complex Dirac matrix corresponding to the geometric number \( g \). In particular,

\[
[\gamma_0] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
[\gamma] = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

(7)

and

\[
\gamma_2 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

\[
\gamma_3 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

It is interesting to see what the representation is of the basis vectors of \( G_3 \). We find that for \( k = 1, 2, 3 \),

\[
[e_k]_4 = [\gamma_k][\gamma_0] = \begin{pmatrix}
0 \\
[e_k]_2 \\
[0]_2
\end{pmatrix}
\]

and \([e_{123}]_4 = i\begin{pmatrix}
[0]_2 \\
[1]_2 \\
[0]_2
\end{pmatrix}\), where the outer subscripts denote the order of the matrices and, in particular, \([0]_2, [1]_2\) are the \( 2 \times 2 \) zero and unit matrices, respectively. The last relationship shows that the \( I := e_{123} \) occurring in the Pauli matrix representation, which represents the oriented unit of volume, is different than the \( i = \sqrt{-1} \) which occurs in the complex matrix representation (7) of the of Dirac algebra. In particular, \([e_2]_2 := \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}\), which is not the Pauli matrix for \( e_2 \in G_3 \) since \( i \neq I \). We will have more to say about this important matter later.

A Dirac spinor is a 4-component column matrix \([\varphi]_4\),

\[
[\varphi]_4 := \begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4
\end{pmatrix}
\]

for \( \varphi_k = x_k + iy_k \in \mathbb{C} \).

(8)
Just as in [4], from the Dirac spinor $|\varphi\rangle$, using (6), we construct its equivalent $S \in \mathbb{C}_{1,3}$ as an element of the minimal left ideal generated by $u_{++}$,

$$
|\varphi\rangle_4 = \begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4
\end{pmatrix} \leftrightarrow \begin{pmatrix}
\varphi_1 & 0 & 0 & 0 \\
\varphi_2 & 0 & 0 & 0 \\
\varphi_3 & 0 & 0 & 0 \\
\varphi_4 & 0 & 0 & 0
\end{pmatrix} \leftrightarrow S := (\varphi_1 + \varphi_2 e_{13} + \varphi_3 e_3 + \varphi_4 e_1)u_{++}.
$$

(9)

Because of its close relationship to a Dirac spinor, we shall refer to $S$ as a geometric Dirac spinor or a Dirac g-spinor.

Noting that

$$
u_{++} \gamma_{21} = \frac{1}{4} (1 + \gamma_0) (\gamma_{21} + i \gamma_2 \gamma_{21}) = \frac{1}{4} (1 + \gamma_0) (i - \gamma_2) = i u_{++} = \gamma_{21} u_{++},
$$

it follows that $\varphi_0 u_{++} = (x_k + \gamma_{21} y_k) u_{++} = u_{++} (x_k + \gamma_{21} y_k)$ and hence

$$
S = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) u_{++} = (\alpha_1 + \alpha_2^j e_{13} + e_3 \alpha_3 + \alpha_1^j e_1) u_{++},
$$

(10)

where each of the elements $\alpha_k$ in $S$ is defined by $\alpha_k = \varphi_k |_{\gamma_{21} \rightarrow -\gamma_{21}}$, and $\alpha_k := \varphi_k |_{\gamma_{21} \rightarrow -\gamma_{21}}$.

Expanding out the terms in (10),

$$
S = \left( (x_1 + x_4 e_1 + y_4 e_2 + x_3 e_3) + I (y_3 + y_2 e_1 - x_2 e_2 + y_1 e_3) \right) u_{++}.
$$

(11)

This suggest the substitution

$$
\varphi_1 \rightarrow x_0 + i y_3, \quad \varphi_2 \rightarrow -y_2 + i y_1, \quad \varphi_3 \rightarrow x_3 + i y_0, \quad \varphi_4 \rightarrow x_1 + i x_2,
$$

(12)

in which the geometric Dirac spinor $S$ takes the more perspicuous forms

$$
S = (X + I Y)u_{++} = (X + I Y) \gamma_{0u_{++}} = (x + I y) u_{++},
$$

(13)

for $X := x_0 + x, \ Y := y_0 + y \in \mathbb{C}_{3,3}$ where $x = x_1 e_1 + x_2 e_2 + x_3 e_3, \ y = y_1 e_1 + y_2 e_2 + y_3 e_3,$ and $x := \sum_{\mu = 0}^{3} \gamma_{\mu}\gamma_{\mu}, \ y := \sum_{\mu = 0}^{3} \gamma_{\mu} \gamma_{\mu} \in \mathbb{C}_{1,3}$.

We now calculate

$$
\bar{S} = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) u_{--},
$$

where $\bar{S}$ is the complex conjugate of $S$, defined by $i \rightarrow -i$,

$$
S^\# = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) u_{--},
$$

where $S^\#$ is the parity transformation defined by $\gamma_{\mu} \rightarrow -\gamma_{\mu}$, and

$$
S^* := (\bar{S})^\# = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) u_{--}.
$$

Using (4), we then define the even spinor operator

$$
\psi := S + \bar{S} + S^\# + S^* = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) (u_{++} + u_{--} + u_{++} + u_{--})
$$
and the odd spinor operator

\[ \Phi := \bar{S} + \bar{S} - S^0 - S^* = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) (u_{++} + u_{+-} - u_{--} - u_{-+}) \]

\[ = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) \gamma_0 = x + i y \in G_{1,3}^+. \] (15)

In addition to the two real even and odd spinor operators in \( G_{1,3} \), we have two complex spinor operators in \( G_{1,3}(\mathbb{C}) \), given by

\[ Z_+ := S - \bar{S} + S^0 - S^* = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) (u_{++} - u_{+-} + u_{--} + u_{-+}) \]

\[ = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) \gamma_3 E_3 = (x + i y) E_3 \in G_{1,3}^+(\mathbb{C}) \] (16),

where \( E_3 := -i e_3 \), and

\[ Z_- := S - \bar{S} + S^0 + S^* = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) (u_{++} - u_{+-} - u_{--} + u_{-+}) \]

\[ = (\alpha_1 + e_{13} \alpha_2 + e_3 \alpha_3 + e_1 \alpha_4) \gamma_0 E_3 = (x + i y) E_3 \in G_{1,3}^+(\mathbb{C}) \] (17).

Using (5), the matrix \([\psi] \) of the even spinor operator \( \psi \) is found to be

\[ [\psi] = \begin{pmatrix}
\varphi_1 & -\varphi_2 & \varphi_3 & \varphi_4 \\
\varphi_2 & \varphi_1 & -\varphi_4 & +\varphi_3 \\
\varphi_3 & -\varphi_4 & \varphi_1 & +\varphi_2 \\
\varphi_4 & +\varphi_3 & -\varphi_2 & -\varphi_1
\end{pmatrix}, \] (18)

\[ [\Phi] = \begin{pmatrix}
\varphi_1 & -\varphi_2 & -\varphi_3 & -\varphi_4 \\
\varphi_2 & \varphi_1 & -\varphi_4 & +\varphi_3 \\
\varphi_3 & -\varphi_4 & \varphi_1 & +\varphi_2 \\
\varphi_4 & +\varphi_3 & -\varphi_2 & -\varphi_1
\end{pmatrix}. \] (19)

Unlike the Dirac spinor \([\varphi]\), the even spinor operator \([\psi]\) is invertible iff \( \det[\psi] \neq 0 \). We find that

\[ \det[\psi] = r^2 + 4a^2 \geq 0, \] (20)

where

\[ r = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 \] and \( a = \text{im} \left( \varphi_1 \varphi_3 + \varphi_2 \varphi_4 \right) \).

Whereas the even spinor operator \([\psi]\) obviously contains the same information as the Dirac spinor \([\varphi]\), it acquires in (14) the geometric interpretation of an even multivector in \( G_{1,3}^+ \). With the substitution (12), the expansion of the determinant (20) takes the interesting form

\[ \det[\psi] = \left( (x_0^2 - x_1^2 - x_2^2 - x_3^2) - (y_0^2 - y_1^2 - y_2^2 - y_3^2) \right)^2 + 4(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)^2 \]

\[ = (x^2 - y^2)^2 + 4(x \cdot y)^2 \text{ for } x, y \in G_{1,3}^+. \] (21)

Since the matrix \([\Phi]\) of the odd spinor operator is the same as the matrix \([\psi]\) of the even spinor operator, except for the change of sign in the last two columns, \( \det[\Phi] = \)
det[ψ]. Indeed, similar arguments apply to the matrices \([Z_+]\) and \([Z_-]\), and so \(\det[ψ] = \det[Z_+] = \det[Z_-]\).

As an even geometric number in \(G_{1,3}^+\), \(ψ\) generates Lorentz boosts in addition to ordinary rotations in the Minkowski space \(\mathbb{R}^{1,3}\). Whereas we started by formally introducing the complex number \(i = \sqrt{-1}\), in order to represent the Dirac gamma matrices, we have ended up with the real spinor operators \(ψ, Φ \in G_{1,3}^+\), in which the role of \(i = \sqrt{-1}\) is taken over by the \(γ_1 = e_1 ∈ G_{1,3}\). This is the key idea in the Hestenes representation of the Dirac equation [9]. However, in the process, the crucial role played by the mutually annihilating idempotents \(u_{±±}\) has been obscured, and the role played by \(i = \sqrt{-1}\) in the definition of the Dirac matrices has been buried. Idempotents are slippery objects which can change the identities of everything they touch. As such, they should always be treated gingerly with care. Idempotents naturally arise in the study of number systems that have zero divisors, [10].

2 Geometric Dirac Spinors to Geometric E-Spinors

Recall from equation (14) that the real, even, Dirac spinor operator is

\[ψ = α_1 + e_1α_2 + e_3α_3 + e_4α_4 = X + IY ∈ G_{1,3}^+\]

for the geometric Dirac spinor,

\[S = ψu_{++} = (φ_1 + φ_2e_{13} + φ_3e_3 + φ_4e_1)u_{++} = (α_1 + e_1α_2 + e_3α_3 + e_4α_4)u_{++}\]

as an element in the minimal left ideal \(\{G_{1,3}^+u_{++}\}\).

Defining \(J := -iI\), we can express

\[u_{++} = γ_0^ε E_3^ε, \quad \text{where} \quad γ_0^± = \frac{1}{2}(1 ± γ_0), \quad E_3^± = \frac{1}{2}(1 ± Je_3).\]

Noting that \(Je_3u_{++} = u_{++}\),

\[S = (φ_1 + φ_2e_{13} + φ_3e_3 + φ_4e_1)u_{++} = (φ_1 + φ_2e_{13} + φ_3e_3 + φ_4e_1)u_{++}\]

\[= (φ_1 + φ_3J) + (φ_4 + φ_2J)e_1)u_{++} = Ωu_{++}, \quad (22)\]

where \(Ω := Ω_0 + Ω_1e_1\) for \(Ω_0 := (φ_1 + φ_3J)\) and \(Ω_1 := (φ_4 + φ_2J)\). Alternatively, \(Ω_0\) and \(Ω_1\) can be defined in the highly useful, but equivalent, way

\[Ω_0 = z_1 + Jz_3, \quad \text{and} \quad Ω_1 = z_4 + z_2I, \quad (23)\]

for \(z_1 := x_3 + y_3I, \ z_3 := x_3 + y_3I, \ z_4 := x_4 + y_2I, \ z_2 := x_2 + y_2I\) are all in \(G_{1,3}^{0+4}\). With the substitution variables (12), the \(z_k\’s\ become

\[z_1 = x_0 + y_0I, \ z_2 = -y_2 + x_2I, \ z_3 = x_3 + y_3I, \ z_4 = x_4 + y_4I. \quad (24)\]
Once again, we calculate
\[ S = (\tilde{\psi}_1 - \tilde{\psi}_3 J) + (\tilde{\psi}_4 - \tilde{\psi}_2 J) e_1 \] \[ u_{++} = \Omega u_{++}, \]
where \( S \) is the complex conjugate of \( S \) defined by \( i \rightarrow -i \).
\[ S^\dagger = (\phi_1 + \phi_3 J) + (\phi_4 + \phi_2 J) e_1 \] \[ u_{--} = \Omega u_{--}, \]
where \( S^\dagger \) is the parity transformation defined by \( \gamma_\mu \rightarrow -\gamma_\mu \), and
\[ S^* := (S)^\dagger = (\tilde{\psi}_1 - \tilde{\psi}_3 J) + (\tilde{\psi}_4 - \tilde{\psi}_2 J) e_1 \] \[ u_{--} = \Omega u_{--}. \]

The even spinor operator (14), satisfies
\[ \psi = S + S^\dagger + S^* = \Omega E_3^+ + \Omega E_3^- = \frac{1}{2} (\Omega + \overline{\Omega}) + \frac{1}{2} (\Omega - \overline{\Omega}) E_3, \]
where as before \( E_3 := J e_3 = E_3^+ - E_3^- \). Equations (14) and (25) express the even spinor operator \( \psi \) in two very different, but equivalent ways, the first as an element of \( G_{1,3}^+ \), and the second by multiplication in the complex geometric algebra \( G_{1,3}^+ (\mathbb{C}) \).

The second approach is closely related to the twistor theory of Roger Penrose [11, p.974].

It follows from (22) and (25), that any even spinor operator \( \psi \in G_{1,3}^+ \) can be written in the matrix form
\[ \psi = (1 \ e_1) E_3^+ \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix} \quad \Leftrightarrow \quad \psi' = (1 \ e_1) E_3^+ \begin{pmatrix} \overline{\Omega}_0^t & \Omega_1^t \\ \Omega_1^t & \overline{\Omega}_0^t \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix} \]
(26)
where the matrix \( [\psi]_\Omega := \begin{pmatrix} \Omega_0 & \overline{\Omega}_1 \\ \Omega_1 & \overline{\Omega}_0 \end{pmatrix} \), and the matrix \( [\psi']_\Omega := \begin{pmatrix} \overline{\Omega}_0^t & \Omega_1^t \\ \Omega_1^t & \overline{\Omega}_0^t \end{pmatrix} \). Analogous to the Pauli matrices, we have
\[ [1]_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [e_1]_\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2]_\Omega = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [e_3]_\Omega = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \]
(27)
which can be obtained from the Pauli matrix representation by multiplying the Pauli matrices for \( e_2 \) and \( e_3 \) by \( J = -iI \). Alternatively, the Pauli matrices can be obtained from the above Dirac-like representation, simply by replacing \( i \) by \( J \), in which case \( J \rightarrow 1 \). Indeed, if we let \( i \rightarrow I \) in (26), and using the substitution variables \( z_k \) defined in (24), we get
\[ \psi = (1 \ e_1) u_+ \begin{pmatrix} z_1 + z_3 \\ z_4 + z_2 \\ z_4 - z_2 \\ z_1 - z_3 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}, \]
(28)
where \( u_+ = \frac{1}{2} (1 + e_1) \), which is exactly the Pauli algebra representation of the geometric number \( \psi \in G_3 \). Recently, I was surprised to discover that I am not the first to consider such a representation of the Pauli matrices [12].
Calculating the determinant of the matrix $[\Omega] := [\psi]_\Omega$ of $\psi$,

$$\det[\Omega] = \det\left(\begin{array}{cc} \Omega_0 & \Omega_1 \\ \Omega_1 & \Omega_0 \end{array}\right) = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 + 2i\text{m}(\varphi, \varphi_0 + \varphi_2 \varphi_4)I,$$  
(29)

or in the more elegant alternative form in the spacetime algebra $\mathbb{G}_{1,3}$,

$$\det[\Omega] = x^2 - y^2 + 2I(x \cdot y) = (x + Iy)(x - Iy) = (X + IY)(\bar{X} + I\bar{Y}),$$  
(30)

where $x = X\gamma_0$ and $y = Y\gamma_0$, and $\bar{A}$ denotes the operation of reverse of the element $A \in \mathbb{G}_{1,3}$. Note that the determinant $\det[\psi]$, found in (20), is related to (29) or (30), by $\det[\psi] = |\det[\psi]|_\Omega^2$ for the matrix of the even spinor operator $\psi$ defined in (18). The spinor operator $\psi$ will have an inverse only when $\det[\psi] \neq 0$.

We have already noted that the geometric algebra $\mathbb{G}_3$ can be algebraically identified with the even sub-algebra $\mathbb{G}^+_{1,3} \subset \mathbb{G}_{1,3}$. Each timelike Dirac vector $\gamma_0$ determines a different rest frame (1) of spacelike bivectors $\{e_1, e_2, e_3\}$. For $\psi \in \mathbb{G}^+_{1,3}$, the matrix $[\psi]_\Omega$ is Hermitian with respect to the rest-frame $\{e_1, e_2, e_3\}$ of $\gamma_0$ if

$$\overline{[\psi]^T} = \overline{[\psi]}_\Omega,$$

or equivalently, $\psi^\dagger = \psi$, where $\dagger$ is the conjugation of reverse in the Pauli algebra $\mathbb{G}_3 \equiv \mathbb{G}^+_{1,3}$. Using the more transparent variables (12), for a Hermitian $[\psi]_\Omega$,

$$[\psi]_\Omega = \left(\begin{array}{cc} \Omega_0 & \Omega_1 \\ \Omega_1 & \Omega_0 \end{array}\right) = \left(\begin{array}{cc} x_0 + Jx_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - Jx_3 \end{array}\right) = \overline{[\psi]^T}_\Omega,$$

with $\det[\psi]_\Omega = x_0^2 - x_1^2 - x_2^2 - x_3^2$, and for anti-Hermitian $[\psi]_\Omega$,

$$[\psi]_\Omega = \left(\begin{array}{cc} \Omega_0 & \Omega_1 \\ \Omega_1 & \Omega_0 \end{array}\right) = \left(\begin{array}{cc} y_0 + Jy_3 & y_1 - iy_2 \\ y_1 + iy_2 & y_0 - Jy_3 \end{array}\right) = -\overline{[\psi]^T}_\Omega$$

with $\det[\psi]_\Omega = -y_0^2 + y_1^2 + y_2^2 + y_3^2$.

Thus, a Hermitian Dirac g-spinor, corresponding to $X = x_0 + x \in \mathbb{G}^3$, has the form

$$S := (\Omega_0 + \Omega_1 e_1)u_{++} = \left(x_0 + x_3 J\right) + \left(x_1 + ix_2 \right)e_1)u_{++} = Xu_{++},$$

for $\Omega_0 = x_0 + x_3 J$ and $\Omega_1 = x_1 + ix_2$, and an anti-Hermitian Dirac g-spinor, corresponding to $IY = I(y_0 + y) \in \mathbb{G}_3$, has the form

$$Q := (\Omega_0 + \Omega_1 e_1)u_{++} = I\left(y_0 + y_3 J\right) + \left(y_1 + iy_2 \right)e_1)u_{++} = IYu_{++},$$

for $\Omega_0 = I(y_0 + y_3 J)$ and $\Omega_1 = I(y_1 + iy_2)$.

In the bra-ket notation, a Dirac g-spinor (9), and its conjugate Dirac g-spinor, in the spacetime algebra $\mathbb{G}^+_{1,3}(\mathbb{C})$, takes the form

$$[\Omega] := 2S = 2(\Omega_0 + \Omega_1 e_1)u_{++} = 2(x + Iy)u_{++},$$  
(31)
and

\[
\langle \Omega \rangle := \bar{\phi} = 2u_+ (\bar{\Omega}_0 - \bar{\Omega}_1 e_1) = 2u_+ (x - iy),
\]

respectively. Taking the product of the conjugate of a g-spinor \( \Phi \) = 2(r + is)\( u_+ \), with a g-spinor \( \Omega \), gives

\[
\langle \Phi | \Omega \rangle = 4 u_+ (\bar{\phi}_0 - e_1 \bar{\phi}_1) (\Omega_0 + e_1 \Omega_1) u_+ = 4 u_+ (\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2 - \bar{\phi}_3 \phi_3 - \bar{\phi}_4 \phi_4)
\]

\[
= 4 u_+ \left( r \cdot x - s \cdot y + \gamma_1 (\gamma_2 \cdot (r \wedge x - s \wedge y) + \gamma_0 \cdot (r \wedge y - s \wedge x)) \right).
\]

The complex \textit{sesquilinear inner product} in \( G^{0+3}_3 \) of the two g-spinors \( |\Phi\rangle \) and \( |\Omega\rangle \) is then defined by

\[
\langle \Phi | \Omega \rangle := \left( \langle \Phi | \Omega \rangle \right) = \left( \bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2 - \bar{\phi}_3 \phi_3 - \bar{\phi}_4 \phi_4 \right) = a + ib \in \mathbb{C}.
\]

Defining \( |\Omega\rangle = 2(\Omega_0 + \Omega_1 e_1)u_+ \) and \( |\Phi\rangle = 2(\Phi_0 + \Phi_1 e_1)u_+ \) for the substituted variables (12), with \( |\Phi\rangle \) being defined in the corresponding variable \( r \) and \( s \), the inner product (34) takes the form

\[
\langle \Phi | \Omega \rangle = (r_0 x_0 - r_1 x_1 - r_2 x_2 - r_3 x_3) - (s_0 y_0 - s_1 y_1 - s_2 y_2 - s_3 y_3)
\]

\[
+ i (r_0 y_3 - r_3 y_0 + s_0 x_3 - s_3 x_0 + r_2 x_1 - r_1 x_2 + s_1 y_2 - s_2 y_1).
\]

Alternatively, writing \( |\Omega\rangle = 2(x + iy)u_+ \) and \( |\Phi\rangle = 2u_+(r - is) \), we have

\[
\langle \Phi | \Omega \rangle = (r \cdot x - s \cdot y) + i (\gamma_2 \cdot (r \wedge x - s \wedge y) + \gamma_0 \cdot (r \wedge y - s \wedge x)) \in \mathbb{C}.
\]

Consider now the parity invariant part \( S_E \) of the geometric Dirac spinor \( S \)

\[
S_E := S + S^\dagger = (\Omega_0 + \Omega_1 e_1)E_3^+ = \Omega E_3^+ = (X + iY)E_3^+.
\]

Writing \( S_E = \Omega_0 (1 + \Omega_0^{-1} \Omega_1 e_1) E_3^+ = \Omega_0 T_E^+ \),

for \( T_E^+ := (1 + \Lambda e_1)E_3^+ \) and \( \Lambda := \Omega_0^{-1} \Omega_1 \), we then find that \( T_E^+ \) is an idempotent. We now study idempotents \( P \in G_3(\mathbb{C}) \) of the form

\[
P = \frac{1}{2} (1 + JM + iN),
\]

for \( N, M \in G^{2+1}_3 \) and \( (JM + iN)^2 = 1 \).

Defining the \textit{symmetric product} \( M \circ N := \frac{1}{2}(MN + NM) \), the condition

\[
(MJ + iN)^2 = 1 \iff M^2 - N^2 = 1 \quad \text{and} \quad M \circ N = 0,
\]

implies that

\[
P = \frac{1}{2} (1 + JM + iN) = JM \frac{1}{2} \left( 1 + \frac{J}{M^2} (M + iMN) \right) = JM \frac{1}{2} (1 + J\hat{b}),
\]

10
for \( \hat{b} := \frac{1}{M} (M + iMN) \in G_{1,3}^\tau \). For the idempotent \( T_E^+ \), defined in (36), we have

\[
T_E^+ = (1 + \Lambda e_1) E_3^+ = \frac{1}{2} (1 + JM + IN) = JME_3^+ = Me_3E_3^+, \tag{38}
\]

since in this case \( \hat{b} = e_3 \). Because \( T_E^+ \) is an idempotent, it further follows that

\[
T_E^+ = JME_3^+ = M^2(\hat{M}E_3^+ \hat{M})E_3^+ = M^2 \hat{A}_+ E_3^+, \tag{39}
\]

where \( \hat{A}_+ := (\hat{M}E_3^+ \hat{M}) \), and \( \hat{M} := \frac{M}{\sqrt{M^2}} \). We also identify \( J\hat{a} \) in \( \hat{A}_+ \), by \( \hat{a} = \hat{M}e_3 \hat{M} \in G_{1,3}^\tau \).

Now decompose the complex unit vector \( \hat{M} = m_1 + im_2 \in G_{1,2}^\tau \), by writing

\[
\hat{M} = \hat{m}_1 \cosh \phi + i\hat{m}_2 \sinh \phi = e^{\hat{m}_1 \times \hat{m}_2}, \tag{40}
\]

where \( \cosh \phi = |m_1| \) and \( \sinh \phi = |m_2| \), and note that since \( \hat{M}^2 = 1 \),

\[
\hat{m}_1 \hat{m}_2 = -\hat{m}_2 \hat{m}_1 = I(\hat{m}_1 \times \hat{m}_2),
\]

so \( I = e_{123} = \hat{m}_1 \hat{m}_2 (\hat{m}_1 \times \hat{m}_2) \). It then follows that the expression for \( \hat{a} \), given after equation (39), becomes

\[
\hat{a} = \hat{M}e_3 \hat{M} = e^{\hat{m}_1 \times \hat{m}_2} e^{\hat{m}_3 e_3} e^{-\hat{m}_1 \times \hat{m}_2}, \tag{41}
\]

which expresses that the North Pole \( e_3 \) of the Riemann sphere is rotated by \( I\hat{m}_1 \) into the point \( \hat{m}_1 e_3 \hat{m}_1 \), and then undergoes the Lorentz Boost defined by the unit vector \( \hat{m}_1 \times \hat{m}_2 \), with velocity \( \tanh \phi = v/c \), to give the unit vector \( \hat{a} \). We can think of the Riemann sphere, itself, as undergoing a Lorentz boost with velocity \( \hat{m}_1 \times \hat{m}_2 \tanh \phi \).

We can now decompose the parity invariant part \( S_E \) of the geometric Dirac Spinor \( S \), into

\[
S_E = \Omega_0 T_E^+ = J\Omega_0 ME_3^+ = \Omega_0 M^2 \hat{A}_+ E_3^+, \tag{42}
\]

where \( \Omega_0 = \varphi_1 + J\varphi_3 \), and

\[
T_E^+ = (1 + \Lambda e_1) E_3^+ = \frac{1}{2} (1 + JM + IN),
\]

for \( \Lambda = \Omega_0^{-1} \Omega_1 \). We then have

\[
\Lambda e_1 - J\Lambda e_2 + Je_3 = JM + IN, \quad \text{and} \quad \Lambda e_3 + J\Lambda e_2 - Je_3 = -JM + IN.
\]

Solving these equations for \( M \), gives

\[
M = \frac{\Lambda - \bar{\Lambda}}{2} e_1 - \frac{\Lambda + \bar{\Lambda}}{2} e_2 + e_3, \quad e_3 \circ M = 1, \quad \text{and} \quad \hat{M}^2 = 1 - \Lambda \hat{X}. \tag{43}
\]

Writing \( M = x + e_3 = \sum_{k=1}^{3} \alpha_k e_k \), for \( \alpha_k \in G_3^{0,3} \), where \( x = \alpha_1 e_1 + \alpha_2 e_2 \) and \( \alpha_3 = 1 \), the \( \Omega \)-matrix (27) of \( M \) takes the form

\[
[M]_\Omega = \left( \begin{array}{cc}
\alpha_1 + i\alpha_2 & J \\
\alpha_1 - i\alpha_2 & -J
\end{array} \right). \]
If for the complex idempotent \( P \), given in (37), we make the substitution \( M = m \) and \( N = n \) for the vectors \( m, n \in G_3 \), then the idempotent \( P = \frac{1}{2}(1 + Jm + In) \). Letting \( i \rightarrow I \), so that \( J \rightarrow 1 \), the idempotent \( P \) becomes \( \frac{1}{2}(1 + m + In) \), which is the idempotent that was studied in the case of the representation of the Pauli spinors on the Riemann sphere, [1, (23)].

The complex unit vector \( \hat{a} \), defined in (41), can be directly expressed in terms of \( \Lambda \). Using (43), we find that

\[
\hat{a} = \hat{M}e_3\hat{M} = (e_3\hat{M} + 2e_3 \circ \hat{M})\hat{M} = -e_3 + 2(e_3 \circ \hat{M})\hat{M}
\]

\[
= \frac{2}{1 - \Lambda^2} M - e_3,
\]

which in turn gives the projective relation

\[
M = x + e_3 = \frac{1 - \Lambda^2}{2} (\hat{a} + e_3) \iff M = \frac{2}{\hat{a} + e_3}
\]

(44)

showing that the projection \( x \) of \( M \) onto the complex hyperplane defined by \( e_1, e_2 \), is on the complex ray extending from the south pole \( -e_3 \). This is equivalent to saying that \( M \) is a complex multiple of the complex vector \( \hat{a} + e_3 \).

Using (42) and (43), we find that

\[
S_E^2 = \Omega_0^2 T_E^+ = \Omega_0^2 J M E_3^+
\]

or

\[
S_E = J\Omega_0 M E_3^+ = J\Omega_0 \sqrt{M^2} \hat{M} E_3^+ = J \frac{\Omega_0}{\sqrt{\Omega_0^2 - \Omega_1^2 E_3^+}} \Omega_0^2 \Omega_1 \hat{M} E_3^+,
\]

so that

\[
S_E = Je^{\omega t} \hat{M} E_3^+ = \hat{M} e^{\omega t + \omega_3 e_3} E_3^+ = e^{\omega t + \omega_3 \hat{a} \hat{M} E_3^+} = Je^{\omega_3 \hat{a} \hat{M} E_3^+},
\]

(45)

for \( \omega = \omega_1 + \omega_2 J \) where \( \omega_k = \phi_k + \theta_k I \in G_{0+3}^2 \) for \( k = 1, 2 \), and

\[
\hat{a} := \hat{M} e_3 \hat{M}, \quad A_3^+ := \hat{M} E_3^+ \hat{M}.
\]

The \( \phi_k, \theta_k \in \mathbb{R} \) are defined in such a way that \( e^{\omega t} = \sqrt{\Omega_0^2 - \Omega_1^2 E_3^+} = \sqrt{\det[\Omega]} \), and \( e^{J\omega_2} := \frac{\Omega_2}{\sqrt{\Omega_0^2 - \Omega_1^2}} \). Note that \( \hat{M} \) can be further decomposed using (40).

There are two additional canonical forms, derived from (45), that are interesting. We have

\[
S_E = e^{\omega t} e^{i\hat{b}_c z_c} e^{\omega_3 e_3} E_3^+ = e^{\omega t} e^{i\hat{b}_c z_c} E_3^+,
\]

(46)

where

\[
\hat{M} e_3 = \hat{M} \circ e_3 + \hat{M} \otimes e_3 = \cos z_c + i \hat{b}_c \sin z_c = e^{i\hat{b}_c z_c}
\]

for \( z_c \in G_{1+3}^0, \hat{b}_c \in G_{1+3}^2, \) and \( \hat{M} \otimes e_3 := \frac{1}{2}(\hat{M} e_3 - e_3 \hat{M}) \) is the anti-symmetric product. Similarly,

\[
\hat{a} \hat{M} = \hat{a} \circ \hat{M} + \hat{a} \otimes \hat{M} = \cos z_a + i \hat{b}_a \sin z_a = e^{i\hat{b}_a z_a}
\]
for \( z_\alpha \in \mathbb{C}_{1,3} \) and \( \mathbf{b}_\alpha \in \mathbb{C}_{1,3} \). Note that \( \mathbf{b}_\alpha \) and \( \mathbf{e}_3 \) anti-commute, as do \( \mathbf{b}_\alpha \) and \( \mathbf{a} \). In our decomposition, it is the unit bivector \( \mathbf{M} \) that defines the Lorentz transformation (41) associated with a Dirac spinor.

Recalling (26), (42), and (45), we define a Pauli E-spinor by

\[
|\Omega\rangle_E := \sqrt{2}(\Omega_0 + \Omega_1 \mathbf{e}_1)E^+_3 = \sqrt{2}J e^{\omega_0}\hat{\mathbf{M}}E^+_3,
\]

(47)

Given the E-spinor \(|\Phi\rangle_E = \sqrt{2}J e^{\omega_0}\hat{\mathbf{M}}E^+_3\), its conjugate is specified by

\[
\langle \Phi |_E = \overline{|\Phi\rangle}_E = \sqrt{2}J e^{\omega_0}\hat{\mathbf{M}}^{-1}_3,
\]

which we use to calculate

\[
\langle \Phi |_E |\Omega\rangle_E = 2e^{\omega_0}\hat{\mathbf{M}}\hat{\mathbf{M}}^{-1}_3 = 2e^{\omega_0}\hat{\mathbf{M}}^{-1}_3 \left( \hat{\mathbf{M}}' \circ \hat{\mathbf{M}} + J(\hat{\mathbf{M}}' \circ \hat{\mathbf{M}}) \circ \mathbf{e}_3 \right),
\]

(48)

which can be expressed in the alternative form,

\[
\langle \Phi |_E |\Omega\rangle_E = 2e^{\omega_0}\hat{\mathbf{M}}\hat{\mathbf{M}}^{-1}_3 \left( \hat{\mathbf{M}}' \circ \hat{\mathbf{M}} + J(\hat{\mathbf{M}}' \circ \hat{\mathbf{M}}) \circ \mathbf{e}_3 \right),
\]

(49)

and

\[
|\Omega\rangle_E \langle \Phi |_E = 2e^{2\omega_0}\hat{\mathbf{M}}\hat{\mathbf{M}}^{-1}_3 = 2e^{2\omega_0}\hat{\mathbf{M}}^{-1}_3.
\]

(50)

The result (49) is in agreement with (29) when \( \Phi = \Omega \), and identical in terms of the complex components \( \phi_k \) and \( \phi_k \), for which

\[
\langle \Phi |_E |\Omega\rangle_E = 2E^+_3 \left( \overline{\phi}_1 \phi_1 + \overline{\phi}_2 \phi_2 - \overline{\phi}_3 \phi_3 - \overline{\phi}_4 \phi_4 + 2i\overline{\Omega}_0 \phi_3 + \overline{\phi}_2 \phi_4 \right),
\]

so the Dirac inner product (34) can also be expressed by

\[
\langle \Phi |_E |\Omega\rangle_E = \frac{1}{4} \left( \langle \Phi |_E |\Omega\rangle_E + \langle \Phi |_E |\Omega\rangle_E^\dagger + \langle \Phi |_E |\Omega\rangle_E^\dagger + \langle \Phi |_E |\Omega\rangle_E \right).
\]

(51)

Of course, the \( \langle \cdot | \rangle \) and \( \langle \cdot | \rangle^\dagger \) conjugation operators all depend upon the decomposition of \( I = \mathbf{e}_{123} \) into the rest-frame \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \in \mathbb{C}_{1,3} \). The importance of this result is that we are able to directly define the Dirac inner product in terms of conjugations of the Pauli E-inner product given in (49). It is also interesting to compare this result with (33).

By a *gauge transformation* of an E-spinor \(|\Omega\rangle_E \), we mean the spinor \( e^{\alpha \theta} |\Omega\rangle_E \), where \( \alpha = i\theta + J\Phi \) for \( \theta, \Phi \in \mathbb{R} \). Now note that

\[
\det[e^{\alpha \theta}] = \det \begin{pmatrix} e^{\alpha \Omega_0} & e^{\alpha \Omega_1} \\ e^{\alpha \Omega_1} & e^{\alpha \Omega_0} \end{pmatrix} = e^{2i\theta} \det[\Omega] = e^{2i\theta} \Omega_0 \Omega_0 \mathbf{M}^2,
\]

since \( \det[\Omega] = \Omega_0 \Omega_0 \mathbf{M}^2 \). We say that \( |\Omega\rangle_E = e^{\alpha \theta} |\Omega\rangle_E \) is *gauge normalized* if

\[
\det[\Omega] \in \mathbb{R}, \quad e^{i\theta} := \frac{\Omega_0}{\sqrt{\Omega_0 \Omega_0}} \in \mathbb{C}.
\]

(52)
The definition of an E-spinor is closely related to the Dirac g-spinor definition (31),

\[ |\Omega \rangle = \sqrt{2} |\Omega \rangle_E Y^+_0 = 2Je^{\theta_0} \hat{M}E^+_3 \gamma^+_0. \tag{53} \]

Given the g-spinor \( |\Phi \rangle = 2Je^{\theta_0} \hat{M}E^+_3 \gamma^+_0 \), its conjugate is

\[ \langle \Phi | = \sqrt{2} \gamma^+_0 \langle \Phi |_E = 2\gamma^+_0 Je^{\theta_0} E^+_3 \hat{M}. \]

Using (49) and (51), we calculate

\[ \langle \Phi | \Omega \rangle = 2\gamma^+_0 \langle \Phi |_E \rangle Y^+_0 = 4\gamma^+_0 e^{2\theta_0 E^+_3 \hat{M}} E^+_3 \gamma^+_0. \tag{54} \]

The Dirac inner product, expressed in (51), shows that

\[ \langle \Phi | \Omega \rangle = 2\gamma^+_0 \langle \Phi |_E \rangle Y^+_0 = 2\gamma^+_0 e^{2\theta_0 E^+_3 \hat{M}} E^+_3 \gamma^+_0. \tag{55} \]

Using (29), (45), and (49) for \( |\Omega \rangle_E = \sqrt{2}Je^{\theta_0} \hat{M}E^+_3 \),

\[ \langle \Omega |_E \rangle \langle \Phi |_E \rangle = 2e^{2\theta_0 E^+_3 \hat{M}} E^+_3 \gamma^+_0 = 2 \det |\Omega \rangle E^+_3, \tag{56} \]

so

\[ \langle \Omega | \Omega \rangle_E := \rho^2 e^{2\theta_0} = \det |\Omega \rangle, \]

where \( \rho_1 := e^{\theta_0} \). When |\Omega \rangle_E is gauge normalized to

\[ |\Omega' \rangle_E := e^{-2\theta_0 - \frac{1}{2} I_2} |\Omega \rangle_E, \]

and similarly, |\Phi \rangle_E is gauge normalized to |\Phi' \rangle_E, then the inner products (55) and (56) become more simply related. For gauge normalized E-spinors |\Omega \rangle_E and |\Phi \rangle_E, the relation (54) simplifies to

\[ \langle \Phi | \Omega \rangle = 2\gamma^+_0 \langle \Phi |_E \rangle Y^+_0 = 4e^{2\theta_0} E^+_3 \hat{M} E^+_3 \gamma^+_0. \tag{57} \]

A geometric Dirac spinor \(|\Omega \rangle \) represents the physical state of a spin \( \frac{1}{2} \)-particle if \( \det |\Omega \rangle = 1 \).

Before our next calculation, for \( \hat{A}_+ = \frac{1}{2} (1 + J \hat{a}) \) and \( \hat{B} = \frac{1}{2} (1 + J \hat{b}) \) for \( \hat{a}, \hat{b} \in G^+_1 \), we find that

\[ \hat{A}_+ \hat{B}_+ \hat{A}_+ = \frac{1}{4} \hat{A}_+ (1 + \hat{B}_E) (1 + \hat{A}_E) = \frac{1}{4} \hat{A}_+ (1 + \hat{B}_E + \hat{A}_E + \hat{B}_E \hat{A}_E), \]

\[ = \frac{1}{2} \hat{A}_+ (1 + \hat{B}_E + \hat{A}_E) - \frac{1}{2} \hat{A}_+ \hat{B}_E + 2 \hat{A}_E \hat{B}_E = \frac{1}{2} \hat{A}_+ (1 + \hat{a} \circ \hat{b}). \tag{58} \]

Using this result, (49) and (51), we calculate

\[ \langle \Omega_1 | \Phi \rangle_E \langle \Phi | \Omega \rangle_E = 4e^{2(\theta_0 + \theta_1)} \hat{M}_\Omega \hat{A}_+ \hat{A}_+ \hat{A}_+ \hat{A}_+ \hat{M}_\Omega \]

\[ = 2e^{2(\theta_0 + \theta_1)} (1 + \hat{a}_0 \circ \hat{a}_0) E^+_3 = 2 \det |\Omega \rangle \det |\Phi \rangle (1 + \hat{a}_0 \circ \hat{a}_0) E^+_3. \tag{59} \]
Using (51), we would now like to calculate $\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle$ for gauge normalized physical states $| \Omega \rangle_E$ and $| \Phi \rangle_E$. Here, we consider a special case where the formula (51) simplifies to

$$
\langle \Phi | \Omega \rangle = \frac{1}{2} \left( \langle \Phi | \Omega \rangle_E + \langle \Phi | \Omega \rangle_E^\dagger \right).
$$

(60)

Referring to (48), this will occur when

$$
\hat{M}_\Phi \circ \hat{M}_\Omega = (\hat{M}_\Phi \circ \hat{M}_\Omega)^\dagger \quad \text{and} \quad (\hat{M}_\Phi \otimes \hat{M}_\Omega) \circ \mathbf{e}_3 = - \left( (\hat{M}_\Phi \otimes \hat{M}_\Omega) \circ \mathbf{e}_3 \right)^\dagger.
$$

(61)

Indeed, if $| \Omega \rangle_E$ and $| \Phi \rangle_E$ are gauge normalized spinor states, then using (51) and (59), we have

$$
\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle = \frac{1}{4} \left( \langle \Omega | \Phi \rangle_E + \langle \Omega | \Phi \rangle_E^\dagger \right) \left( \langle \Phi | \Omega \rangle_E + \langle \Phi | \Omega \rangle_E^\dagger \right)
$$

$$
= \frac{1}{2} \det[\Omega] \det[\Phi] (1 + \hat{a}_\Phi \circ \hat{a}_\Omega).
$$

(62)

If $\det[\Omega] = \det[\Phi] = 1$, then the probability of finding the spin $\frac{1}{2}$ particle in the state $| \Phi \rangle$, having prepared the particle in the state $| \Omega \rangle$, is given by $\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle$. There is a particularly simple formula for evaluating (62), directly in terms of $\hat{M}_\Omega$ and $\hat{M}_\Phi$, which follows from (44). We find that

$$
\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle = \frac{1}{2} (1 + \hat{a}_\Phi \circ \hat{a}_\Omega) = 1 - \frac{(M_\Omega - M_\Phi)^2}{M_\Phi^2 M_\Omega^2}.
$$

(63)

Let us consider a couple of examples of gauge normalized states for which (63) applies. The usual up state, with $\Omega_0 = 1$, $\Omega_1 = 0$ and $\hat{M}_\Omega = \hat{a}_\Omega = \mathbf{e}_3$, is gauge normalized. For our first example, let us compare this state to the gauge normalized state defined by $| \Phi \rangle = \frac{2}{\sqrt{1 + x^2 + y^2}} \left( 1 + (x + iy) e_1 \right) u_{++}$, for which

$$
\hat{M}_\Phi = x \mathbf{e}_1 + y \mathbf{e}_2 + \mathbf{e}_3, \quad \text{and} \quad \hat{a}_\Phi = \hat{M}_\Phi \mathbf{e}_3 \hat{M}_\Phi = \hat{a}_\Omega^\dagger.
$$

We find that $\det[\Omega] = \det[\Phi] = 1$, and the formula (63) reduces to the classical result for a Pauli spinor

$$
\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle = \frac{1}{2} (1 + \hat{a}_\Phi \circ \mathbf{e}_3) = \frac{1}{1 + x^2 + y^2} \leq 1,
$$

as expected [1]. See Figure 1.

If, instead, we let $| \Phi \rangle = \frac{2}{\sqrt{1 - x^2 - y^2}} \left( 1 + (x + iy) e_1 \right) u_{++}$, then

$$
\hat{M}_\Phi = I y \mathbf{e}_1 - I x \mathbf{e}_2 + \mathbf{e}_3, \quad \text{and} \quad \hat{a}_\Phi = \hat{M}_\Phi \mathbf{e}_3 \hat{M}_\Phi = \frac{2I y e_1 - 2I x e_2 + (1 + x^2 + y^2) e_3}{1 - x^2 - y^2},
$$

and the formula (63) gives

$$
\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle = \frac{1}{2} (1 + \hat{a}_\Phi \circ \mathbf{e}_3) = \frac{1}{1 - x^2 - y^2} \geq 1,
$$

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Figure 1: Shown is the discrete probability distribution for observing a spin 1/2 particle in different physical spin states, given that the particle was prepared in the spin up state immediately preceding.

Figure 2: Shown is the discrete distribution for observing a particle in different physical states, given that the particle was prepared in the spin up state immediately preceding. The values within the unit disk are all greater than or equal to one.
Figure 3: The larger dish shows unit direction velocities on the Riemann sphere for gauge normalized states. The inner dish shows the velocity vectors of the gauge normalized states, measured from the origin.

which as a probability only makes sense when $x = y = 0$ and $\hat{a} = e_3$. See Figure 2.

Let us now find the probability of finding the spin $1/2$ particle, prepared in the spin up state $|\Omega\rangle = 2u_+e_3$, in a gauge normalized state $|\Phi\rangle$ where $\phi_0 = 1$ and $\phi_1 = \phi_4 + \phi_2 J$ and $\det[\Phi] = 1$. It follows that

$$\det[\Phi] = \det \begin{pmatrix} 1 & -\phi_4 + \phi_2 J \\ \phi_4 + \phi_2 J & 1 \end{pmatrix} = 1 - (\phi_4 + \phi_2 J)(\phi_4 - \phi_2 J)$$

$$= 1 - \phi_4 \phi_4 + \phi_2 \phi_2 + J(\phi_4 - \phi_2 \phi_4).$$

For $\det[\Phi] = 1$, we must have $\phi_4 \phi_4 = \phi_2 \phi_2$ and $(\phi_2 \phi_4 - \phi_2 \phi_4) = 0$. We can satisfy this condition by setting $\phi_4 = \phi_2$. Letting $\phi_2 = x + iy$, $\phi_1 = (1 + J)(x + iy)$, and using (43) with $\Lambda = \phi_1$,

$$M_\Phi = \frac{\phi_1 - \phi_1^*}{2}e_1 - \frac{\phi_1 + \phi_1^*}{2}e_2 + e_3 = (x + iy)e_1 + (y - ix)e_2 + e_3 = \hat{M}_\Phi.$$  

We then calculate

$$\hat{a}_\Phi = M_\Phi e_3 \hat{M}_\Phi = -e_3 + 2(e_1 \circ \hat{M}_\Phi) \hat{M}_\Phi = 2(x + iy)e_1 + 2(y - ix)e_2 + e_3,$$

from which follows the surprising prediction that the probability of finding the spin $1/2$ particle in the state $|\Phi\rangle$ is

$$\langle \Omega|\Phi\rangle \langle \Phi|\Omega\rangle = \frac{1}{2} (1 + \hat{a}_\Phi \circ e_3) = 1. \quad (64)$$

To better understand the significance of this prediction, we use (40) and (41) to write $\hat{a}_\Phi$ in the form

$$\hat{a}_\Phi = \hat{M}_\Phi e_3 \hat{M}_\Phi = e^{\phi_1 \hat{m}_1 \times \hat{m}_2} e_3 \hat{m}_1 e_3 e^{-\phi_1 \hat{m}_1 \times \hat{m}_2}.$$
Figure 4: Shown is the discrete distribution for observing a particle in different physical states, given that the particle was prepared in the spin up state immediately preceding. In this case, the physical significance of the results are unclear.

The prediction means that in the preferred rest frame defined by

\[
\{ \epsilon'_k \}_{k=1}^3 = e^{\phi \hat{m}_1 \times \hat{m}_2} \{ \epsilon_k \}_{k=1}^3 e^{-\phi \hat{m}_1 \times \hat{m}_2},
\]

moving at a velocity \( \hat{m}_1 \times \hat{m}_2 \tanh 2\phi \), the probability of finding the spin 1/2 particle in the spin state \( \hat{a}_\Phi \) is certain. In other words, what was spin up in the inertial system \( \{ \epsilon_k \} \) is spin \( \hat{a}_\Phi \) in the inertial system \( \{ \epsilon'_k \} \). See Figure 3.

We give one more example to try to get a better understanding of what is going on. In this case, we measure a particle prepared in the spin-up state defined by \( M_\Omega = e_3 \), in the spin state defined by \( M_\Phi = (x_2 + I y_2) e_1 + (x_2 - I y_2) e_2 + e_1 \). We find that

\[
\langle \Omega | \Phi \rangle \langle \Phi | \Omega \rangle = \frac{1}{2} (1 + \hat{a}_\Phi \circ \hat{a}_\Omega) = 1 - \frac{(M_\Omega - M_\Phi)^2}{M_\Omega^2 M_\Phi^2} = \frac{1}{1 + 2(x_2^2 - y_2^2)}.
\]

The discrete distribution is shown in Figure 4.

### 3 Fierz Identities

The so-called Fierz identities are quadratic relations between the physical observables of a Dirac spinor. The identities are most easily calculated in terms of the spinor operator \( \psi \) of a Dirac spinor. Whereas spinors are usually classified using irreducible representations of the Lorentz group \( SO^+_{1,3} \), Pertti Lounesto has developed a classification scheme based upon the Fierz identities [5, P.152,162]. A Dirac spinor, which describes an electron, the subject of this paper, is characterized by the property that \( \det [\psi]_\Omega \neq 0 \). Other types of spinors, such as Majorana and Weyl spinors, and even Lounesto’s boomerang spinor, can all be classified by bilinear covariants of their spinor operators \( \psi \in \mathbb{G}_3 \equiv \mathbb{G}_{1,3}^+ \).
For \( g \in \mathbb{G}_3 \), let \( g^- \) and \( g^\dagger \), and \( g^* \) denote the conjugations of inversion, reversion, and their composition \( g^* = (g^-)^\dagger \), respectively. Thus for \( g = \alpha + x + Iy \), where \( \alpha \in \mathbb{G}^{0+3} \),

\[
g^- = \alpha^\dagger - x + Iy, \quad g^\dagger = \alpha^\dagger + x - Iy, \quad \text{and} \quad g^* = \alpha - x - Iy \quad (65)
\]

Any element \( \omega \in \mathbb{G}_{1,3} \) can be written \( \omega = g_1 + g_2 \gamma_0 \) for some \( g_1, g_2 \in \mathbb{G}_3 \). For \( \omega \in \mathbb{G}_{1,3} \), as before, let \( \bar{\omega} \) denote the conjugation of reverse in \( \mathbb{G}_{1,3} \). Since \( g^\dagger = \gamma_0 g^* \gamma_0 \),

\[
\bar{\omega} = \bar{g}_1 + \gamma_0 \bar{g}_2 = g_1^\dagger + \gamma_0 g_2 = g_1^\dagger + g_2^\dagger \gamma_0 \gamma_2
\]

Each unit timelike vector \( \gamma_0 \in \mathbb{G}_{1,3} \) determines a unique Pauli sub-algebra \( \mathbb{G}_{3} = \mathbb{G}_{1,3}^\gamma = G_{1,3} \),

and its corresponding rest-frame \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \), satisfying \( \mathbf{e}_k \gamma_0 = -\gamma_0 \mathbf{e}_k \) for \( k = 1, 2, 3 \). The relative conjugations of the Pauli algebra \( \mathbb{G}_3 \) can be defined directly in terms of the relative conjugations of the larger algebra \( \mathbb{G}_{1,3} \). For \( g \in \mathbb{G}_3 \),

\[
g^\dagger := \gamma_0 g^* \gamma_0, \quad g^\dagger := \gamma_0 g^* \gamma_0, \quad g^* := (g^-)^\dagger = \tilde{g}.
\]

Using (65), we calculate

\[
\begin{align*}
g g^\dagger &= (\alpha^\dagger + x^2 + y^2 + (\alpha + \alpha^\dagger)x - (\alpha - \alpha^\dagger)y), \quad \text{where} \quad \text{det}[g]_\Omega = \mathbb{G}_{0+3}, \\
g g^\dagger &= \alpha^\dagger (\alpha + x + Iy)(\alpha - x - Iy) = \alpha^2 - (x - Iy)^2 = \text{det}[g]_\Omega = \mathbb{G}_{0+3},
\end{align*}
\]

or \( gg^* = R_1 + IR_2 \) for \( R_1, R_2 \in \mathbb{R} \).

Define

\[
J := g\gamma_0 g^* = g\gamma_0 g^* g_0 g_0 = g g^\dagger = \theta_0 + \phi \hat{a}, \quad \text{where} \quad \hat{a} = -1 \quad \text{and} \quad \gamma_0, \hat{a} = 0,
\]

\[
S := g\gamma_2 g^* = -I g e_3 g^* \quad \text{and} \quad K := g\gamma_3 g^* = g e_3 g^\dagger.
\]

We then find

\[
J^2 = g\gamma_0 g^* g\gamma_0 g^* = (R_1 + IR_2)(R_1 - IR_2) = R_1^2 + R_2^2 = r^2 + s^2 \geq 0,
\]

\( S \) is a bivector in \( \mathbb{G}_{1,3} \) since \( \bar{S} = S^* = -S \), and

\[
S^2 = (-I g e_3 g^*) (-I g e_3 g^*) = -g e_3 R e_3 g^* = -R^2
\]

where \( R = R_1 + IR_2 \). Similarly, \( \bar{K} = K \) and the inverse of \( K \in \mathbb{G}_{1,3} \) is \( -K \), from which it follows that \( K \in \mathbb{G}_{1,3} \), and

\[
K^2 = g\gamma_3 g^* g\gamma_3 g^* = g\gamma_0 R g^* = -R R^\dagger = -J^2 \leq 0.
\]

We also find that

\[
KJ = g\gamma_3 g^* g\gamma_0 g^* = g\gamma_0 R g^* = g e_3 g^* R^\dagger = I S R^\dagger = K \wedge J,
\]

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and consequently, $\mathbf{K} \cdot \mathbf{J} = 0$. Note also that

$$JS = -Ig\gamma_0 g^* e_3 e^* = -Ig\gamma_0 Re_3 e^* = IR^I\mathbf{K},$$

and

$$JSK = IR^I K^2 = -I|R|^2 R^+. $$

Writing $g = (x_0 + x) + (y_0 + y)$, $g^\dagger = (x_0 + x) - (y_0 + y)$, and $g^* = (x_0 - x) - (y_0 - y)$, calculate

$$gg^\dagger = (x_0^2 + y_0^2 + x^2 + y^2) + 2(x_0 x + y_0 y + x \times y),$$

$$g^\dagger g = (x_0^2 + y_0^2 + x^2 + y^2) + 2(x_0 x + y_0 y - x \times y),$$

and

$$gg^* = g^* g = x_0^2 - y_0^2 + y^2 - x^2 + 2I(x_0 y_0 - x \cdot y) = R.$$ We see that

$$\mathbf{J} = gg^\dagger y_0 = \theta y_0 + \phi \hat{a} = (x_0^2 + y_0^2 + x^2 + y^2) y_0 + 2(x_0 x + y_0 y + x \times y) y_0.$$

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