Matrix Gateway to Geometric Algebra, Spacetime and Spinors

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Preface

Gometric algebra has been presented in many different guises since its invention by William Kingdon Clifford shortly before his death in 1879. The guiding principle of this book is that it should be fully integrated into the foundations of mathematics, and in this regard nothing is more fundamental than the concept of number itself. Since I acquired this conviction as a graduate student at Arizona State University more than 50 years ago, much work has been done in applying geometric algebra to problems in mathematics, physics, engineering and computer science, in the gradual recognition of importance of geometric algebra and in the truth of this principle.

Two important areas of mathematics that lie at its foundations are geometry and algebra. If we consider the human skeleton to define the geometry of the human body, and the circulatory system to be its algebra, then it is not far fetched to say that geometric algebra is the life blood of the human body.

Matrix linear algebra, like the circulatory system, reaches across much of the mathematical skeleton that has been developed over the centuries of written human history. In this book we fully integrate the ideas of geometric algebra directly into the fabric of matrix linear algebra. A geometric matrix is a matrix which is identified with a unique geometric number in a geometric algebra. The matrix product of two geometric matrices is just the product of the corresponding geometric numbers. Matrix linear algebra is a prerequisite for reading this book. Any equation can be either interpreted as a matrix equation or an equation in geometric algebra, thus fully unifying the two languages. This is the motivating idea behind the title of this book.

Chapter 1 introduces the hyperbolic numbers alongside the familiar complex numbers. At the heart of the complex numbers is the so called unit imaginary number $i := \mathrm{sqrt}(-1)$. In contrast, the hyperbolic numbers are based upon the introduction of a new square root of $+1$, called a unipotent, [62], [69, Ch.2]. The hyperbolic numbers are applied to solving the cubic equation, and to the Lorentzian geometry of relativity theory.

Chapter 2 reviews basic ideas about vectors as directed line segments, and the addition and scalar multiplication of vectors. We then introduce a general scheme for the multiplication of vectors based upon the greatest theorem from antiquity - the Pythagorean Theorem. The material of this Chapter has not been previously published and represents the work the present author, Eng.

Jose Alfonso Juarez Gonzalez, and Eng. Sergio Ramos Ramirez.
Chapter 3 introduces 2×2 real and complex square matrices in a way which uncovers the underlying geometric interpretations of such matrices. We call such matrices, together with their inherited geometric interpretations, geometric matrices, and study their properties.

Chapter 4 undertakes a more standard classification of all geometric algebras and their basic properties. By the geometrization of the real number system we mean the extension of the real number system to include an arbitrary number of new, anti-commuting square roots of ±1. Higher dimensional geometric algebras are constructed in terms of lower dimensional geometric algebra building blocks. The famous Bott periodicity of geometric algebras is discussed.

Chapter 5 is a continuation of the work begun in Chapter 3. The concept of a geometric matrix is extended to higher dimensional square matrices by introducing the directed Kronecker product of matrices. The permutation group algebra of the symmetric group S_n is introduced, including other ideas from representation theory.

Chapter 6 introduces the famous Plücker Relations in geometric algebra. The Plücker relations are at the heart Grassmannians and fundamental ideas of algebraic geometry.

Chapters 7 - 9 explore more advanced topics in the application of geometric algebras to Pauli and Dirac spinors, relativity and Maxwell's equations, quaternions and split quaternions, and group manifolds. They are included to highlight the great variety of topics that are imbued with new geometric insight when expressed in geometric algebra. Additional references to this more advanced material are provided. The usefulness of these later chapters will depend heavily on the background and previous knowledge of the reader.

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