Geometric Spinors, Relativity and the Hopf Fibration

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Abstract

This article explores geometric number systems that are obtained by extending the real number system to include new *anticommuting* square roots of ± 1, each such new square root representing the direction of a unit vector along orthogonal coordinate axes of a Euclidean or pesudoEuclidean space. These new number systems can be thought of as being nothing more than a geometric basis for tables of numbers, called matrices. At the same time, the consistency of matrix algebras prove the consistency of our geometric number systems. The flexibility of this new concept of geometric numbers opens the door to new understanding of the nature of spacetime, the concept of Pauli and Dirac spinors, and the famous Hopf fibration.

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1 Introduction

The concept of number has played a decisive role in the ebb and flow of civilizations across centuries. Each more advanced civilization has made its singular contributions to the further development, starting with the natural "counting numbers" of ancient peoples, to the quest of the Pythagoreans' idea that (rational) numbers are everything, to the heroic development of the "imaginary" numbers to gain insight into the solution of cubic and quartic polynomials, which underlies much of modern mathematics, used extensively by engineers, physicists and mathematicians of today [4]. I maintain that the culmination of this development is the *geometrization* of the number concept:

> **Axiom:** The real number system can be geometrically extended to include new, anti-commutative square roots of ± 1, each new such square root representing the direction of a unit vector along orthogonal coordinate axes

of a Euclidean or pseudo-Euclidean space $\mathbb{R}^{p,q}$. The resulting associative geometric algebra is denoted by $\mathbb{G}_{p,q}$.

Whereas we have stated our Axiom for finite dimensional geometric algebras, there is no impediment to applying it to infinite dimensional Hilbert spaces as well. But mainly we shall be interested in developing the ideas as they apply to the 8 dimensional real geometric algebra $\mathbb{G}_3$ of space, which is the natural generalization of the familiar Gibbs-Heaviside vector algebra of the dot and cross products, the complex numbers of $i = \sqrt{-1}$, and Hamilton's quaternions. The antecedents of our *geometric algebras* can be found in the works of W. K. Clifford [2], H. Grassmann [6], and W. Hamilton [7].

2 The geometric algebra of space $\mathbb{G}_3$

The most direct way of obtaining the geometric algebra $\mathbb{G}_3$ of the Euclidean space $\mathbb{R}^3$ is to extend the real number system $\mathbb{R}$ to include three new *anti-commutating* square roots $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $+1$, that represent *unit vectors* along the respective *xyz*-coordinate axes. Thus, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \notin \mathbb{R}$ and $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$. The resulting associative geometric algebra $\mathbb{G}_3 := \mathbb{R}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, as a real linear space, has the $2^3 = 8$-dimensional *standard basis*

$$\mathbb{G}_3 = span_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\},$$

where $\mathbf{e}_{jk} := \mathbf{e}_j \mathbf{e}_k = -\mathbf{e}_k \mathbf{e}_j$ represent unit *bivectors* in the three xy, xz, yz-coordinate planes, for $j \neq k$, and $I := \mathbf{e}_{123} = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ represents the oriented *directed trivector*, or *pseudoscalar* of element of space. The even sub-algebra $\mathbb{G}_3^+ = span_{\mathbb{R}}\{1, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$ of $\mathbb{G}_3$ is algebraically isomorphic to the algebra of quaternions. The geometric numbers of 3-dimensional space $\mathbb{R}^3$ are pictured in Figure 1.

To see that the rules of our geometric algebra are consistent, we relate it immediately to the famous Pauli algebra $\mathscr{P}$ of square 2×2 matrices over the complex numbers $\mathbb{C}$, [11]. The most intuitive way of doing this is to introduce the *mutually annihilating idempotents* $u_\pm := \frac{1}{2}(1 \pm \mathbf{e}_3)$, which satisfy the rules

$$u_+^2 = u_+, u_-^2 = u_-, u_+ u_- = 0, u_+ + u_- = 1, u_+ - u_- = \mathbf{e}_3.$$

In addition, $\mathbf{e}_1 u_+ = u_- \mathbf{e}_1$. All these rules are easily verified and left to the reader. Another important property of the geometric algebra $\mathbb{G}_3$ is that the pseudoscalar element $I = \mathbf{e}_{123}$ is in the *center* of the algebra commuting with all elements, and $I^2 = -1$. Thus, I can take over the roll of the unit imaginary $i = \sqrt{-1}$.

By the *spectral basis* $\mathscr{S}_{\mathbb{G}}$ of the geometric algebra $\mathbb{G}_3$, we mean

$$\mathscr{S}_{\mathbb{G}} := \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}, \tag{1}$$

where I is taking over the roll of $i = \sqrt{-1}$ in the usual Pauli algebra. Any geometric number $g = s + It + \mathbf{a} + I\mathbf{b} \in \mathbb{G}_3$ corresponds directly to a Pauli matrix $[g]$, by the simple rule

$$g = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}.$$

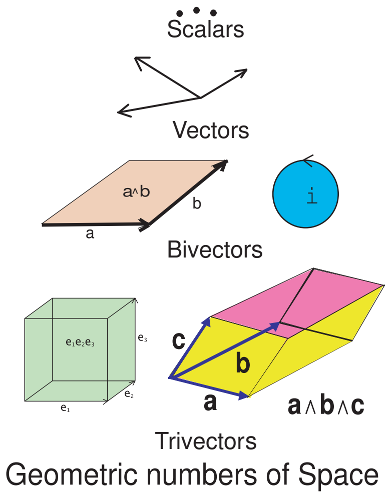

Figure 1: Scalars, vectors, bivectors and trivectors (pseudoscalars) make up the geometric numbers of space.

Since the rules of geometric addition and multiplication are fully compatible with the rules of matrix addition and multiplication, matrices over the geometric algebra $\mathbb{G}_3$ are well defined.

For example, the famous Pauli matrices $[\mathbf{e}_1], [\mathbf{e}_2], [\mathbf{e}_3]$ are specified by

$$[\mathbf{e}_1] := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, [\mathbf{e}_2] := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, [\mathbf{e}_3] := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

as can be easily checked. For $[\mathbf{e}_2]$, we have

$$\mathbf{e}_2 = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = I\mathbf{e}_1(u_+ - u_-) = I\mathbf{e}_1\mathbf{e}_3.$$

The Pauli algebra $\mathscr{P}$ and the geometric algebra $\mathbb{G}_3$ are isomorphic algebras over the complex numbers defined by $\mathbb{C} := \mathbb{R}(I)$. The geometric product of $g_1 g_2 \in \mathbb{G}_3$ corresponds to the product of the matrices $[g_1][g_2] = [g_1 g_2]$. The great advantage of the geometric algebra $\mathbb{G}_3$ over the Pauli algebra $\mathscr{P}$, is that the geometric numbers are liberated from their coordinate representations as 2×2 complex matrices, as well as being endowed with a comprehensive geometric interpretation. On the other hand, the consistency of the rules of the geometric algebra $\mathbb{G}_3$ follow from the known consistency of rules of matrix algebra, and offer a computational tool for computing the product of geometric numbers. More generally, every geometric algebra $\mathbb{G}_{p,q}$ is algebraically isomorphic to a real matrix algebra, or a sub-algebra of a real matrix algebra of sufficiently high dimension [13].

It is worthwhile to offer a summary of the deep relationship between *pre-relativistic* Gibbs-Heaviside vector algebra [3], and the geometric algebra $\mathbb{G}_3$. The geometric product of two vectors $\mathbf{a},\mathbf{b}\in\mathbb{G}_3^1$ is given by

$$\mathbf{ab}=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})+\frac{1}{2}(\mathbf{ab}-\mathbf{ba})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}, \tag{3}$$

where the *inner product* $\mathbf{a}\cdot\mathbf{b}:=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})$, and the *outer product* $\mathbf{a}\wedge\mathbf{b}:=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$ has the interpretation (due to Grassmann) of the *bivector* in the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$. The bivector $\mathbf{a}\wedge\mathbf{b}=I(\mathbf{a}\times\mathbf{b})$, where $\mathbf{a}\times\mathbf{b}$ is the vector normal to the plane of $\mathbf{a}\wedge\mathbf{b}$, and is its *dual*.

Another advantage of the *unified* geometric product $\mathbf{ab}$ over the inner product $\mathbf{a}\cdot\mathbf{b}$ and the cross product $\mathbf{a}\times\mathbf{b}$, is the powerful cancellation rule

$$\mathbf{ab}=\mathbf{ac} \quad\iff\quad \mathbf{a}^2\mathbf{b}=\mathbf{a}^2\mathbf{c} \quad\iff \mathbf{b}=\mathbf{c},$$

provided of course that $\mathbf{a}^2\neq 0$. It takes knowledge of *both* $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$ (or $\mathbf{a}\times\mathbf{b}$ in $\mathbb{R}^3$), to uniquely determine the relative directions of the vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^{p,q}$, the pseudo-Euclidean space of $p+q$ dimensions. Another unique advantage of the geometric algebra is the *Euler formula* made possible by (3),

$$\mathbf{ab}=|\mathbf{a}|\mathbf{b}|e^{I\hat{\mathbf{c}}\theta}=|\mathbf{a}||\mathbf{b}|(\cos\theta+I\hat{\mathbf{c}}\sin\theta),$$

where $|\mathbf{a}|:=\sqrt{\mathbf{a}^2}$ and similarly for $|\mathbf{b}|$, and θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. The unit vector $\hat{\mathbf{c}}$ can be defined by $\hat{\mathbf{c}}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$. The Euler formula, for the bivector $I\hat{\mathbf{c}}$ which has square -1, is the generator of rotations in the plane of the bivector $\mathbf{a}\wedge\mathbf{b}$. Later, when talking about *Lorentz boosts*, we will also utilize the *hyperbolic Euler form*

$$e^{\phi\hat{\mathbf{c}}}=\cosh\phi+\hat{\mathbf{c}}\sinh\phi, \tag{4}$$

where $\tanh\phi=\frac{v}{c}\in\mathbb{R}$ determines the *rapidity* of the boost in the direction of the unit vector $\hat{\mathbf{c}}\in\mathbb{G}_3^1$. The complex numbers $\mathbb{C}:=\mathbb{R}(i)$ and *hyperbolic numbers* $\mathscr{H}=\mathbb{R}(u)$, where $u\notin R$ and $u^2=1$, have many properties in common [14].

A couple more formulas, relating vector cross and dot products to the geometric product, are

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}:=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})I=I\det[\mathbf{a},\mathbf{b},\mathbf{c}],$$

and

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}):=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}=-\mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

The triple vector products $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$ are directly related to the geometric product by the identity

$$\mathbf{a}(\mathbf{b}\wedge\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})+\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}),$$

where

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}):=\frac{1}{2}\Big(\mathbf{a}(\mathbf{b}\wedge\mathbf{c})-(\mathbf{b}\wedge\mathbf{c})\mathbf{a}\Big)$$

and

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}):=\frac{1}{2}\Big(\mathbf{a}(\mathbf{b}\wedge\mathbf{c})+(\mathbf{b}\wedge\mathbf{c})\mathbf{a}\Big).$$

Detailed discussions and proofs of these identities, and their generalizations to higher dimensional geometric algebras, can be found in [16].

We now return to beautiful results which depend in large part only upon the geometric product. Thus the reader can relax and fall back upon the familiar rules of matrix algebra, which are equally valid in the isomorphic geometric algebra.

3 Stereographic projection in $\mathbb{R}^3$

Consider the equation

$$\mathbf{m} = \frac{2}{\hat{\mathbf{a}}+\mathbf{e}_3} = \frac{2(\hat{\mathbf{a}}+\mathbf{e}_3)}{(\hat{\mathbf{a}}+\mathbf{e}_3)^2} = \frac{\hat{\mathbf{a}}+\mathbf{e}_3}{1+\hat{\mathbf{a}}\cdot\mathbf{e}_3}, \tag{5}$$

where $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$ is a unit vector for the vector $\mathbf{a} \in \mathbb{G}_3^1$. Clearly, this equation is well defined except when $\hat{\mathbf{a}} = -\mathbf{e}_3$. Let us solve this equation for $\hat{\mathbf{a}}$, but first we find that

$$\mathbf{m}\cdot\mathbf{e}_3 = \Big(\frac{2}{\hat{\mathbf{a}}+\mathbf{e}_3}\Big)\cdot\mathbf{e}_3 = \frac{(\hat{\mathbf{a}}+\mathbf{e}_3)\cdot\mathbf{e}_3}{1+\hat{\mathbf{a}}\cdot\mathbf{e}_3} = 1.$$

Returning to equation (5),

$$\hat{\mathbf{a}} = \frac{2}{\mathbf{m}} - \mathbf{e}_3 = \frac{1}{\mathbf{m}}\big(2-\mathbf{m}\mathbf{e}_3\big)$$

$$= \frac{1}{\mathbf{m}}\big(2+\mathbf{e}_3\mathbf{m}-2\mathbf{e}_3\cdot\mathbf{m}\big) = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}}. \tag{6}$$

Equation (5) can be equivalently expressed by

$$\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = (\hat{\mathbf{m}}\mathbf{e}_3)\mathbf{e}_3(\mathbf{e}_3\hat{\mathbf{m}}) = (-I\hat{\mathbf{m}})\mathbf{e}_3(I\hat{\mathbf{m}}), \tag{7}$$

showing that $\hat{\mathbf{a}}$ is obtained by a rotation of $\mathbf{e}_3$ in the plane of $\hat{\mathbf{m}}\wedge\mathbf{e}_3$ through an angle of 2θ where $\cos\theta := \mathbf{e}_3\cdot\hat{\mathbf{m}}$, or equivalently, by a rotation of $\mathbf{e}_3$ in the plane of $I\hat{\mathbf{m}}$ through an angle of π .

It is easily shown that the most general idempotent in $\mathbb{G}_3$ has the form

$$s = \frac{1}{2}(1+\mathbf{m}+I\mathbf{n}) \tag{8}$$

where

$$(\mathbf{m}+I\mathbf{n})^2 = 1 \quad\Longleftrightarrow\quad \mathbf{m}^2-\mathbf{n}^2 = 1 \ \text{ and } \ \mathbf{m}\cdot\mathbf{n} = 0,$$

and $I := \mathbf{e}_{123}$ is the unit pseudoscalar element in $\mathbb{G}_3^3$. Consider now idempotents of the form $p = (1+\lambda\mathbf{e}_1)u_+$, where $\lambda \in \mathbb{G}_3^{0+3}$. Equating $s = p$, we find that

$$(1+\mathbf{m}+I\mathbf{n}) = (1+\lambda\mathbf{e}_1)(1+\mathbf{e}_3) = 1+\lambda\mathbf{e}_1 - I\lambda\mathbf{e}_2+\mathbf{e}_3.$$

Reversing the parity (a vector $\mathbf{v}\to-\mathbf{v}$) of this equation, gives

$$(1-\mathbf{m}+I\mathbf{n}) = (1-\lambda^\dagger\mathbf{e}_1)(1-\mathbf{e}_3) = 1-\lambda^\dagger\mathbf{e}_1 - I\lambda^\dagger\mathbf{e}_2-\mathbf{e}_3,$$

since the *parity change* λ^- of λ is identical to the *reverse* (reversing the order of the geometric product of vectors) $\lambda^\dagger$ of λ.

We can now solve these last two equations for $\mathbf{m}$ and $I\mathbf{n}$ in terms of λ, getting

$$\mathbf{m} = \frac{\lambda+\lambda^\dagger}{2}\mathbf{e}_1 + \frac{\lambda-\lambda^\dagger}{2I}\mathbf{e}_2 + \mathbf{e}_3,$$

and

$$I\mathbf{n} = \frac{\lambda-\lambda^\dagger}{2}\mathbf{e}_1 - \frac{\lambda+\lambda^\dagger}{2I}\mathbf{e}_2 = \mathbf{m}\wedge\mathbf{e}_3,$$

or

$$\mathbf{m} = \mathbf{x}+\mathbf{e}_3 \quad \text{and} \quad \mathbf{n} = -I(\mathbf{m}\wedge\mathbf{e}_3) = \mathbf{m}\times\mathbf{e}_3, \tag{9}$$

where $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 \in \mathbb{R}^2$, the xy-plane. From (8) and (9), it immediately follows that

$$s = \frac{1}{2}(1+\mathbf{m}+I\mathbf{n}) = \frac{1}{2}(1+\mathbf{m}+\mathbf{m}\wedge\mathbf{e}_3) = \mathbf{m}u_+. \tag{10}$$

We also easily find that

$$\mathbf{m}^2 = 1+\lambda\lambda^\dagger = 1+\mathbf{x}^2 \geq 1$$

so that

$$|\mathbf{m}| = \sqrt{1+\lambda\lambda^\dagger} = \sqrt{1+\mathbf{x}^2}.$$

A *Pauli spinor* is a column matrix of two complex components, which we denote by $[\alpha]_2 := \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$. Each Pauli spinor $[\alpha]_2$ corresponds to a *minimal left ideal* $[\alpha]_L$, which in turn corresponds to *geometric Pauli spinor*, or *Pauli g-spinor* α in the geometric algebra $\mathbb{G}_3$, [17]. Using the spectral basis (1), we have

$$[\alpha]_2 = \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \quad \longleftrightarrow \quad [\alpha]_L := \begin{pmatrix}\alpha_0 & 0\\ \alpha_1 & 0\end{pmatrix}$$

$$\longleftrightarrow \quad \alpha := (\alpha_0+\alpha_1\mathbf{e}_1)u_+ \in \mathbb{G}_3. \tag{11}$$

By factoring out α_0 from g-spinor α, we get

$$\alpha = (\alpha_0+\alpha_1\mathbf{e}_1)u_+ = \alpha_0(1+\frac{\alpha_1}{\alpha_0}\mathbf{e}_1)u_+ = \alpha_0 p = \alpha_0\mathbf{m}u_+,$$

where $p = s$ was the idempotent defined above for $\lambda = \frac{\alpha_1}{\alpha_0}$.

By the *norm* $|\alpha|$ of the g-spinor α, we mean

$$|\alpha| := \sqrt{2\langle\alpha^\dagger\alpha\rangle_0} = \sqrt{\alpha_0\alpha_0^\dagger+\alpha_1\alpha_1^\dagger} \geq 0, \tag{12}$$

where $\langle g\rangle_0$ means the *real number part* of the geometric number $g \in \mathbb{G}_3$. More generally, we define the *sesquilinear inner product* between the spinors α,β to be

$$\langle\alpha|\beta\rangle := 2\langle\alpha^\dagger\beta\rangle_{0+3} = \alpha_0^\dagger\beta_0+\alpha_1^\dagger\beta_1,$$

where $\langle g\rangle_{0+3}$ means the *scalar and pseudo-scalar parts* of the geometric number $g \in \mathbb{G}_3$.

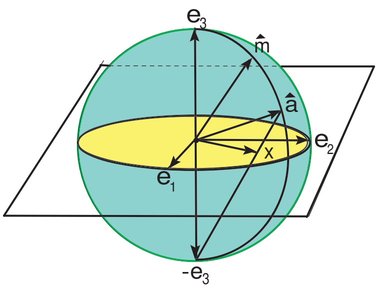

Figure 2: Stereographic Projection from the South Pole to the xy-plane.

From equations (8) and (12), it follows that for $\lambda = \alpha_0^{-1}\alpha_1$,

$$\alpha = \alpha_0 s = \alpha_0\sqrt{1+\lambda\lambda^\dagger}\hat{\mathbf{m}}u_+ = \rho e^{I\theta}\hat{\mathbf{m}}u_+ = \rho e^{I\theta}\hat{\mathbf{a}}_+\hat{\mathbf{m}},$$

where

$$e^{I\theta} := \frac{\alpha_0}{\sqrt{\alpha_0\alpha_0^\dagger}}, \quad \rho := \sqrt{\alpha_0\alpha_0^\dagger + \alpha_1\alpha_1^\dagger},$$

and $\hat{\mathbf{a}}_+ := \hat{\mathbf{m}}u_+\hat{\mathbf{m}}$. Equation (5) has an immediate interpretation on the Riemann sphere centered at the origin. Figure 2 shows a cross-section of the Riemann 2-sphere, taken in the plane of the bivector $\mathbf{m}\wedge\mathbf{e}_3$, through the origin. We see that the stereographic projection from the South pole at the point $-\mathbf{e}_3$, to the point $\hat{\mathbf{a}}$ on the Riemann sphere, passes through the point $\mathbf{x} = proj(\mathbf{m})$ of the point $\mathbf{m}$ onto the xy-plane through the origin with the normal vector $\mathbf{e}_3$. Stereographic projection is just one example of conformal mappings, which have important generalizations to higher dimensions [15, 16].

We can now simply answer a basic question in quantum mechanics. If a spin $\frac{1}{2}$-particle is prepared in a Pauli g-state α, what is the probability of finding it in a Pauli g-state β immediately thereafter? We calculate

$$\begin{aligned}\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle &= 2\Big\langle(\alpha^\dagger\beta)^\dagger(\alpha^\dagger\beta)\Big\rangle_{0+3}\\ &= 2\Big\langle u_+\hat{\mathbf{m}}_b\hat{\mathbf{m}}_a u_+\hat{\mathbf{m}}_a\hat{\mathbf{m}}_b u_+\Big\rangle_{0+3}\\ &= 2\langle\hat{\mathbf{m}}_b\hat{\mathbf{b}}_+\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\hat{\mathbf{m}}_b\rangle_{0+3}\\ &= \langle(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})u_+\rangle_{0+3} = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}).\end{aligned} \tag{13}$$

This relationship can be directly expressed in terms of $\mathbf{m}_a$ and $\mathbf{m}_b$. Using (6), for $\hat{\mathbf{a}} = \frac{2}{\mathbf{m}_a} - \mathbf{e}_3$ and $\hat{\mathbf{b}} = \frac{2}{\mathbf{m}_b} - \mathbf{e}_3$, a short calculation gives the result

$$1 - \frac{(\mathbf{m}_a-\mathbf{m}_b)^2}{\mathbf{m}_a^2\mathbf{m}_b^2} = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})$$

$$\iff \quad \frac{(\mathbf{m}_a - \mathbf{m}_b)^2}{\mathbf{m}_a^2 \mathbf{m}_b^2} = \frac{1}{2}(1 - \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}), \tag{14}$$

showing that the probability of finding the particle in that Pauli g-state β is directly related to the Euclidean distance between the points $\mathbf{m}_a$ and $\mathbf{m}_b$.

Clearly, when $\hat{\mathbf{b}} = -\hat{\mathbf{a}}$, the expression in (14) simplifies to

$$\frac{(\mathbf{m}_a - \mathbf{m}_b)^2}{\mathbf{m}_a^2 \mathbf{m}_b^2} = 1.$$

This will occur when $\mathbf{m}_b := \frac{1}{\mathbf{m}_a \wedge \mathbf{e}_3} \mathbf{m}_a$, for which case

$$\hat{\mathbf{b}} = \hat{\mathbf{m}}_b \mathbf{e}_3 \hat{\mathbf{m}}_b = -\hat{\mathbf{m}}_a \mathbf{e}_3 \hat{\mathbf{m}}_a = -\hat{\mathbf{a}} \ \text{ and } \ \mathbf{m}_b \cdot \mathbf{m}_a = 0.$$

Writing $\mathbf{m}_a = \mathbf{x}_a + \mathbf{e}_3$ for $\mathbf{x}_a \in \mathbb{R}^2$,

$$\mathbf{m}_b = \mathbf{x}_b + \mathbf{e}_3 = \frac{1}{\mathbf{x}_a \mathbf{e}_3}(\mathbf{x}_a + \mathbf{e}_3)$$

$$= \frac{\mathbf{e}_3 \mathbf{x}}{\mathbf{x}^2}(\mathbf{x}_a + \mathbf{e}_3) = -\frac{1}{\mathbf{x}_a} + \mathbf{e}_3. \tag{15}$$

This means that a spin $\frac{1}{2}$-particle prepared in the state $\mathbf{m}_a$ will have a zero probability of being found in the state $\mathbf{m}_b$, for a measurement taken immediately afterwards. It is worthwhile mentioning that these ideas can be naturally generalized to Dirac spinors [18], [19].

4 First course in special relativity

Let us now try to fit together all the pieces we have been dealing with into a coherent language for special relativity. Special relativity was first put forth as a basic theory of the physical universe by Albert Einstein in his famous 1905 paper [5, p. 35-65], although the basic ideas had been put forward in different forms by Lorentz and Poincare.

By an *inertial system* or *rest frame* of an observer, we mean a set of three orthonormal (anti-commuting) geometric numbers $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathbb{G}_3$, oriented by the property that

$$\mathbf{e}_{123} := \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = I.$$

These numbers represent three orthonormal *relative vectors* along the *xyz*-coordinate axes of an observer, and generate the *relative geometric algebra* $\mathbb{G}_3 := \mathbb{R}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of that observer. The relative geometric algebra $\mathbb{G}_3' = \mathbb{R}(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ of any other observer in another rest-frame $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$, is no more than a *repartitioning* of the elements of $\mathbb{G}_3$ into relative vectors and relative bivectors. In other words, what one observer calls a vector in his relative geometric algebra represents a mixture of a vector and a bivector in the geometric algebra of an observer in a different inertial system. But all of the observers agree on the element called the unit pseudoscalar $I := \mathbf{e}_{123} = \mathbf{e}_{123}'$, the oriented volume element of space-time [12]. In Figure 3, a *Lorentz boost* in the direction of the

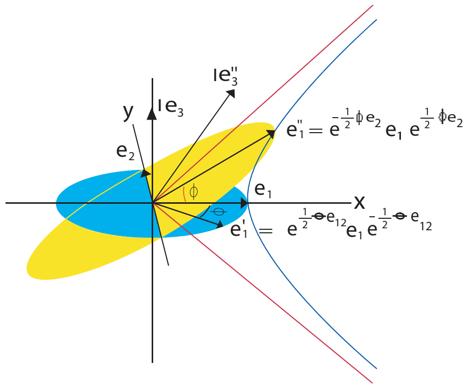

Figure 3: The vector $\mathbf{e}_1$ is rotated in the xy-plane of $\mathbf{e}_{12}$ into the vector $\mathbf{e}'_1$. The vector $\mathbf{e}_1$ is *boosted* into the *relative vector* $\mathbf{e}''_1$ of the *relative plane* of the *relative bivector* $\mathbf{e}''_{12} = \mathbf{e}''_1\mathbf{e}_2$. The relative plane of $\mathbf{e}''_{12}$ has the relative velocity of $\frac{\mathbf{v}}{c} = \mathbf{e}_2 \tanh\phi$ with respect to the plane of $\mathbf{e}_{12}$, where c is the speed of light. Note that the bivector axes $I\mathbf{e}_3 = \mathbf{e}_{12}$ is shown and not the z-axes $\mathbf{e}_3$.

vector $\mathbf{e}_2$ boosts the relative Euclidean plane $\mathbf{e}_{12}$ of the first observer into the relative Euclidean plane $\mathbf{e}''_{12}$ of the second observer.

A *space-time event* is named by a point $X = ct + \mathbf{x} \in \mathbb{G}_3^{0+1}$, where the real number t is the *time* of the event and the position vector $\mathbf{x}$ is the *place* of the event. The constant c is the velocity of light in empty space. The set $\mathscr{H} := \{X| \ X = ct + \mathbf{x}\}$ of all events, both past $t < 0$, present $t = 0$ and future $t > 0$, is called the *event horizon* with respect to any observer in that *rest frame*. The *world line* of an observer at the *spacial origin* $\mathbf{x} = 0$ of his rest frame is $X_0 = ct$. It should be mentioned that there is another way in employing the geometric algebra $\mathbb{G}_3$ in the formulation of special relativity. In the article [1], the approach used here is referred to as the "absolute" *Algebra of Physical Space* (APS), in contrast to the alternative approach called the "relative" APS.

Suppose now that we have a second observer $X'_0 = ct'$ who is moving at a constant velocity $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, as measured by the first observer, and suppose that at $t = 0$, $X' = 0$, the *origin* of *space-time*. Then the world line of the second observer is

$$X = ct + t\mathbf{v} = ct\left(1 + \frac{\mathbf{v}}{c}\right) = ct\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}e^{\phi\hat{\mathbf{v}}}$$

$$= ct'e^{\phi\hat{\mathbf{v}}} = X'_0 e^{\phi\hat{\mathbf{v}}},$$

or equivalently,

$$t = \frac{t'}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \quad \text{and} \quad \frac{\mathbf{v}}{c} = \hat{\mathbf{v}}\tanh\phi. \tag{16}$$

This means that $X'_0 = ct' = Xe^{-\phi\hat{\mathbf{v}}}$. Multiplying X on the right by $e^{-\phi\hat{\mathbf{v}}}$, transforms away the velocity $\mathbf{v}$ as measured in the first inertial system.

The defining principle of special relativity is that the *space-time* interval, is independent of the observer that is measuring it, regardless of any relative velocity. The space-time interval $|X_1 - X_2|$ between two events $X_1 = ct_1 + \mathbf{x}_1$ and $X_2 = ct_2 + \mathbf{x}_2$ is

$$\begin{aligned} |X_1 - X_2| &:= \sqrt{|(X_1 - X_2)(X_1 - X_2)^*|} \\ &= \sqrt{|c^2(t_1 - t_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2|}. \end{aligned} \tag{17}$$

Note that the space-time interval between two events X_1 and X_2 reduces to the ordinary Euclidean distance $|\mathbf{x}_1 - \mathbf{x}_2|$, if the events are *simultaneous* at the same time t. Note also that the *time dilation* of t with respect to t', by the factor $\gamma := \cosh\phi = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$, is at the heart of the so-called *twin paradox* of special relativity.

We have seen in (16) that the mapping

$$X' = Xe^{-\phi\hat{\mathbf{v}}} = \gamma(ct + \mathbf{x})(1 - \frac{\mathbf{v}}{c}), \tag{18}$$

transforms away the velocity for a particle with world line $X = ct + t\mathbf{v}$. More generally, for two events $X_1 = ct_1 + \mathbf{x}_1$ and $X_2 = ct_2 + \mathbf{x}_2$ in $\mathscr{H}$, and corresponding events $X_1' = ct_1' + \mathbf{x}_1'$ and $X_2' = ct_2' + \mathbf{x}_2'$ in $\mathscr{H}'$, related by the mapping (18), we find that

$$(X_1' - X_2')(X_1' - X_2')^* = (X_1 - X_2)(X_1 - X_2)^*,$$

so the space-time interval between two events is independent of the observer who is measuring it.

Let us study the mapping $X' = Xe^{-\phi\hat{\mathbf{v}}}$ more closely. Decomposing

$$\mathbf{x} = (\mathbf{x}\hat{\mathbf{v}})\hat{\mathbf{v}} = (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} + (\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}} = \mathbf{x}_\parallel + \mathbf{x}_\perp,$$

we find that

$$\begin{aligned} ct' + \mathbf{x}' &= \gamma(ct + \mathbf{x}_\parallel + \mathbf{x}_\perp)(1 - \frac{\mathbf{v}}{c}) \\ &= \gamma(ct - \mathbf{x}\cdot\frac{\mathbf{v}}{c}) + \gamma\big(\mathbf{x}_\parallel - t\mathbf{v}\big) + \mathbf{x}_\perp\gamma\big(1 - \frac{\mathbf{v}}{c}\big), \end{aligned}$$

so that $t' = \gamma(t - \frac{\mathbf{x}\cdot\mathbf{v}}{c^2})$ and

$$\begin{aligned} \mathbf{x}' &= \gamma\Big(\mathbf{x} - t\mathbf{v} - I(\mathbf{x} - t\mathbf{v})\times\frac{\mathbf{v}}{c}\Big) \\ &= e^{\frac{1}{2}\phi\hat{\mathbf{v}}}\Big(\gamma(\mathbf{x}_\parallel - t\mathbf{v}) + \mathbf{x}_\perp\Big)e^{-\frac{1}{2}\phi\hat{\mathbf{v}}}. \end{aligned} \tag{19}$$

The last expression for $\mathbf{x}'$ shows that $\mathbf{x}'$ is a position vector in the rest-frame of $\mathscr{H}'$, as would be expected.

Introducing the new variable $\mathbf{r} = \mathbf{x} - t\mathbf{v}$, the expression (19) takes the form

$$\mathbf{x}' = \gamma\Big(\mathbf{x} - t\mathbf{v} - I(\mathbf{x} - t\mathbf{v})\times\frac{\mathbf{v}}{c}\Big) = \gamma\Big(\mathbf{r} - I\mathbf{r}\times\frac{\mathbf{v}}{c}\Big), \tag{20}$$

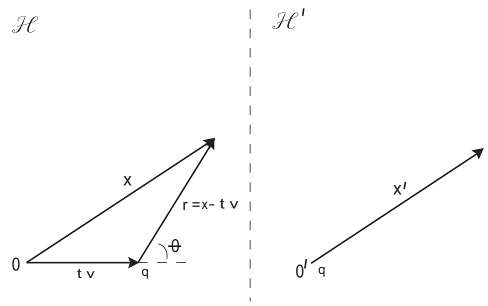

Figure 4: In the event horizon $\mathscr{H}$, the charge q is located at the point $t\mathbf{v}$, and the electromagnetic field is observed at the point $\mathbf{x}$. In $\mathscr{H}'$ the charge is located at the spacial origin $0'$, and the electric field is observed at the point $\mathbf{x}'$. The time coordinates ct and ct' are suppressed.

which we use to calculate

$$|\mathbf{x}'| = \sqrt{|\mathbf{x}'|^2} = \gamma|\mathbf{r}|\sqrt{1-\frac{\mathbf{v}^2}{c^2}\sin^2\theta},$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{v}$.

The equation (20) is interesting from the point of a physicist. Suppose that a moving electric point charge is located on the world line $X = ct(1+\frac{\mathbf{v}}{c})$, then an observer on the world line $X' = ct' + \mathbf{x}'$, at rest in the space-time horizon $\mathscr{H}'$, will only see a Coulomb electric field $E' = \kappa\frac{\mathbf{x}'}{|\mathbf{x}'|^3}$, whereas the observer on the world line $X = ct + \mathbf{x}$, at rest in the space-time horizon $\mathscr{H}$, will see the electromagnetic field

$$F = \kappa\frac{\mathbf{x}'}{|\mathbf{x}'|^3} = E + IB,$$

where

$$E = \frac{\kappa\mathbf{r}}{\gamma^2|\mathbf{r}|^3\left(1-\frac{\mathbf{v}^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}, \text{ and } B = -E\times\frac{\mathbf{v}}{c},$$

and κ is a constant depending upon the charge and the system of units used, [9, pp. 2,381]. This is the relativistic generalization of the classical *Biot-Savart Law*, see Figure 4.

5 Composition of Lorentz boosts and rotations

We now give an important formula for the product of Lorentz boosts of the form

$$e^{\phi_1\hat{\mathbf{a}}_1} = \frac{1+\mathbf{a}_1}{\sqrt{1-\mathbf{a}_1^2}} \quad \text{and} \quad e^{\phi_2\hat{\mathbf{a}}_2} = \frac{1+\mathbf{a}_2}{\sqrt{1-\mathbf{a}_2^2}}$$

where $|\mathbf{a}_1| < 1$ and $|\mathbf{a}_2| < 1$, and $\phi_1 > 0,\ \phi_2 > 0$. We now show that

$$e^{\phi_1 \hat{\mathbf{a}}_1} e^{\phi_2 \hat{\mathbf{a}}_2} = e^{\phi \hat{\mathbf{b}}_1} e^{\theta I \hat{\mathbf{b}}_2}, \tag{21}$$

where

$$\theta = \tan^{-1} \frac{|\mathbf{a}_1 \times \mathbf{a}_2|}{\sqrt{1 + \mathbf{a}_1 \cdot \mathbf{a}_2}}, \quad \phi = \tanh^{-1} \frac{|\mathbf{a}_1 + \mathbf{a}_2|}{\sqrt{\mathbf{a}_1^2 \mathbf{a}_2^2 + 2\mathbf{a}_1 \cdot \mathbf{a}_2 + 1}},$$

$$\hat{\mathbf{b}}_1 = \frac{1 + \mathbf{a}_2^2 + 2\mathbf{a}_1 \cdot \mathbf{a}_2}{1 + \mathbf{a}_1^2 \mathbf{a}_2^2 + 2\mathbf{a}_1 \cdot \mathbf{a}_2} \mathbf{a}_1 + \frac{1 - \mathbf{a}_1^2}{1 + \mathbf{a}_1^2 \mathbf{a}_2^2 + 2\mathbf{a}_1 \cdot \mathbf{a}_2} \mathbf{a}_2$$

and

$$\hat{\mathbf{b}}_2 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}.$$

The proof of (21) follows by expanding out both sides of this equation and equating scalar, vector, and bivector parts. The formulas for the angles θ and ϕ follow directly from

$$\tan^2 \theta = \frac{(\mathbf{a}_1 \times \mathbf{a}_2)^2}{(1 + \mathbf{a}_1 \cdot \mathbf{a}_2)^2}$$

and

$$(1 + \tan^2 \theta) \tanh^2 \phi = \frac{(\mathbf{a}_1 + \mathbf{a}_2)^2}{(1 + \mathbf{a}_1 \cdot \mathbf{a}_2)^2}.$$

We also calculate

$$\cosh \phi = \frac{\sqrt{\mathbf{a}_1^2 \mathbf{a}_2^2 + 2\mathbf{a}_1 \cdot \mathbf{a}_2 + 1}}{\sqrt{(1 - \mathbf{a}_1^2)(1 - \mathbf{a}_2^2)}},$$

and

$$\sinh \phi = \frac{|\mathbf{a}_1 + \mathbf{a}_2|}{\sqrt{(1 - \mathbf{a}_1^2)(1 - \mathbf{a}_2^2)}}.$$

Note also that

$$e^{\phi \hat{\mathbf{b}}_1} e^{\theta I \hat{\mathbf{b}}_2} = e^{\theta I \hat{\mathbf{b}}_2} (e^{-\theta I \hat{\mathbf{b}}_2} e^{\phi \hat{\mathbf{b}}_1} e^{\theta I \hat{\mathbf{b}}_2}) = e^{\theta I \hat{\mathbf{b}}_2} e^{\phi \hat{\mathbf{b}}_1'},$$

for $\hat{\mathbf{b}}_1' = \hat{\mathbf{b}}_1 e^{2\theta I \hat{\mathbf{b}}_2}$.

Similarly, the product of two Euclidean rotations is found to be

$$e^{I \hat{\mathbf{a}}_1 \theta_1} e^{I \hat{\mathbf{a}}_2 \theta_2} = e^{I \hat{\mathbf{b}} \theta}, \tag{22}$$

where

$$\theta = \cos^{-1} \left(\cos\theta_1 \cos\theta_2 - \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2 \sin\theta_1 \sin\theta_2 \right),$$

and

$$\hat{\mathbf{b}} = \frac{\hat{\mathbf{a}}_1 \sin\theta_1 \cos\theta_2 + \hat{\mathbf{a}}_2 \cos\theta_1 \sin\theta_2 - \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2 \sin\theta_1 \sin\theta_2}{(\cos\theta_1 \cos\theta_2 - \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2 \sin\theta_1 \sin\theta_2) \tan\theta_3}. \tag{23}$$

6 Minkowski spacetime

There is a direct and very beautiful relationship between the geometric algebra $\mathbb{G}_3 = \mathbb{R}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ of *any* observer $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and the *Dirac algebra*

$$\mathbb{G}_{1,3} = \mathbb{R}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$$

of *spacetime*, also known as the *spacetime algebra* of *Minkowski spacetime* [8]. The Minkowski spacetime vectors γ_μ obey the rules

$$\gamma_0^2 = 1,\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1$$

and are mutually anticommuting, $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ for $\mu \neq \nu$ over the values $\mu,\nu \in \{0,1,2,3\}$.

Noting that a vector $\mathbf{v} \in \mathbb{G}_3^1$ in the space of an observer, with the ticking of the local clock, is sweeping out a *bivector* in spacetime, we *factor* the space vector $\mathbf{v} \in \mathbb{G}_3^1$ into the spacetime bivector $\mathbf{v} = v \wedge \gamma_0 \in \mathbb{G}_{1,3}^2$ in Minkowski spacetime. Each local inertial system corresponds to a unique timelike vector γ_0. Thus the local inertial system $\{\mathbf{e}_k = \gamma_k \wedge \gamma_0\}_{k=1}^3$ is related to the local inertial system $\{\mathbf{e}'_k = \gamma'_k \wedge \gamma'_0\}_{k=1}^3$ by the mapping (19), so that

$$\gamma'_\mu = e^{\frac{1}{2}\phi\hat{\mathbf{v}}}\gamma_\mu e^{-\frac{1}{2}\phi\hat{\mathbf{v}}}, \quad \text{for} \quad \mu \in \{0,1,2,3\}.$$

In summary,

$$\mathbb{G}_3 = \mathbb{R}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = \mathbb{R}(\gamma_1\gamma_0,\gamma_2\gamma_0,\gamma_3\gamma_0) = \mathbb{G}_{1,3}^+$$

where $\mathbb{G}_{1,3}^+$ denotes the even sub-algebra of the spacetime algebra $\mathbb{G}_{1,3}$.

7 Hopf Fibration

The Hopf fibration has generated considerable interest in the mathematics and physics communities. A nice treatment using quaternions can be found in [10]. Let us see how it can be expressed in the even more powerful geometric algebra $\mathbb{G}_3$ whose even sub-algebra $\mathbb{G}_3^+$ is isomorphic to the quaternions, as has been previously noted.

The unit sphere S^3 can be defined by

$$S^3 = \{\hat{R} \in \mathbb{G}_3^{0+2} |\ \hat{R} = r_0 + I\mathbf{r}, \text{ and } \hat{R}\hat{R}^\dagger = 1\},$$

for $r_0 \in \mathbb{R}$, $\mathbf{r} = \sum_{k=1}^3 r_k\mathbf{e}_k \in \mathbb{R}^3$, and $I = \mathbf{e}_{123}$. For $\mathbf{e}_3 \in \mathbb{G}_3^1$, a mapping $S^3 \to S^2$ is defined by

$$S^2 = \{\hat{R}\mathbf{e}_3\hat{R}^\dagger |\ \hat{R} \in S^3\},$$

which is just the statement that any unit vector $\hat{\mathbf{a}} \in S^2$ can be obtained by a rotation of the vector $\mathbf{e}_3$. Now note that the set of all points $\hat{S} \in S^3$ which map to $\hat{R}\mathbf{e}_3\hat{R}^\dagger$, given by $\hat{S} = \hat{R}e^{I\mathbf{e}_3\theta}$, for $0 \le \theta \le \pi$, is a circle in S^3, since clearly $\hat{S}\hat{S}^\dagger = 1$, and

$$\hat{S}\mathbf{e}_3\hat{S}^\dagger = (-\hat{S})\mathbf{e}_3(-\hat{S}^\dagger) = \hat{R}e^{i\mathbf{e}_3\theta}\mathbf{e}_3e^{-I\mathbf{e}_3\theta}\hat{R}^\dagger = \hat{R}\mathbf{e}_3\hat{R}^\dagger.$$

We have seen in (7) that

$$\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = -I\hat{\mathbf{m}}\mathbf{e}_3I\hat{\mathbf{m}} = (\hat{\mathbf{m}}\mathbf{e}_3)\mathbf{e}_3(\mathbf{e}_3\hat{\mathbf{m}}),$$

is a way of expressing *stereographic projection* from the South Pole at $-\mathbf{e}_3$ to the point $\mathbf{x} \in \mathbb{R}^2$ for $\mathbf{m} = \mathbf{x} + \mathbf{e}_3$, where $\hat{\mathbf{a}} \in S^2$ is on the ray connecting $-\mathbf{e}_3$ and $\mathbf{x}$. The points $\mathbf{m}$ and $\hat{\mathbf{a}}$ are related by the equation

$$\mathbf{m} = \frac{2}{\hat{\mathbf{a}} + \mathbf{e}_3}, \tag{24}$$

for all points $\hat{\mathbf{a}} \neq -\mathbf{e}_3$.

The unit sphere S^3 is also defined in $\mathbb{G}_4 = \mathbb{R}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, by

$$S^3 = \{\mathbf{s} = \sum_{\mu=0}^{3} s_\mu \mathbf{e}_\mu | \;\; \mathbf{s} \in \mathbb{G}^1_{4,0} \;\text{ and }\; \mathbf{s}^2 = 1\},$$

and (24) applied to $S^3 \to \mathbb{R}^3$ gives

$$\mathbf{M} = \frac{2}{\hat{\mathbf{s}} + \mathbf{e}_0} = \mathbf{x} + \mathbf{e}_0 \quad \Longleftrightarrow \quad \hat{\mathbf{s}} = \hat{\mathbf{M}}\mathbf{e}_0\hat{\mathbf{M}},$$

where $\hat{\mathbf{s}} \in S^3$, and

$$\mathbf{x} = (\mathbf{M} \wedge \mathbf{e}_0) \cdot \mathbf{e}_0 = \frac{(\hat{\mathbf{s}} \wedge \mathbf{e}_0) \cdot \mathbf{e}_0}{1 + \mathbf{e}_0 \cdot \hat{\mathbf{s}}} \in \mathbb{R}^3 \tag{25}$$

is the stereographic projection of $\mathbf{M} \in S^3$ from the south pole at $-\mathbf{e}_0 \in S^3$ into $\mathbf{x} \in \mathbb{R}_3$.

Let $\hat{\mathbf{r}} \in S^2$, $\hat{\mathbf{r}} \neq -\mathbf{e}_3$. Then

$$\hat{R} = \sqrt{\hat{\mathbf{r}}\mathbf{e}_3} = \frac{1 + \hat{\mathbf{r}}\mathbf{e}_3}{\sqrt{2(1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)}} \in S^3$$

satisfies the property that

$$\hat{R}\mathbf{e}_3\hat{R}^\dagger = \hat{R}^2\mathbf{e}_3 = \hat{\mathbf{r}}\mathbf{e}_3\mathbf{e}_3 = \hat{\mathbf{r}}, \tag{26}$$

and the most general element $\pm\hat{S} \in S^3$ which has this property is the element

$$\hat{S} = \hat{R}e^{I\mathbf{e}_3\theta} = \frac{1 + \hat{\mathbf{r}}\mathbf{e}_3}{\sqrt{2(1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)}}(\cos\theta + I\mathbf{e}_3\sin\theta)$$

$$= \frac{(1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)\cos\theta + I\big(\hat{\mathbf{r}} \times \mathbf{e}_3\cos\theta + (\hat{\mathbf{r}} + \mathbf{e}_3)\sin\theta\big)}{\sqrt{2(1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)}}, \tag{27}$$

for $0 \leq \theta < \pi$. If we now apply the stereographic projection (25) to $\hat{\mathbf{s}} = \hat{S} \in S^3$, we obtain

$$\mathbf{x} = \frac{\hat{\mathbf{r}} \times \mathbf{e}_3\cos\theta + (\hat{\mathbf{r}} + \mathbf{e}_3)\sin\theta}{\sqrt{2(1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)} + (1 + \hat{\mathbf{r}} \cdot \mathbf{e}_3)\cos\theta}. \tag{28}$$

In defining the mappings (26) and (27), we excluded the point $\hat{\mathbf{r}} = -\mathbf{e}_3$. In this special case, we define

$$\hat{R} = \sqrt{(-\mathbf{e}_3)\mathbf{e}_3} = \sqrt{-1} := I\mathbf{e}_1,$$

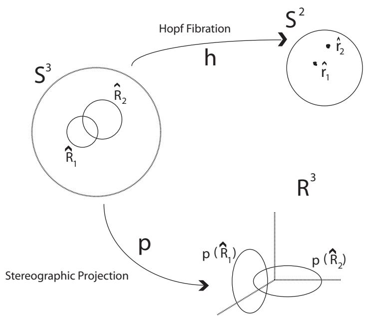

Figure 5: The Hopf fibers are linked circles on the S^3, each circle corresponding to a single point in S^2 under the Hopf mapping $h(S^3) \to S^2$. Under stereographic projection $p(S^3) \to \mathbb{R}^3$, the linked circles in S^3 are projected to linked circles in $\mathbb{R}^3$.

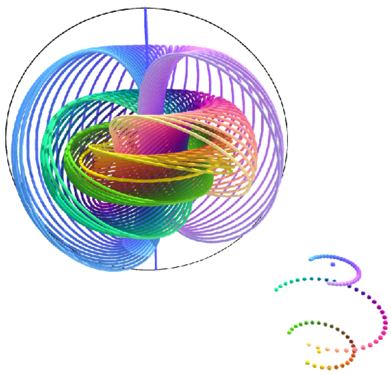

Figure 6: Pictured is the Hopf fibration in R^3, which is the stereographic projection from S^3. Each colored point on the Riemann sphere, shown in the lower right, corresponds to a circle (or straight line) of the same color in $\mathbb{R}^3$. In particular, the blue point at the north pole, corresponds to the blue vertical line (a circle of infinite diameter), and the yellow point at the south pole, corresponds to the yellow circle at the equator.

so that

$$\hat{S} = I\mathbf{e}_1 e^{I\mathbf{e}_3\theta} = I(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta). \tag{29}$$

Clearly, $\hat{S}\mathbf{e}_3\hat{S}^\dagger = -\mathbf{e}_3$ for each $0 \le \theta < \pi$. As a stereographic projection, the mapping (28) takes circles into circles, except for $\hat{\mathbf{r}} = \mathbf{e}_3$ when the unit circle (29) in S^3 projects to the line $\mathbf{x} = t\mathbf{e}_3$ through the north pole of S^2. Now note that any unit circle (27), which projects to (28), can be mapped to the unit circle (29) simply by multiplying (27) on the left by $I\mathbf{e}_1\hat{R}^\dagger$, giving

$$I\mathbf{e}_1\hat{R}^\dagger(\hat{R}e^{I\mathbf{e}_3\theta}) = I\mathbf{e}_1 e^{I\mathbf{e}_3\theta}.$$

The circle in (28) crosses the $\mathbf{e}_{12}$ plane when $\theta = 0, \pi$, or when

$$\mathbf{x}^2 = \frac{1-r_3}{r_3+3\pm 2\sqrt{2(1+r_3)}}.$$

A closer analysis shows that the circle defined by (28) is *linked* to the unit circle in the xy-plane, because when $\theta = 0$, $\mathbf{x}^2 < 1$ and when $\theta = \pi$, $\mathbf{x}^2 > 1$. This follows from the fact that for $-1 < r_3 < 1$,

$$\frac{1-r_3}{r_3+3+2\sqrt{2(1+r_3)}} = \frac{r_3+3-2\sqrt{2(1+r_3)}}{1-r_3}.$$

As a result of these observations, it follows that any two circles on the hypersphere S^3 are linked because the mapping that takes one of them to the unit circle when stereographically projected into $\mathbb{R}^3$, also takes the other one to a circle which when stereographically projected into $\mathbb{R}^3$ is linked to the unit circle in $\mathbb{R}^2$. An overview of the mappings which make up the Hopf fibration is given in Figure 5, and the Hopf fibration in $\mathbb{R}^3$ is pictured in Figure 6.

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