Geometry of Spin $\frac{1}{2}$ Particles

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Abstract

The geometric algebras of space and spacetime are derived by sucessively extending the real number system to include new mutually anticommuting square roots of ± 1. The quantum mechanics of spin 1/2 particles are then expressed in these geometric algebras. Classical 2 and 4 component spinors are represented by geometric numbers which have parity, providing new insight into the familiar bra-ket formalism of Dirac. The classical Dirac Equation is shown to be equivalent to the Dirac-Hestenes equation, so long as the issue of parity is not taken into consideration, the latter quantity being constructed in such a way that it is parity invarient.

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1 Geometric Concept of Number

In his book "Number: The Language of Science" Tobias Dantzig, in describing the invention of matrices has this to say:

> (It is) a theory in which a whole array of elements is regarded as a number-individual. These "filing cabinets" are added and multiplied, and a whole calculus of matrices has been established which may be regarded as a continuation of the algebra of complex numbers. These abstract beings have lately found a remarkable interpretation in the quantum theory of the atom, and in man's other scientific fields, [2].

We now show how these "filing cabinets" take on the interpretation of geometric numbers in a geometric number system called *geometric algebra*. It was William Kingdon Clifford (1845-1879), himself, who first referred to his famous algebras as *geometric algebras*, but some authors still call these algebras *Clifford algebras* [3]. David

Hestenes and other theoretical physicists over the last half-century have shown this comprehensive geometric interpretation imbues the equations of physics with new meaning [4]. We shall restrict our attention to developing geometric algebras of the plane, 3-dimensional space, and Minkowski spacetime and their application to quantum mechanics, but the ideas apply to higher dimensional spaces as well [5].

1.1 Geometric algebra of the plane

We extend the real number system $\mathbb{R}$ to include two new anticommuting square roots $\mathbf{e}_1, \mathbf{e}_2$ of $+1$, which we identify as unit vectors along the x and y axis of the coordinate plane $\mathbb{R}^2$. Thus,

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \quad \text{and} \quad \mathbf{e}_{12} := \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1.$$

We assume that the associative and distributive laws of multiplication of real numbers remain valid in our geometrically extended number system, and give the new quantity $i := \mathbf{e}_{12}$ the geometric interpretation of a *directed plane segment* or *bivector*. Note that

$$i^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1$$

so that i has the same algebraic property as the imaginary unit of the complex numbers $\mathbb{C}$. Indeed, every unit bivector of the n-dimensional Euclidean space $\mathbb{R}^n$ shares this property and is the generator of rotations in the vector plane of that bivector.

The Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for $\theta \in \mathbb{R}$ and the unit bivector $i = \mathbf{e}_{12}$, and the *hyperbolic Euler identity*

$$e^{\mathbf{e}_1\phi} = \cosh\theta + \mathbf{e}_1\sinh\theta$$

for $\phi \in \mathbb{R}$ and the unit vector $\mathbf{e}_1$, are easily established as special cases of the general algebraic definition of the exponential function

$$e^X \equiv \sum_{n=0}^{\infty}\frac{X^n}{n!} = \cosh X + \sinh X.$$

In [5, Chp. 2] and [6], the author shows that the hyperbolic number plane, regarded as the extension of the real number system to include a **new** sqrare root $\mathbf{e}_1 := \sqrt{+1}$ of $+1$, has much in common with the more famous complex number plane. Since it took centuries for the complex numbers to be recognised as bonified numbers, I suppose that it is not surprising that the sister hyperbolic numbers are discriminated against even to this day. After all, it was Leopold Kronecker (1823-1891) who said, “God made the integers, all the rest is the work of man”.

The *standard basis* over the real numbers of the geometric algebra $\mathbb{G}_2 := \mathbb{G}(\mathbb{R}^2)$ is

$$\mathbb{G}_2 = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}. \tag{1}$$

With respect to this basis a geometric number $g \in \mathbb{G}_2$ can be written

$$g = \alpha + \mathbf{v} = (a + bi) + (c\mathbf{e}_1 + d\mathbf{e}_2) \tag{2}$$

where $\alpha = a + bi = re^{i\theta}$ behaves formally like a complex number and $\mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2 \in \mathbb{R}^2$ is a vector in the Euclidean plane of the bivector $i = \mathbf{e}_{12}$. It can also be easily checked that $\mathbf{v}\alpha = \overline{\alpha}\mathbf{v}$, where $\overline{\alpha} = a - \beta i$ is the (complex) *conjugate* of α. It is this property that gives α the geometric significance as the generators of rotations in the plane of $\mathbf{e}_{12}$, [5, p.55].

Another basis of $\mathbb{G}_2$, the *spectral basis*, gives the algebra of real 2×2 matrices the interpretation of representing geometric numbers in $\mathbb{G}_2$. We write

$$\mathbb{G}_2 = span\{\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix}\} = span\{\begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}\}, \tag{3}$$

or $\mathbb{G}_2 = span\{u_+, \mathbf{e}_1 u_+, \mathbf{e}_1 u_-, u_-\}$, where the idempotents $u_\pm := \frac{1}{2}(1 \pm \mathbf{e}_2)$. The relationship between the standard basis (1) and the spectral basis (3) is directly expressed by

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_+ \\ \mathbf{e}_1 u_+ \\ \mathbf{e}_1 u_- \\ u_- \end{pmatrix}.$$

Noting $\mathbf{e}_1 u_+ = u_- \mathbf{e}_1$, it follows that

$$\begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = u_+ + \mathbf{e}_1 u_+ \mathbf{e}_1 = u_+ + u_- = 1.$$

Using this last relationship, and the fact that

$$u_+^2 = u_+ = \mathbf{e}_2 u_+, \quad \text{and} \quad u_+ \mathbf{e}_1 u_+ = 0 = u_+ \mathbf{e}_{12} u_+,$$

with similar relationships for $u_- g u_+$, $u_+ g u_-$ and $u_- g u_-$, we calculate for the geometric number g given in (2),

$$\begin{aligned} g &= \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} g \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} \alpha + \mathbf{v} & (\alpha + \mathbf{v})\mathbf{e}_1 \\ \mathbf{e}_1(\alpha + \mathbf{v}) & \mathbf{e}_1(\alpha + \mathbf{v})\mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} a + d & c - b \\ c + b & a - d \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \qquad (4) \\ &= (a + d)u_+ + (c - b)\mathbf{e}_1 u_- + (c + b)\mathbf{e}_1 u_+ + (a - d)u_-. \qquad (5) \end{aligned}$$

We say that $[g] = \begin{pmatrix} a + d & c - b \\ c + b & a - d \end{pmatrix}$ is the matrix of the geometric number

$$g = (a + b\mathbf{e}_{12}) + (c\mathbf{e}_1 + d\mathbf{e}_2) \in \mathbb{G}_2$$

with respect to the spectral basis (3). From (4) it follows that

$$g = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}. \tag{6}$$

The interesting thing about equation (6) is that it can be turned around and solved directly for for the matrix of $[g]$ of g. To accomplish this we define the $\mathbf{e}_1$-conjugate of $g \in \mathbb{G}_2$ by

$$g^{\mathbf{e}_1} := \mathbf{e}_1 g \mathbf{e}_1.$$

Multiplying equation (6) on the left and right by $u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+$, respectively, gives

$$u_+ \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} g \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ = u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \\ \mathbf{e}_1 & 1 \end{pmatrix} u_+ [g] \begin{pmatrix} 1 & \mathbf{e}_1 \\ \mathbf{e}_1 & 1 \end{pmatrix} u_+ = u_+ [g].$$

By taking the $\mathbf{e}_1$-conjugate of the last equation, we get

$$u_- \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} g^{\mathbf{e}_1} \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_- = u_- [g].$$

Adding the last two equations gives the desired result

$$[g] = u_+ \begin{pmatrix} g & g\mathbf{e}_1 \\ \mathbf{e}_1 g & g^{\mathbf{e}_1} \end{pmatrix} u_+ + u_- \begin{pmatrix} g^{\mathbf{e}_1} & \mathbf{e}_1 g \\ g\mathbf{e}_1 & g \end{pmatrix} u_-. \tag{7}$$

Let $g_k = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g_k] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}$ where $[g_k]$ is the matrix of the geometric number $g_k \in \mathbb{G}_2$ for $k = 1, 2$ as given in (2), (4) and (6). The algebra of real 2×2 matrices $Mat_{\mathbb{R}}(2)$ is algebraically isomorphic to the geometric algebra $\mathbb{G}_2$ because $[g_1 + g_2] = [g_1] + [g_2]$ and $[g_1 g_2] = [g_1][g_2]$ for all 2×2 matrices $[g_1], [g_2] \in Mat_{\mathbb{R}}(2)$ and their corresponding geometric numbers $g_1, g_2 \in \mathbb{G}_2$. For example, using (6) and $u_+ \mathbf{e}_1 u_+ = 0$,

$$g_1 g_2 = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g_1] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g_2] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} [g_1] \begin{pmatrix} u_+ & u_+ \mathbf{e}_1 u_+ \\ u_+ \mathbf{e}_1 u_+ & u_+ \end{pmatrix} [g_2] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g_1][g_2] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [g_1 g_2] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix},$$

which shows that $[g_1 g_2] = [g_1][g_2]$.

Extending the real number system $\mathbb{R}$ to include the new anticommuting square roots $\mathbf{e}_1, \mathbf{e}_2$ of $+1$ is well defined since the resulting geometric number system $\mathbb{G}_2$ is algebraically isomorphic to the 2×2 matrix algebra $Mat_{\mathbb{R}}(2)$. Indeed, each geometric algebra $\mathbb{G}_{p,q}$ is either algebraically isomorphic to a real or complex square matrix algebra, or matrix subalgebra, of the appropriate dimension, [5, p.205]. Next, we consider the case of the geometric algebra $\mathbb{G}_3$ of space.

1.2 Geometric algebra of space

The geometric algebra $\mathbb{G}_2$ of the *xy*-plane in $\mathbb{R}^2$ can be extended to the geometric algebra $\mathbb{G}_3$ of the space $\mathbb{R}^3$ simply by extending $\mathbb{G}_2$ to include $\mathbf{e}_3$, a new square root of $+1$, which *anticommutes* with both $\mathbf{e}_1$ and $\mathbf{e}_2$. Naturally, $\mathbf{e}_3$ has the geometric interpretation of a unit vector along the *z*-axis. The *standard basis* of $\mathbb{G}_3$, with respect to the *coordinate frame* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Euclidean space $\mathbb{R}^3$, is

$$\mathbb{G}_3 := \mathbb{G}(\mathbb{R}^3) = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}. \tag{8}$$

The elements $\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}$ are *unit bivectors* in the *xy*-, *xz*-, and *yz*-planes and generate rotations in those planes, and $i := \mathbf{e}_{123} := \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the *unit trivector* or *oriented volume element* of $\mathbb{R}^3$. The unit trivector i also has the property that $i^2 = -1$, as follows from

$$i^2 = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_3)\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1^2(\mathbf{e}_2\mathbf{e}_3)^2 = -1.$$

Any geometric number $g \in \mathbb{G}_3$ can be written in the form

$$g = \sum_{k=0}^{3} \alpha_k \mathbf{e}_k \tag{9}$$

where $\mathbf{e}_0 := 1$ and $\alpha_k = a_k + ib_k$ for $a_k, b_k \in \mathbb{R}$ and where $0 \leq k \leq 3$. The conjugation known as the *reverse* $g^\dagger$ of the geometric number g, is defined by reversing the orders of all the products of the vectors that make up g, giving

$$g^\dagger = \sum_{k=0}^{3} \overline{\alpha}_k \mathbf{e}_k. \tag{10}$$

In particular, writing $g = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$, the sum of a real number $s \in \mathbb{G}_3^0$, a vector $\mathbf{v} \in \mathbb{G}_3^1$, a bivector $\mathbf{B} \in \mathbb{G}_3^2$ and a trivector $\mathbf{T} \in \mathbb{G}_3^3$, $g^\dagger = s + \mathbf{v} - \mathbf{B} - \mathbf{T}$ as can be easily checked.

Another conjugation widely used in geometric algebra is the *grade inversion*, obtained by replacing each vector in a product by its negative. It corresponds to an inversion in the origin, otherwise known as a *parity inversion*. For the geometric number g given in (9), the grade inversion is

$$g^- := \overline{\alpha}_0 - \sum_{k=1}^{3} \overline{\alpha}_k \mathbf{e}_k. \tag{11}$$

When g is written in the form $g = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$, the grade inversion takes the form $g^- = s - \mathbf{v} + \mathbf{B} - \mathbf{T}$.

Alternatively, we can obtain the geometric algebra $\mathbb{G}_3$ by extending or *complexifying* $\mathbb{G}_2$ to include a commuting imaginary i, which we subsequently identify as the unit trivector. We write

$$\mathbb{G}_3 = \mathbb{G}_2(i) := span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{21}i, \mathbf{e}_{12}, -\mathbf{e}_2 i, \mathbf{e}_1 i, i\}, \tag{12}$$

and then define $\mathbf{e}_3 := \mathbf{e}_{21}i, \mathbf{e}_{13} = -\mathbf{e}_2 i$ and $\mathbf{e}_{23} = \mathbf{e}_1 i$.

Since the unit trivector i commutes with all the elements of $\mathbb{G}_3$, we can simply complexify the spectral basis (3) of $\mathbb{G}_2$ to obtain the spectral basis of $\mathbb{G}_3$,

$$\mathbb{G}_3 = span\{\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix}\} = span\{\begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}\}, \tag{13}$$

where in this case $u_\pm := \frac{1}{2}(1 \pm \mathbf{e}_3)$. We can then directly apply the analogous formulas (4),(5),(6), and (7) to elements $g \in \mathbb{G}_3$ merely by allowing the values of a,b,c,d to be of the form $s+it$ where $s,t \in \mathbb{R}$ and $i = \mathbf{e}_{123}$. The famous *Pauli matrices* of the coordinate frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are simply obtained using the spectral basis (13), with $u_\pm = \frac{1}{2}(1 \pm \mathbf{e}_3)$, getting

$$[\mathbf{e}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ [\mathbf{e}_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ [\mathbf{e}_3] = -i[\mathbf{e}_1][\mathbf{e}_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

In this representation, the imaginary unit i acquires the geometric interpretation of the oriented volume element in $\mathbb{G}_3$, as follows from

$$[\mathbf{e}_{123}] = [\mathbf{e}_1][\mathbf{e}_2][\mathbf{e}_3] = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

In terms of the geometric product, all other products in the geometric algebra are defined. For example, given vectors $\mathbf{a}, \mathbf{b} \in \mathbb{G}_3^1 \equiv \mathbb{R}^3$,

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \in \mathbb{G}_3^{0+2}, \tag{16}$$

where $\mathbf{a} \cdot \mathbf{b} := \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \in \mathbb{R}$ is the symmetric *inner product* and $\mathbf{a} \wedge \mathbf{b} := \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$ is the antisymmetric *outer product* of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. The outer product satisfies $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$, expressing the duality relationship between the standard Gibbs-Heaviside cross product $\mathbf{a} \times \mathbf{b}$ and the outer product $\mathbf{a} \wedge \mathbf{b} \in \mathbb{G}_3^2$. In what follows the outer product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ of three vectors is also used. Similar to (16), we write

$$\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) + \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}), \tag{17}$$

where

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) := \frac{1}{2}\big(\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) - (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\big) = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \in \mathbb{G}_3^1,$$

and

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) := \frac{1}{2}\big(\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\big) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) i \in \mathbb{G}_3^3.$$

A much more detailed treatment of $\mathbb{G}_3$ is given in [5, Chp.3], and in [7] I explore the close relationship that exists between geometric algebras and their matrix counterparts. Geometric algebra is developed in [5] as the *geometric completion* of the real number system, providing a new foundation for much of mathematics and physics.

1.3 Properties of idempotents and nilpotents

An *idempotent* $p \in \mathbb{G}_3$ has the defining property $p^2 = p$. Of course $+1, 0$ are idempotents, and so is $p = \frac{1}{2}(1+\sqrt{2}\mathbf{e}_1 + i\,\mathbf{e}_2)$. For a general geometric number $g = \omega + \mathbf{m} + i\mathbf{n} \in \mathbb{G}_3$ to be an idempotent, we must have

$$g^2 = \omega^2 + \mathbf{m}^2 - \mathbf{n}^2 + 2i\,\mathbf{m}\cdot\mathbf{n} + 2\omega(\mathbf{m}+i\,\mathbf{n}) = \omega + \mathbf{m} + i\,\mathbf{n}, \tag{18}$$

or equivalently,

$$\omega^2 + \mathbf{m}^2 - \mathbf{n}^2 + 2i\,\mathbf{m}\cdot\mathbf{n} = \omega \quad \text{and} \quad (2\omega-1)(\mathbf{m}+i\,\mathbf{n}) = 0.$$

If $\mathbf{m}+i\,\mathbf{n} \neq 0$, it follows that $\omega = \frac{1}{2}$, $\mathbf{m}\cdot\mathbf{n} = 0$, and $\mathbf{m}^2 - \mathbf{n}^2 = \frac{1}{4}$. Equivalently, a geometric number A, not equal to 0 or 1, is an idempotent iff

$$A = \frac{1}{2}(1+\mathbf{m}+i\,\mathbf{n}), \quad \text{where} \quad \mathbf{m}^2 - \mathbf{n}^2 = 1, \quad \text{and} \quad \mathbf{m}\cdot\mathbf{n} = 0, \tag{19}$$

for $\mathbf{m}, \mathbf{n} \in \mathbb{G}_3^1$. It follows immediately that any idempotent A has the important cannonical form

$$A = \frac{1}{2}(1+\mathbf{m}+i\,\mathbf{n}) = \mathbf{m}\frac{1}{2}(1+\mathbf{m}^{-1}+i\,\mathbf{m}^{-1}\mathbf{n}) = \mathbf{m}\frac{1}{2}(1+\hat{\mathbf{a}}) = \mathbf{m}\hat{\mathbf{a}}_+, \tag{20}$$

where $\hat{\mathbf{a}} = \mathbf{m}^{-1} + i\,\mathbf{m}^{-1}\mathbf{n} \in \mathbb{G}_3^1$, $\hat{\mathbf{a}}_+ = \frac{1}{2}(1+\hat{\mathbf{a}})$ and $\hat{\mathbf{a}}\cdot\mathbf{m} = 1$. We use the notation $\hat{\mathbf{a}}_\pm = \frac{1}{2}(1\pm\hat{\mathbf{a}})$ for the mutually annihiliating idempotents $\hat{\mathbf{a}}_+$ and $\hat{\mathbf{a}}_-$.

For the idempotents $\hat{\mathbf{a}}_\pm$, note that

$$\hat{\mathbf{a}}\hat{\mathbf{a}}_+ = \hat{\mathbf{a}}_+ \quad \text{and} \quad \hat{\mathbf{a}}\hat{\mathbf{a}}_- = -\hat{\mathbf{a}}_-.$$

Because of these properties, we would like to identify $\hat{\mathbf{a}}_\pm$ as the two *pure eigenprojector spin states* of an electron. Unfortunately, as is discussed in the next section, this is not quite corret.

Given a second idempotent $\hat{\mathbf{b}}_+$, we calculate

$$2\langle\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\rangle_0 = \frac{1}{2}\langle(1+\hat{\mathbf{a}})(1+\hat{\mathbf{b}})\rangle_0 = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}), \tag{21}$$

where $\langle\ \_\ \rangle_0$ denotes real scalar part of the enclosed expression, and using $\mathbf{a}\cdot\mathbf{b} = \langle\mathbf{ab}\rangle_0$ that we found in (16). A closely related calculation gives *almost* the same answer

$$(\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+)^\dagger(\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+) = (\hat{\mathbf{b}}_+\hat{\mathbf{a}}_+)(\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+) = \hat{\mathbf{b}}_+\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+ = \frac{\hat{\mathbf{b}}_+}{4}(1+\hat{\mathbf{a}})(1+\hat{\mathbf{b}}) =$$

$$= \frac{\hat{\mathbf{b}}_+}{4}(1+\hat{\mathbf{a}}+\hat{\mathbf{b}}+\hat{\mathbf{a}}\hat{\mathbf{b}}) = \frac{\hat{\mathbf{b}}_+}{4}(1+\hat{\mathbf{a}}+\hat{\mathbf{b}}-\hat{\mathbf{b}}\hat{\mathbf{a}}+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}_+,$$

but multiplied by the idempotent $\hat{\mathbf{b}}_+$. In summary,

$$(\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+)^\dagger(\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+) = \hat{\mathbf{b}}_+\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+ = Prob(\hat{\mathbf{a}}_+|\hat{\mathbf{b}}_+)\hat{\mathbf{b}}_+, \tag{22}$$

where $Prob(\hat{\mathbf{a}}_+|\hat{\mathbf{b}}_+) := \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})$ and $(g_1g_2)^\dagger = g_2^\dagger g_1^\dagger$ for all $g_1, g_2 \in \mathbb{G}_3$ is the operation of reverse defined in (10). Another important property that easily follows is

$$\hat{\mathbf{b}}_+\mathbf{m}\hat{\mathbf{b}}_+ = \frac{1}{2}\hat{\mathbf{b}}_+\mathbf{m}(1+\hat{\mathbf{b}}) = \frac{1}{2}\hat{\mathbf{b}}_+(\mathbf{m}-\hat{\mathbf{b}}\mathbf{m}+2\hat{\mathbf{b}}\cdot\hat{\mathbf{a}}) = (\hat{\mathbf{b}}\cdot\mathbf{m})\hat{\mathbf{b}}_+. \tag{23}$$

A geometric number g is a *nilpotent* if $g^2 = 0$. A similar analysis to (18) for a general idempotent shows that a nilpotent N has the canonical form

$$N = \frac{1}{2}(\mathbf{r}+i\mathbf{s}) = \mathbf{r}\frac{1}{2}(1+i\mathbf{r}^{-1}\mathbf{s}) = \mathbf{r}\hat{\mathbf{n}}_+ = \hat{\mathbf{n}}_-\mathbf{r}, \tag{24}$$

where $\hat{\mathbf{n}} := i\mathbf{r}^{-1}\mathbf{s} \in \mathbb{G}_3^1$ for the orthogonal vectors $\mathbf{r}\cdot\mathbf{s} = 0$ and $\mathbf{r}^2 = \mathbf{s}^2$. Thus, every nilpotent N has associated with it a corresponding idempotent $\hat{\mathbf{n}}_+$.

2 Quantum mechanics of spin

The *observable* corresponding to the *spin* of an electron in a coordinate frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a *unit vector* $\hat{\mathbf{m}} = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3$ in the Euclidean space $\mathbb{R}^3$ of that frame. The *Hermitian matrix* of the observable $\hat{\mathbf{m}}$ is

$$[\hat{\mathbf{m}}] = \begin{pmatrix} m_3 & m_1 - im_2 \\ m_1 + im_2 & -m_3 \end{pmatrix} = \begin{pmatrix} m_3 & m_- \\ m_+ & -m_3 \end{pmatrix},$$

as given in (13) and (14), with $u_\pm = \frac{1}{2}(1 \pm \mathbf{e}_3)$ and $m_\pm = m_1 \pm im_2$, where

$$\hat{\mathbf{m}} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+[\hat{\mathbf{m}}]\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = m_3u_+ + m_-\mathbf{e}_1u_- + m_+\mathbf{e}_1u_+ - m_3u_-. \tag{25}$$

By a *geometric spinor* $|\alpha\rangle$ *with respect to* $u_+ = \frac{1}{2}(1+\mathbf{e}_3)$, we mean

$$|\alpha\rangle := \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+ \quad \Longleftrightarrow \quad \langle\alpha| := |\alpha\rangle^\dagger = \sqrt{2}u_+(\overline{\alpha}_0 + \overline{\alpha}_1\mathbf{e}_1) \tag{26}$$

where $\alpha_0, \alpha_1 \in \mathbb{G}_3^{0+3}$. The spinor $|\alpha\rangle$ is said to be *non-degenerate* if in addition $\overline{\alpha}_0\alpha_0 + \overline{\alpha}_1\alpha_1 \neq 0$. Spinors have the important *superposition property* that if $|\alpha\rangle$ and $|\beta\rangle$ are spinors with respect to u_+ then

$$|\omega\rangle := \lambda_1|\alpha\rangle + \lambda_2|\beta\rangle \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{G}_3^{0+3} \tag{27}$$

is also a spinor with respect to u_+. This is precisely the property that naked idempotents $\mathbf{m}_+, \mathbf{n}_+$ do not have.

Given two spinors $|\alpha\rangle$ and $|\beta\rangle$, the geometric product

$$\langle\alpha||\beta\rangle = 2u_+(\overline{\alpha}_0 + \overline{\alpha}_1\mathbf{e}_1)(\beta_0 + \beta_1\mathbf{e}_1)u_+ = 2(\overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1)u_+ \tag{28}$$

is also a spinor. The *inner product* $\langle\alpha|\beta\rangle$ is defined to be the scalar and pseudoscalar parts of the geometric product of the spinors,

$$\langle\alpha|\beta\rangle := \Big\langle \langle\alpha||\beta\rangle \Big\rangle_{0+3} = \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1. \tag{29}$$

An important observation which will be used later is that

$$\Big\langle \big(\langle\alpha||\beta\rangle\big)^\dagger \big(\langle\alpha||\beta\rangle\big)\Big\rangle_{0+3} = 2\overline{\langle\alpha|\beta\rangle}\langle\alpha|\beta\rangle, \tag{30}$$

as can be easily verified. The spinor $|\alpha\rangle$ is said to be a *normalized spinor* if

$$\langle\alpha|\alpha\rangle = \overline{\alpha}_0\alpha_0 + \overline{\alpha}_1\alpha_1 = 1. \tag{31}$$

Traditionally, a 2-component column spinor is

$$[\alpha]_2 := \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \quad \text{where} \quad \alpha_0, \alpha_1 \in \mathbb{C}. \tag{32}$$

In general, for a spinor $|\alpha\rangle := \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+$ with $\alpha_0 \neq 0$, we can write

$$|\alpha\rangle = \sqrt{2}\alpha_0\Big(1 + \frac{\alpha_1}{\alpha_0}\mathbf{e}_1\Big)u_+ = \sqrt{2}\alpha_0 A \tag{33}$$

for $A = \Big(1 + \frac{\alpha_1}{\alpha_0}\mathbf{e}_1\Big)u_+$. It is easily checked that A is an idempotent, so that A can be expressed in the cannonical form (20),

$$A = \frac{1}{2}\Big(1 + \frac{\alpha_1}{\alpha_0}\mathbf{e}_1 - i\frac{\alpha_1}{\alpha_0}\mathbf{e}_2 + \mathbf{e}_3\Big) = \frac{1}{2}(1 + \mathbf{m} + i\mathbf{n}) = \mathbf{m}\frac{1}{2}(1 + \hat{\mathbf{a}}) = \mathbf{m}u_+, \tag{34}$$

since $\hat{\mathbf{a}} = \frac{\mathbf{m}+i\mathbf{mn}}{\mathbf{m}^2} = \mathbf{e}_3$. Another closely related canonical form for the idempotent A, with the help of (23), is

$$A = \mathbf{m}u_+ = \mathbf{m}\hat{\mathbf{m}}\hat{\mathbf{m}}u_+\mathbf{m}u_+ = \mathbf{m}^2\hat{\mathbf{a}}_+u_+, \tag{35}$$

where we have redefined the unit vector $\hat{\mathbf{a}} := \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}}$.

All of the above quantities can be expressed in terms of α_0 and α_1. We find that

$$\mathbf{m} = \frac{1}{2\alpha_0\overline{\alpha}_0}\Big((\alpha_1\overline{\alpha}_0 + \alpha_0\overline{\alpha}_1)\mathbf{e}_1 + i(\alpha_0\overline{\alpha}_1 - \alpha_1\overline{\alpha}_0)\mathbf{e}_2\Big) + \mathbf{e}_3, \tag{36}$$

$$\mathbf{n} = \frac{1}{2\alpha_0\overline{\alpha}_0}\Big(i(\alpha_0\overline{\alpha}_1 - \alpha_1\overline{\alpha}_0)\mathbf{e}_1 - (\alpha_0\overline{\alpha}_1 + \alpha_1\overline{\alpha}_0)\mathbf{e}_2\Big), \tag{37}$$

and the unit vector $\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ is specified by

$$a_1 = \frac{\overline{\alpha}_0\alpha_1 + \alpha_0\overline{\alpha}_1}{\alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1},\ \ a_2 = \frac{i(\alpha_0\overline{\alpha}_1 - \alpha_1\overline{\alpha}_0)}{\alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1},\ \ a_3 = \frac{\alpha_0\overline{\alpha}_0 - \alpha_1\overline{\alpha}_1}{\alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1}. \tag{38}$$

For a normalized spinor (31),

$$\mathbf{m}^2 = \frac{2}{1 + \hat{\mathbf{a}}\cdot\mathbf{e}_3} = \frac{\alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1}{\alpha_0\overline{\alpha}_0} = \frac{1}{\alpha_0\overline{\alpha}_0}. \tag{39}$$

Using the above information (34) and (35) for the idempotent A, the spinor $|\alpha\rangle$ takes the canonical form

$$|\alpha\rangle = \sqrt{2}\alpha_0 A = \sqrt{2}\alpha_0\mathbf{m}u_+ = \sqrt{2}\alpha_0\mathbf{m}^2\hat{\mathbf{a}}_+u_+. \tag{40}$$

The matrix $[\hat{\mathbf{a}}_+]$ of $\hat{\mathbf{a}}_+$ is the *density matrix* of the pure state $|\alpha\rangle$. By taking the reverse of the spinor equation (40), we get

$$\langle\alpha| = \sqrt{2}\overline{\alpha}_0 u_+ \mathbf{m} = \sqrt{2}\overline{\alpha}_0 \mathbf{m}^2 u_+ \hat{\mathbf{a}}_+, \tag{41}$$

and find that

$$\frac{1}{2}|\alpha\rangle\langle\alpha| = \alpha_0\overline{\alpha}_0\mathbf{m}^2\hat{\mathbf{m}}u_+\hat{\mathbf{m}} = \hat{\mathbf{a}}_+. \tag{42}$$

It is interesting to note that a *Cartan spinor* is specifed by

$$|\alpha\rangle\mathbf{e}_1|\alpha\rangle^* = |\alpha\rangle\mathbf{e}_1\langle\alpha|^- = (\alpha_0^2-\alpha_1^2)\mathbf{e}_1 + i(\alpha_0^2+\alpha_1^2)\mathbf{e}_2 - 2\alpha_0\alpha_1\mathbf{e}_3$$
$$= -\frac{\alpha_0}{\overline{\alpha}_0}\hat{\mathbf{m}}(\mathbf{e}_1+i\mathbf{e}_2)\hat{\mathbf{m}} = -\frac{2}{\overline{\alpha}_0^2}\hat{\mathbf{a}}_+\hat{\mathbf{m}}\mathbf{e}_1\hat{\mathbf{m}}$$

and represents a null complex vector, [8, p.41].

There are several other operations on spinors which are important. By multiplying the spinor equation (40) on the left and right by $\mathbf{e}_3$, we find that

$$|\alpha\rangle^{\mathbf{e}_3} := \mathbf{e}_3|\alpha\rangle\mathbf{e}_3 = \sqrt{2}\alpha_0\mathbf{e}_3\mathbf{m}\mathbf{e}_3u_+ = \sqrt{2}\alpha_0\mathbf{m}^2\mathbf{e}_3\hat{\mathbf{a}}_+\mathbf{e}_3u_+. \tag{43}$$

Multiplying the spinor equation (40) on the left and right by $\mathbf{e}_1$, we get

$$|\alpha\rangle^{\mathbf{e}_1} := \mathbf{e}_1|\alpha\rangle\mathbf{e}_1 = \sqrt{2}\alpha_0\mathbf{e}_1\mathbf{m}\mathbf{e}_1u_- = \sqrt{2}\alpha_0\mathbf{m}^2\mathbf{e}_1\hat{\mathbf{a}}_+\mathbf{e}_1u_-. \tag{44}$$

The spinor $|\alpha\rangle^{\mathbf{e}_3}$ is called the $\mathbf{e}_3$-*conjugate* of $|\alpha\rangle$, and the spinor $|\alpha\rangle^{\mathbf{e}_1}$ is called the $\mathbf{e}_1$-*conjugate* of $|\alpha\rangle$. Using (11) and (40) the *grade inversion* $|\alpha\rangle^-$ of the spinor $|\alpha\rangle\in\mathbb{G}_3$ is

$$|\alpha\rangle^- := \sqrt{2}\Big((\alpha_0+\alpha_1\mathbf{e}_1)u_+\Big)^- = \sqrt{2}(\overline{\alpha}_0-\overline{\alpha}_1\mathbf{e}_1)u_- = -\sqrt{2}\overline{\alpha}_0\mathbf{m}u_- = \sqrt{2}\overline{\alpha}_0\mathbf{m}^2\hat{\mathbf{a}}_-u_-. \tag{45}$$

Given normalized spinors $|\alpha\rangle, |\beta\rangle$, we calculate with the help of (40) and (23),

$$\langle\alpha|\beta\rangle = 2\overline{\alpha}\beta_0u_+\mathbf{m}\mathbf{n}u_+ = 2\overline{\alpha}_0\beta_0\Big(\mathbf{m}\cdot\mathbf{n}+i(\mathbf{m}\times\mathbf{n})\cdot\mathbf{e}_3\Big)u_+ = 0$$

iff $\mathbf{m}\cdot\mathbf{n}=0$ and $(\mathbf{m}\times\mathbf{n})\cdot\mathbf{e}_3=0$. Expressed in terms of the associated spinor directions $\hat{\mathbf{m}},\hat{\mathbf{n}}$, the inner product $\langle\alpha|\beta\rangle$ is quite different than the component expression (29). A great simplification occurs when we calculate $\overline{\langle\alpha|\beta\rangle}\langle\alpha|\beta\rangle$, finding with the help of (30) and (39),

$$\overline{\langle\alpha|\beta\rangle}\langle\alpha|\beta\rangle = \frac{1}{2}\Big\langle\big(\langle\alpha||\beta\rangle\big)^\dagger\big(\langle\alpha||\beta\rangle\big)\Big\rangle_{0+3} = \frac{1}{2}\Big\langle\langle\beta|\big(|\alpha\rangle\langle\alpha|\big)|\beta\rangle\Big\rangle_{0+3}$$
$$= \Big\langle\langle\beta|\hat{\mathbf{a}}_+|\beta\rangle\Big\rangle_{0+3} = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}).$$

Given an observable $\hat{\mathbf{a}}$, suppose that an electron has been prepared in the pure spin state $|\alpha\rangle$. Then the probability of finding the electron in the pure spin state $|\beta\rangle$ is

$$Prob(|\alpha\rangle,|\beta\rangle) := \overline{\langle\alpha|\beta\rangle}\langle\alpha|\beta\rangle = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) = 2\langle\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\rangle_0, \tag{46}$$

just as we have already seen in (21) and (22). Immediately after the measurment is carried out, the electron will be in the pure state $|\beta\rangle$. Identity (46) is independent of the basis spinor u_+ that was used in the definition (26) of the spinors $|\alpha\rangle$ and $|\beta\rangle$. Clearly, the spinor states $|\alpha\rangle$ and $|\beta\rangle$ are *orthogonal* iff $\hat{\mathbf{a}} = -\hat{\mathbf{b}}$.

Given a spinor state $|\alpha\rangle$, the *average value* of the observable $\hat{\mathbf{b}}$ in that state is defined by $\langle\alpha|\hat{\mathbf{b}}|\alpha\rangle$. Using (22), (23), (39), (40), we calculate

$$\hat{\mathbf{b}}|\alpha\rangle = \sqrt{2}\alpha_0\hat{\mathbf{b}}\mathbf{m}u_+ = \sqrt{2}\alpha_0(\hat{\mathbf{b}}\cdot\mathbf{m} + \hat{\mathbf{b}}\wedge\mathbf{m})u_+, \tag{47}$$

and

$$\langle\alpha|\hat{\mathbf{b}}|\alpha\rangle := 2\langle u_+(\hat{\mathbf{m}}\hat{\mathbf{b}}\hat{\mathbf{m}})u_+\rangle_{0+3} = (\hat{\mathbf{m}}\hat{\mathbf{b}}\hat{\mathbf{m}})\cdot\mathbf{e}_3 = \hat{\mathbf{b}}\cdot(\hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}}) = \hat{\mathbf{b}}\cdot\hat{\mathbf{a}}. \tag{48}$$

It follows that the average value lies in between the values ± 1. It is important to note that the final values in both (46) and (48) in no way depend upon the idempotent u_+, so u_+ could be replaced by any other idempotent $\hat{\mathbf{c}}_+$ defined by any unit vector $\hat{\mathbf{c}}$, but we would then also have to redefine $|\alpha\rangle$.

In the special cases where

$$|\alpha\rangle = |0\rangle := \sqrt{2}u_+ \quad \text{or} \quad |\alpha\rangle = |1\rangle := \sqrt{2}\mathbf{e}_1u_+,$$

we have

$$\hat{\mathbf{b}}|0\rangle = \sqrt{2}(b_3 + b_+\mathbf{e}_1)u_+ \quad \text{and} \quad \hat{\mathbf{b}}|1\rangle = \sqrt{2}(b_- - b_3\mathbf{e}_1)u_+. \tag{49}$$

Using (48) and (49), we also calculate

$$\langle 0|\hat{\mathbf{b}}|0\rangle = b_3, \quad \text{and} \quad \langle 1|\hat{\mathbf{b}}|1\rangle = -b_3, \tag{50}$$

and

$$\langle 1|\hat{\mathbf{b}}|0\rangle = b_+ = b_1 + ib_2, \quad \text{and} \quad \langle 0|\hat{\mathbf{b}}|1\rangle = b_- = b_1 - ib_2. \tag{51}$$

Often an electron in the spinor state $|0\rangle = \sqrt{2}u_+$ is said to be *up*, and an electron in the spinor state $|1\rangle = \sqrt{2}\mathbf{e}_1u_+$ is said to be *down*. Using (50), the average value of the observable $\hat{\mathbf{b}}$ in the state $|0\rangle$ is b_3, and the average value of $\hat{\mathbf{b}}$ in the state $|1\rangle$ is $-b_3$.

We can also carry out the above calculation using spinor operators. Following [10, p.63], given the spinor $|\alpha\rangle$, the *spinor operator* ψ of $|\alpha\rangle$ is defined by

$$\psi := \frac{1}{\sqrt{2}}\Big(|\alpha\rangle + |\alpha\rangle^-\Big) = \mathbf{m}(\alpha_0u_+ - \overline{\alpha}_0u_-) \in \mathsf{SU}(2), \tag{52}$$

where the inversion $|\alpha\rangle^-$ has been defined in (45). The spinor operator is an even multivector in $\mathbb{G}_3^+$ and hence invarient under a parity swap. It retains the normalization property that $\psi\psi^\dagger = \det[\psi] = 1$, and allows us to calculate the associated direction $\hat{\mathbf{a}}$ of $|\alpha\rangle$ as a rotation of the unit vector $\mathbf{e}_3$. With the help of (22), (38) and (52), we find that

$$\psi\mathbf{e}_3\psi^\dagger = \mathbf{m}(\alpha_0u_+ - \overline{\alpha}_0u_-)\mathbf{e}_3(\overline{\alpha}_0u_+ - \alpha_0u_-)\mathbf{m}$$
$$= \mathbf{m}(\alpha_0u_+ + \overline{\alpha}_0u_-)(\overline{\alpha}_0u_+ - \alpha_0u_-)\mathbf{m} = \alpha_0\overline{\alpha}_0\mathbf{m}^2\hat{\mathbf{m}}(u_+ - u_-)\hat{\mathbf{m}} = \hat{\mathbf{a}}, \tag{53}$$

and also

$$\psi u_+\psi^\dagger = \frac{1}{2}\psi(1+\mathbf{e}_3)\psi^\dagger = \hat{\mathbf{a}}_+. \tag{54}$$

The relationship (54) should be compared with (42).

3 Geometric algebra of spacetime

The geometric algebra $\mathbb{G}_3$ of the coordinate frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ in $\mathbb{R}^3$ can be extended to the geometric algebra $\mathbb{G}_{1,3}$ of the pseudoeuclidean space $\mathbb{R}^{1,3}$ of *Minkowski spacetime*, by extending $\mathbb{G}_3$ by an additional square root γ_0 of $+1$ which *anticommutes* with each of the space vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. So γ_0 satisfies $\gamma_0^2=1$ and $\gamma_0\mathbf{e}_k=-\mathbf{e}_k\gamma_0$ for $k=1,2,3$. Equivalently, the *spacelike vectors*

$$\gamma_k := \mathbf{e}_k\gamma_0 \quad \Longleftrightarrow \quad \mathbf{e}_k=\gamma_k\gamma_0=\gamma_{k0},$$

together with the *timelike vector* γ_0, define the *spacetime algebra*

$$\mathbb{G}_{1,3}=gen\{\gamma_0,\gamma_1,\gamma_2,\gamma_3\}$$

of the *orthogonal spacetime frame* $\{\gamma_\mu|\, 0\le\mu\le 3\}$ in $\mathbb{R}^{1,3}$. In summary, the spacetime vectors obey the rules

$$\gamma_0^2=1,\ \gamma_k^2=-1,\ \gamma_\mu\gamma_\nu=-\gamma_\nu\gamma_\mu$$

for $\mu\neq\nu$, $\mu,\nu=0,1,2,3$, and $k=1,2,3$. Note also that *pseudoscalar*

$$\gamma_{0123}=\gamma_{10}\gamma_{20}\gamma_{30}=\mathbf{e}_{123}$$

of $\mathbb{G}_{1,3}$ is the same as the pseudoscalar of the *rest frame* $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of $\mathbb{G}_3$, and it anticommutes with each of the spacetime vectors γ_μ for $\mu=0,1,2,3$.

The forgoing shows that the geometric algebra $\mathbb{G}_3$ is naturally isomorphic to the *even subalgebra* $\mathbb{G}_{1,3}^{+}$ of the spacetime algebra $\mathbb{G}_{1,3}$ generated by the bivectors $\gamma_\mu\gamma_\nu$ of $\mathbb{G}_{1,3}$, [9]. Any element $g\in\mathbb{G}_{1,3}$ can be expressed in the form $g=G_1+G_2\gamma_0$ where $G_1,G_2\in\mathbb{G}_3$. In particular, a unit vector $\hat{\mathbf{m}}=m_1\mathbf{e}_1+m_2\mathbf{e}_2+m_3\mathbf{e}_3$ in the geometric algebra $\mathbb{G}_3$ of the rest frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ becomes the unit spacetime bivector

$$\hat{\mathbf{m}}=(m_1\gamma_1+m_2\gamma_2+m_3\gamma_3)\gamma_0\in\mathbb{G}_{1,3}^2,$$

with the spacelike vector $m_1\gamma_1+m_2\gamma_2+m_3\gamma_3\in\mathbb{G}_{1,3}^1$ as a factor.

In the above, we have carefully distinguished the rest frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ in the spacetime algebra $\mathbb{G}_{1,3}$. Any other rest frame $\{\mathbf{e}_1',\mathbf{e}_2',\mathbf{e}_3'\}$ can be obtained by an ordinary space rotation of the rest frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ followed by a *Lorentz boost*. In the spacetime algebra $\mathbb{G}_{1,3}$, this is equivalent to defining a new frame of spacetime vectors $\{\gamma_\mu'|\, 0\le\mu\le3\}\subset\mathbb{G}_{1,3}$, and the corresponding rest frame $\{\mathbf{e}_k'=\gamma_k'\gamma_0'|\, k=1,2,3\}$ of a Euclidean space $\mathbb{R}^{3'}$ moving with respect to the Euclidean space $\mathbb{R}^3$ defined by the rest frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. The way we introduced the geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_{1,3}$ may appear novel, but they perfectly reflect all the common relativistic concepts [5, Chp.11].

The well-known *Dirac matrices* can be obtained as a subalgebra of the 4×4 matrix algebra $Mat_{\mathbb{C}}(4)$ over the complex numbers. We first define the idempotent

$$u_{++}:=\frac{1}{4}(1+\gamma_0)(1+i\gamma_{12})=\frac{1}{4}(1+i\gamma_{12})(1+\gamma_0), \tag{55}$$

where the unit imaginary $i = \sqrt{-1}$ is assumed to commute with all elements of $\mathbb{G}_{1,3}$. Whereas it would be nice to identify this unit imaginary i with the pseudoscalar element $\gamma_{0123} = \mathbf{e}_{123}$ as we did in $\mathbb{G}_3$, this is no longer possible since γ_{0123} anticommutes with the spacetime vectors γ_μ as previously mentioned.

Noting that

$$\gamma_{12} = \gamma_1\gamma_0\gamma_0\gamma_2 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_{21}$$

and similarly $\gamma_{31} = \mathbf{e}_{13}$, it follows that

$$\mathbf{e}_{13}u_{++} = u_{+-}\mathbf{e}_{13}, \;\; \mathbf{e}_3 u_{++} = u_{-+}\mathbf{e}_3, \;\; \mathbf{e}_1 u_{++} = u_{--}\mathbf{e}_1, \tag{56}$$

where

$$u_{+-} := \frac{1}{4}(1+\gamma_0)(1-i\gamma_{12}), \; u_{-+} := \frac{1}{4}(1-\gamma_0)(1+i\gamma_{12}), \; u_{--} := \frac{1}{4}(1-\gamma_0)(1-i\gamma_{12}).$$

The idempotents $u_{++}, u_{+-}, u_{-+}, u_{--}$ are *mutually annihiliating* in the sense that the product of any two of them is zero, and *partition unity*

$$u_{++} + u_{+-} + u_{-+} + u_{--} = 1. \tag{57}$$

By the *spectral basis* of the Dirac algebra $\mathbb{G}_{1,3}$ we mean the elements of the matrix

$$\begin{pmatrix} 1 \\ \mathbf{e}_{13} \\ \mathbf{e}_3 \\ \mathbf{e}_1 \end{pmatrix} u_{++} \begin{pmatrix} 1 & -\mathbf{e}_{13} & \mathbf{e}_3 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_{++} & -\mathbf{e}_{13}u_{+-} & \mathbf{e}_3 u_{-+} & \mathbf{e}_1 u_{--} \\ \mathbf{e}_{13}u_{++} & u_{+-} & \mathbf{e}_1 u_{-+} & -\mathbf{e}_3 u_{--} \\ \mathbf{e}_3 u_{++} & \mathbf{e}_1 u_{+-} & u_{-+} & -\mathbf{e}_{13}u_{--} \\ \mathbf{e}_1 u_{++} & -\mathbf{e}_3 u_{+-} & \mathbf{e}_{13}u_{-+} & u_{--} \end{pmatrix}. \tag{58}$$

Any geometric number $g \in \mathbb{G}_{1,3}$ can be written in the form

$$g = \begin{pmatrix} 1 & \mathbf{e}_{13} & \mathbf{e}_3 & \mathbf{e}_1 \end{pmatrix} u_{++}[g] \begin{pmatrix} 1 \\ -\mathbf{e}_{13} \\ \mathbf{e}_3 \\ \mathbf{e}_1 \end{pmatrix} \tag{59}$$

where $[g]$ is the *Dirac matrix* corresponding to the geometric number g. In particular,

$$[\gamma_0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, [\gamma_1] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{60}$$

and

$$[\gamma_2] = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, [\gamma_3] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is interesting to see what the representation is of the basis vectors of $\mathbb{G}_3$. We find that for $k = 1,2,3$,

$$[\mathbf{e}_k]_4 = [\gamma_k][\gamma_0] = \begin{pmatrix} [0]_2 & [\mathbf{e}_k]_2 \\ [\mathbf{e}_k]_2 & [0]_2 \end{pmatrix} \quad \text{and} \quad [\mathbf{e}_{123}]_4 = i\begin{pmatrix} [0]_2 & [1]_2 \\ [1]_2 & [0]_2 \end{pmatrix},$$

where the outer subscripts denote the order of the matrices and, in particular, $[0]_2$, $[1]_2$ are the 2×2 zero and unit matrices, respectively. The last relationship shows that the i occuring in the Pauli matrix representation (15), which represents the oriented unit of volume, is different than the $i=\sqrt{-1}$ which occurs in the complex matrix representation (60) of the of Dirac algebra.

A *Dirac spinor* is a 4-component column matrix $[\varphi]_4$,

$$[\varphi]_4 := \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \quad \text{for} \quad \varphi_k \in \mathbb{C}. \tag{61}$$

Following [10, p.143], we use the Dirac spinor to construct the (matrix) *spinor operator*

$$[\psi] = \begin{pmatrix} \varphi_1 & -\overline{\varphi}_2 & \varphi_3 & \overline{\varphi}_4 \\ \varphi_2 & \overline{\varphi}_1 & \varphi_4 & -\overline{\varphi}_3 \\ \varphi_3 & \overline{\varphi}_4 & \varphi_1 & -\overline{\varphi}_2 \\ \varphi_4 & -\overline{\varphi}_3 & \varphi_2 & \overline{\varphi}_1 \end{pmatrix}. \tag{62}$$

Unlike the Dirac spinor $[\varphi]_4$, the spinor operator $[\psi]$ is invertible iff $\det[\psi]\neq 0$. We find that

$$\det[\psi] = r^2 + 4a^2 \geq 0, \tag{63}$$

where

$$r = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 \quad \text{and} \quad a = \Im\left(\overline{\varphi}_1\varphi_3 + \overline{\varphi}_2\varphi_4\right).$$

Where-as the spinor operator $[\psi]$ obviously contains the same information as the Dirac spinor $[\varphi]_4$, we will shortly see that it acquires the geometric interpretation of an even multivector in $\mathbb{G}_{1,3}^+$.

Noting that

$$u_{++}\gamma_{21} = \frac{1}{4}(1+\gamma_0)(\gamma_{21}+i\gamma_{12}\gamma_{21}) = \frac{1}{4}(1+\gamma_0)(i-\gamma_{12}) = iu_{++} = \gamma_{21}u_{++},$$

it follows that $u_{++}\varphi_k = u_{++}\Re(\varphi_k) + u_{++}i\gamma_{12}\Im(\varphi_k)i = u_{++}\big(\Re(\varphi_k)+\gamma_{21}\Im(\varphi_k)\big)$ and hence

$$u_{++}[\psi] = u_{++}[\psi]_\alpha,$$

where each of the elements α_k in $[\psi]_\alpha$ is defined by $\alpha_k = \varphi_k|_{i\to\gamma_{21}}$. Using (59), the geometric number $\psi\in\mathbb{G}_{1,3}^+$ corresponding to the matrix spinor operator $[\psi]$ is

$$\begin{aligned} \psi &= \begin{pmatrix} 1 & \mathbf{e}_{13} & \mathbf{e}_3 & \mathbf{e}_1 \end{pmatrix} u_{++}[\psi]_\alpha \begin{pmatrix} 1 \\ -\mathbf{e}_{13} \\ \mathbf{e}_3 \\ \mathbf{e}_1 \end{pmatrix} \\ &= \alpha_1 + \alpha_2\mathbf{e}_{13} + \alpha_3\mathbf{e}_3 + \alpha_4\mathbf{e}_1 \in \mathbb{G}_{1,3}^+. \end{aligned} \tag{64}$$

To establish the last equality above, with the help of (57), note that

$$\psi = \psi u_{++} + \psi u_{+-} + \psi u_{-+} + \psi u_{--},$$

and then succesively show that

$$\psi u_{\pm\pm} = \Big(\alpha_1 + \alpha_2 \mathbf{e}_{13} + \alpha_3 \mathbf{e}_3 + \alpha_4 \mathbf{e}_1\Big) u_{\pm\pm},$$

for each of different sign combinations $++, +-, -+, --$.

As an even geometric number in $\mathbb{G}_{1,3}^+$, ψ generates Lorentz boosts in addition to ordinary rotations in the Minkowski space $\mathbb{R}^{1,3}$. The amazing property of (64) is that where as we started by formally introducing the complex number $i = \sqrt{-1}$, in order to represent the Dirac gamma matrices, we have ended up with a spinor operator $\psi \in \mathbb{G}_{1,3}^+$ in which the role of $i = \sqrt{-1}$ is taken over by the $\gamma_{21} = \mathbf{e}_{12} \in \mathbb{G}_3$. This is the key idea in the Hestenes representation of the *Dirac equation* [18].

Although we have formally eliminated the need for the artificial $i = \sqrt{-1}$ in the Dirac spinor by replacing it with the bivector $\gamma_{21} = \mathbf{e}_{12}$ in the spinor operator, such a quantity occuring in a real geometric algebra is unacceptable. The imaginary $i = \sqrt{-1}$ has the effect of *complexifying* the geometric algebra $\mathbb{G}_{1,3}$ to the complex geometric algebra $\mathbb{G}_4(\mathbb{C}) \cong Mat_4(\mathbb{C})$, [10, p.217]. We might, instead, search for a larger real geometric algebra containing the geometric algebra $\mathbb{G}_{1,3}$, just as we enlarged the geometric algebra $\mathbb{G}_3$ of space to the geometric algebra $\mathbb{G}_{1,3}$ of spacetime. The real geometric algebra $\mathbb{G}_{2,3}$, obtained from the geometric algebra $\mathbb{G}_{1,3}$ by assuming the existence of a new timelike vector γ_0' which anticommutes with all the spacetime vectors $\gamma_\mu \in \mathbb{G}_{1,3}$, has the required properties. The *unit pseudoscalar* $i' := \gamma_0' \gamma_0 \gamma_1 \gamma_2 \gamma_3 \in \mathbb{G}_{2,3}$ has all the desired algebraic properties of the imaginary $i = \sqrt{-1}$, commuting with all the elements of $\mathbb{G}_{1,3}$ and having square -1. All of the previous arguments in the complex algebra $\mathbb{G}_4(\mathbb{C})$ remain valid in $\mathbb{G}_{2,3}$, but now i' is the oriented element of volume of the 5-dimentional pseudoeclidean space $\mathbb{R}^{2,3}$.

However, the real geometric algebras of the pseudoeuclidean spaces $\mathbb{R}^{4,1}$ and $\mathbb{R}^{0,5}$ also fit the bill, as is seen in **Table 1** in [10, p.217]. In [11, p.326], Hestenes argues that the geometric algebra $\mathbb{G}_{4,1}$ of $\mathbb{R}^{4,1}$ might be an even better choice because the even subalgebra $\mathbb{G}_{4,1}^+$ is isomorphic to $\mathbb{G}_{1,3}$ in exactly the same way that $\mathbb{G}_{1,3}^+$ is isomorphic to $\mathbb{G}_3$. In the paper, "Vector Analysis of Spinors" [12], the author explores the interpretation of a geometric Pauli spinor on the Riemann sphere, ideas which he then generalizes in another paper to geometric Dirac spinors in the complex geometric algebra $\mathbb{G}_3(\mathbb{C})$ [13]. These real and complex geometric algebras open up many new possibilities for the interpretation of relativistic quantum mechanics.

4 Tensor products of geometric numbers

Tensor products of geometric numbers, and their corresponding geometric algebras, are necessary for describing multiparticle systems used, for example, in the construction of a quantum computer, or even in the much simpler "double slit" experiments. The resulting *quantum entanglement*, which implies the "spooky action-at-a distance" that even Einstein couldn't stomach, is discussed in Section 5.

By the *tensor product* $\mathbb{G}_3 \otimes \mathbb{G}_3$ of the geometric algebra $\mathbb{G}_3$ with itself, we essentially mean two distinguishable copies of the *same* geometric algebra $\mathbb{G}_3$. We can express this as

$$\mathbb{G}_3 \otimes \mathbb{G}_3 = \{g_1 \otimes g_2 | \; g_1, g_2 \in \mathbb{G}_3\}. \tag{65}$$

In addition, the subspace consisting of everything which commutes with the entire algebra, called the *center* of $\mathbb{G}_3$, may be identified with the complex numbers

$$\mathbb{C} \equiv span\{1, i = \mathbf{e}_{123}\}.$$

The tensor products of $g_1, g_2, g_3, g_4 \in \mathbb{G}_3$ and $\alpha, \beta \in \mathbb{C}$ satisfies the rules

$$\alpha(g_1 \otimes g_2) = (\alpha g_1) \otimes g_2 = g_1 \otimes (\alpha g_2), \tag{66}$$

$$(\alpha g_1 + \beta g_2) \otimes g_3 = \alpha(g_1 \otimes g_3) + \beta(g_2 \otimes g_3), \tag{67}$$

and

$$g_1 \otimes (\alpha g_2 + \beta g_3) = \alpha(g_1 \otimes g_2) + \beta(g_1 \otimes g_3). \tag{68}$$

Either the standard basis (8) or the spectral basis (13) can be tensored with itself to get the respective *standard basis* or *spectral basis* of the tensor algebra $\mathbb{G}_3 \otimes \mathbb{G}_3$ over $\mathbb{C}$. By the *product* of tensors $g_1 \otimes g_2$ and $g_3 \otimes g_4$, we mean

$$\Big(g_1 \otimes g_2\Big)\Big(g_3 \otimes g_4\Big) = (g_1 g_3) \otimes (g_2 g_4). \tag{69}$$

The tensor product (65) satisfies the same algebraic rules (67) and (68) as the geometric product itself. This suggests extending the tensor product to commuting copies of the geometric algebra $\mathbb{G}_{1,3}$ of spacetime. In [14, 15] the authors identify the tensor product of N copies of $\mathbb{G}_3$ with commuting copies of $\mathbb{G}_{1,3}^{+}$ in the geometric algebra $\mathbb{G}_{N,3N}$ to represent the spinor operators of an N-qubit quantum computer. As explained in [14], in order to get the usual tensor products of complex matrices that is used for multi-distinguishable spin $\frac{1}{2}$ particles, one must introduce a commuting idempotent to project down to only a single set of complex numbers.

5 Non-entangled and entangled states

If $|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1)u_+$ and $|\beta\rangle = \sqrt{2}(\beta_0 + \beta_1)u_+$ are normalized states, then the *product state* is given by the *tensor product*

$$\begin{aligned} |\alpha\rangle \otimes |\beta\rangle &= \Big(\sqrt{2}(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+\Big) \otimes \Big(\sqrt{2}(\beta_0 + \beta_1 \mathbf{e}_1)u_+\Big) \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle, \end{aligned} \tag{70}$$

where we are using the *qubit* notation

$$|00\rangle := 2u_+ \otimes u_+, \quad |01\rangle := 2u_+ \otimes \mathbf{e}_1 u_+$$

and, similarly, for the definitions of $|10\rangle$ and $|11\rangle$. Taking the *reverse* of the product state $|\alpha\rangle \otimes |\beta\rangle$ gives

$$\Big(|\alpha\rangle \otimes |\beta\rangle\Big)^{\dagger} = \langle\alpha| \otimes \langle\beta|$$

and

$$\Big(\langle\alpha|\otimes\langle\beta|\Big)\Big(|\alpha\rangle\otimes|\beta\rangle\Big) := \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle.$$

The product state $|\alpha\rangle\otimes|\beta\rangle$ represents the state of two *unentangled* electrons, where each electron is prepared independently, and can be measured independently. A direct consequence of (48) is that for any values $s,t\in[-1,1]$ chosen, directions $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ can always be found resulting in the average values

$$\langle\alpha|\hat{\mathbf{m}}|\alpha\rangle = s, \quad \text{and} \quad \langle\beta|\hat{\mathbf{n}}|\beta\rangle = t.$$

The states $|S_\pm\rangle = |01\rangle \pm |10\rangle$ are *entangled* states. In an entangled state if one of the electrons is measured to be *up* along a given direction, then the other will definitely be *down* without the need for measurement. For any direction $\hat{\mathbf{m}}$, the averages of the states $|S_\pm\rangle$ measured in the directions $\hat{\mathbf{m}}\otimes 1$ and $1\otimes\hat{\mathbf{m}}$ is 0, meaning that measurements of the first and second electrons along this direction are equally likely to be ± 1. With the help of (49), we easily calculate

$$(\hat{\mathbf{m}}\otimes 1)|S_\pm\rangle = \hat{\mathbf{m}}|u\rangle\otimes|1\rangle \pm \hat{\mathbf{m}}|1\rangle\otimes|0\rangle$$

$$= \Big(m_3|0\rangle + m_+|1\rangle\Big)\otimes|1\rangle \pm \Big(-m_3|1\rangle + m_-|0\rangle\Big)\otimes|0\rangle$$

$$= \Big(m_3|01\rangle + m_+|11\rangle\Big) \pm \Big(-m_3|10\rangle + m_-|00\rangle\Big),$$

from which we find that

$$\langle S_\pm|\hat{\mathbf{m}}\otimes 1|S_\pm\rangle = \Big(\langle 01|\pm\langle 10|\Big)\Big(m_3|01\rangle + m_+|11\rangle \mp m_3|10\rangle \pm m_-|00\rangle\Big)$$

$$= m_3 - m_3 = 0,$$

and a similar calculation shows that $\langle S_\pm|1\otimes\hat{\mathbf{m}}|S_\pm\rangle = 0$. For a further insightful discussion of these results, see [1].

6 The Schrödinger, Pauli, and Dirac Equations

Spin and quantum entanglement have been studied by many authors. The Susskind Lectures [1] give a fascinating account, presenting clearly the issues that separate quantum and classical mechanics. Another elementary presentation is given in [16]. Of course, no account of quantum mechanics can be complete without at least a cursory presentation of the fundamental Schrödinger and Dirac equations, the foundation upon which all of quantum mechanics is built.

The Schrödinger equation is obtained from the classical expression for the *total energy* H of a particle at the position $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$,

$$H = \frac{1}{2}m\mathbf{v}^2 + V = \frac{1}{2m}\mathbf{p}^2 + V, \tag{71}$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the *momentum vector* of the particle and V is the potential, by way of the *substitution*

$$\mathbf{p} \to -i\hbar\nabla, \tag{72}$$

where $\nabla := \nabla_{\mathbf{x}}$ is the standard *gradient* with respect to the position $\mathbf{x} \in \mathbb{R}^3$, $\hbar = h/2\pi$ and h is Planck's constant. The Schrödinger equation thus becomes

$$i\hbar\frac{\partial\varphi}{\partial t} = \hat{H}\varphi = \Big(-\frac{\hbar}{2m}\nabla^2 + V\Big)\varphi, \tag{73}$$

for the *Hamiltonian* $\hat{H} = -\frac{\hbar}{2m}\nabla^2 + V$, where $\varphi := \varphi(\mathbf{x},t) \in \mathbb{C}$.

When the potential V is independent of time, there will be a complete set of *stationary states*

$$\varphi_n(\mathbf{x},t) = e^{-iE_n(\mathbf{x})t/\hbar}$$

where the *spatial wave function* $\varphi_n(\mathbf{x})$ satisfies the *time independent* Schrödinger equation

$$\Big(-\frac{\hbar}{2m}\nabla^2 + V\Big)\varphi_n(\mathbf{x}) = E_n\varphi_n(\mathbf{x}), \tag{74}$$

in terms of which the *time dependent* Schrödinger equation has the general solution

$$\varphi(\mathbf{x},t) = \sum_n c_n\varphi_n(\mathbf{x})e^{-iE_nt/\hbar}, \tag{75}$$

[16, p.122]. According to the Born interpretation of the wave function, the probability of finding the particle in the infinitesimal volume

$$|d^3\mathbf{x}| = dxdydz \quad \text{is} \quad |\varphi(\mathbf{x},t)|^2|d^3\mathbf{x}|.$$

The Schrödinger equation (73) is generalized to include spin in the *Schrödinger-Pauli* equation [10, p. 51]. Expressed in terms of our spinor (26) with respect to u_+,

$$i\hbar\partial_t|\alpha\rangle = \frac{1}{2mc}\Big(i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2|\alpha\rangle + eV|\alpha\rangle \tag{76}$$

for the *electric field potential* V and the *magnetic field* $\mathbf{B} = -i(\nabla\wedge\mathbf{A}) = \nabla\times\mathbf{A}$ of a particle of mass m with *electric charge* e, and where $i = \mathbf{e}_{123}$ is the unit pseudoscalar of $\mathbb{G}_3$. Expanding equation (76), we find that

$$i\hbar\partial_t|\alpha\rangle = \frac{1}{2mc}\Big(-\hbar^2\nabla^2 + \frac{e^2}{c^2}\mathbf{A}^2 + i\hbar\frac{e}{c}\big(\nabla\mathbf{A} + \mathbf{A}\nabla\big)\Big)|\alpha\rangle + eV|\alpha\rangle$$

$$= \frac{1}{2mc}\Big(-\hbar^2\nabla^2 + \frac{e^2}{c^2}\mathbf{A}^2 + i\hbar\frac{e}{c}\big(\nabla\cdot\mathbf{A} + 2\mathbf{A}\cdot\nabla\big) - \hbar\frac{e}{c}\mathbf{B}\Big)|\alpha\rangle + eV|\alpha\rangle$$

This is equivalent to the classical 2×2 matrix form of this equation,

$$i\hbar\frac{\partial[\alpha]_2}{\partial t} = \frac{1}{2mc}\Big(i\hbar[\nabla] + \frac{e}{c}[\mathbf{A}]\Big)^2[\alpha]_2 + eV[\alpha]_2, \tag{77}$$

obtained by taking the matrix of both sides of (76) and extracting the first column, where $[\mathbf{e}_k]$ are the Pauli matrices given in (14) and $[\varphi]_2$ is the 2-component spinor (32). The matrix equation (77) can be expanded in the same way that we expanded equation (76). See reference [17, eqn:(A.4)]. The reader is referred to the Appendix for more technical details.

Recall that the spinor operator (52) is $\psi = \frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\alpha\rangle^-\big)$. If $|\alpha\rangle$ satisfies the Schrödinger-Pauli equation (76), then $|\alpha\rangle^-$ will satisfy the parity inversion of this equation,

$$-i\hbar\partial_t|\alpha\rangle^- = \frac{1}{2mc}\Big(i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)^2|\alpha\rangle^- + eV|\alpha\rangle^-. \tag{78}$$

It then follows that the operator spinor ψ will satisfy the equation obtained by taking $\frac{1}{\sqrt{2}}$ times the sum of equations (76) and (78), giving the parity invarient Schrödinger-Pauli-Hestenes equation

$$i\hbar\partial_t\psi\mathbf{e}_3 = \frac{1}{2mc}\Big(i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2\psi u_+ + \frac{1}{2mc}\Big(i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)^2\psi u_- + eV\psi$$

$$= \frac{1}{2mc}\Big(-\hbar^2\nabla^2 + \frac{e^2}{c^2}\mathbf{A}^2\Big)\psi + \frac{i\hbar e}{2mc^2}\big(\nabla\cdot\mathbf{A} + 2\mathbf{A}\cdot\nabla\big)\psi\mathbf{e}_3 - \frac{\hbar e}{2mc^2}\mathbf{B}\psi + eV\psi,$$

[17, eqn.(1.8)].

The Schrödinger equation (73) applies to all quantum phenomena except magnetism and relativity. The magnetism and spin of an electron are taken care of in the Schrödinger-Pauli equation (76), or equivalently (77). In order to extend quantum mechanics to special relativity, we start with the expression for *relativistic energy* $\frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2$. Following [19, (5.5)] and [10, p.136], we express the Dirac equation in terms of the Dirac spinor operator (64), getting what is known as the *Dirac-Hestenes* equation

$$\hbar\partial\psi\gamma_{21} - e\mathbf{A}\psi = m\psi\gamma_0, \tag{79}$$

where $\partial = \sum_{\mu=0}^{3}\gamma^\mu\frac{\partial}{\partial x^\mu}$, and an interaction with a electromagnetic field is included by way of the spacetime potential $\mathbf{A} = \sum_{\mu=0}^{3}\gamma_\mu A^\mu$.

Just like for the Shrödinger-Pauli equation (76), the *classical Dirac equation* can be retrieved from its spinor operator form (79). First, multiply both sides of the equation (79) on the right by the primitive idempotent u_{++}, using that $u_{++}\gamma_0 = u_{++}$ and $u_{++}\gamma_{21} = iu_{++}$, and using the spectral basis relations (58), (59), and (60). Finally, extracting the first column on each side of the resulting matrix equation, we obtain

$$i\hbar[\partial][\varphi]_4 - e[\mathbf{A}][\varphi]_4 = m[\varphi]_4 \tag{80}$$

where $[\varphi]_4$ is the Dirac spinor (61) and $i = \sqrt{-1}$. To show the full equivalence of (79) and (80), one must also show that (80) implies (79), which is taken up in the Appendix. In [17, 18, 19], and in many other papers, Hestenes explores new geometric features of the Dirac theory of the electron made possible by the spinor operator form (79) of the Dirac equation.

Appendix

In this Appendix, we explore in greater detail how our presentation of quantum mechanics is related to the more usual approaches.

Multiplying the spectral basis (13) on the right by $u_+ = \frac{1}{2}(1+\mathbf{e}_3)$ gives the *left spinor basis*

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix} u_+ = \begin{pmatrix} u_+ & 0 \\ \mathbf{e}_1 u_+ & 0 \end{pmatrix}. \tag{81}$$

Similarly, multiplying the spectral basis (13) on the right by $u_- = \frac{1}{2}(1-\mathbf{e}_3)$ gives the *right spinor basis*

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_- = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix} u_- = \begin{pmatrix} 0 & \mathbf{e}_1 u_- \\ 0 & u_- \end{pmatrix}. \tag{82}$$

Taking the matrix of each side of equation (26), we get

$$\frac{1}{\sqrt{2}}[|\alpha\rangle] = [(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+] = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix},$$

so the *left column spinor* $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix}$. The factor of $\frac{1}{\sqrt{2}}$ is inserted as a matter of convenience in the way that we defined the Hermitian inner product (29). Similarly, taking the matrix of each side of the $\mathbf{e}_1$-*conjugate* (44), we get

$$\frac{1}{\sqrt{2}}[|\beta\rangle^{e_1}] = [(\beta_0 + \beta_1 \mathbf{e}_1)u_-] = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_0 \end{pmatrix},$$

so the *right column spinor* $\begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_0 \end{pmatrix}$. Unlike with ordinary 2-component spinors (32), the full matrix $[g]$ of a geometric number $g \in \mathbb{G}_3$ can be recovered from the corresponding matrices of left and right column spinors,

$$g = gu_+ + gu_- = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle^{e_1}) = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} \alpha_0 & \beta_1 \\ \alpha_1 & \beta_0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}.$$

Taking the matrix representation of (28), we find that

$$[\langle\alpha||\beta\rangle] = 2[u_+][(\overline{\alpha}_0 + \overline{\alpha}_1 \mathbf{e}_1)][(\beta_0 + \beta_1 \mathbf{e}_1)][u_+] = 2(\overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1)[u_+]$$

or

$$[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1 \\ \overline{\alpha}_1 & \overline{\alpha}_0 \end{pmatrix} \begin{pmatrix} \beta_0 & \beta_1 \\ \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1 & 0 \\ 0 & 0 \end{pmatrix},$$

from which it follows that

$$\langle\alpha|\beta\rangle = \Big\langle [\langle\alpha||\beta\rangle] \Big\rangle_{0+3} = Tr \begin{pmatrix} \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1 & 0 \\ 0 & 0 \end{pmatrix} = \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1.$$

The matrix of the spinor operator (52) is given by

$$[\psi] = \frac{1}{\sqrt{2}}\big([|\alpha\rangle] + [|\alpha\rangle^-]\big) = \begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\overline{\alpha}_1 \\ 0 & \overline{\alpha}_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & -\overline{\alpha}_1 \\ \alpha_1 & \overline{\alpha}_0 \end{pmatrix}.$$

Recalling (40), $|\alpha\rangle = \sqrt{2}\alpha_0 \mathbf{m} u_+$, the calculation (42), is equivalent to the matrix calculation

$$\frac{1}{2}[|\alpha\rangle\langle\alpha|] = \alpha_0\overline{\alpha}_0[\mathbf{m}][u_+][\mathbf{m}] = [\hat{\mathbf{a}}_+]$$

where the matrices $[\mathbf{m}]$, $[u_+]$ and $[\hat{\mathbf{a}}_+]$ are specified by

$$[\mathbf{m}] = \begin{pmatrix} 1 & \frac{\overline{\alpha}_1}{\overline{\alpha}_0} \\ \frac{\alpha_1}{\alpha_0} & -1 \end{pmatrix}, \quad [u_+] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad [\hat{\mathbf{a}}_+] = \frac{1}{2}\begin{pmatrix} 1+a_3 & a_- \\ a_+ & 1-a_3 \end{pmatrix}.$$

Analogously to (82) for the spectral basis (13) of the Pauli algebra, we can extract any column of the spectral basis (58) of the Dirac algebra. For example, to extract the third column we multiply (58) on the right by u_{-+} to get

$$\begin{pmatrix} u_{++} & -\mathbf{e}_{13}u_{+-} & \mathbf{e}_3 u_{-+} & \mathbf{e}_1 u_{--} \\ \mathbf{e}_{13}u_{++} & u_{+-} & \mathbf{e}_1 u_{-+} & -\mathbf{e}_3 u_{--} \\ \mathbf{e}_3 u_{++} & \mathbf{e}_1 u_{+-} & u_{-+} & -\mathbf{e}_{13}u_{--} \\ \mathbf{e}_1 u_{++} & -\mathbf{e}_3 u_{+-} & \mathbf{e}_{13}u_{-+} & u_{--} \end{pmatrix} u_{-+} = \begin{pmatrix} 0 & 0 & \mathbf{e}_3 u_{-+} & 0 \\ 0 & 0 & \mathbf{e}_1 u_{-+} & 0 \\ 0 & 0 & u_{-+} & 0 \\ 0 & 0 & \mathbf{e}_{13}u_{-+} & 0 \end{pmatrix}.$$

Multiplying the spinor operator (64) on the right by u_{++} we obtain

$$\psi u_{++} = \varphi_1 u_{++} + \varphi_2 \mathbf{e}_{13} u_{++} + \varphi_3 \mathbf{e}_3 u_{++} + \varphi_4 \mathbf{e}_1 u_{++},$$

whose spinor matrix

$$[\psi u_{++}] = \begin{pmatrix} \varphi_1 & 0 & 0 & 0 \\ \varphi_2 & 0 & 0 & 0 \\ \varphi_3 & 0 & 0 & 0 \\ \varphi_4 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix} Mod(u_{++})$$

corresponds to the Dirac spinor (61).

The classical Dirac equation (80) is equivalent to the Dirac-Hestenes equation (79) multiplied on the right by u_{++},

$$\Big(i\hbar\partial\psi - e\mathbf{A}\psi - m\psi\Big)u_{++} = 0. \tag{83}$$

Taking the *spacetime inversion* of this equation, by replacing all spacetime vectors by their negatives, gives the parity related equation

$$\Big(i\hbar\partial\psi - e\mathbf{A}\psi + m\psi\Big)u_{-+} = 0.$$

Two more equations are obtained, which are parity related to (83), by taking the *complex conjugate* of both of these last two equations, giving

$$\Big(i\hbar\partial\psi + e\mathbf{A}\psi + m\psi\Big)u_{+-} = 0,$$

and

$$\Big(i\hbar\partial\psi + e\mathbf{A}\psi - m\psi\Big)u_{--} = 0.$$

But the sum of these four parity related equations is exactly the parity invarient Dirac-Hestenes equation (79).

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