Group Manifolds in Geometric Algebra

Garret Sobczyk

Abstract. This article explores group manifolds which are efficiently expressed in lower dimensional (Clifford) geometric algebras. The spectral basis of a geometric algebra allows the insightful transition between a geometric algebra of multivectors and its representation as a matrix over the real or complex numbers, or over the quaternions or split quaternions. Whereas almost all of the ground covered is well known, our approach is novel and lays down the fundamental ideas of Lie groups and algebras for group manifolds that are important in mathematics and physics, including the 3-sphere in 4-D Euclidean space, the 3-hyperboloid hypersurface in the neutral 4-D pseudo-Euclidean space, and the Lie group structure $GL(2, C)$ of complex Minkowski spacetime. Other topics covered are the Haar measure of $SU(2)$, and the Riemannian geometry imposed by the group structure at the identity.

Mathematics Subject Classification (2010). 15A66, 22E99, 53C15.

Keywords. geometric algebra, Haar measure, Lie algebra, Lie group, quaternions, Riemann curvature bivector, split quaternions.

1. Introduction

In Cartan differential geometry, and in the Kähler calculus, Klein's Erlangen Program is brought to centerstage, as are the fundamental ideas of Lie groups and Lie algebras [5]. Lie groups and their corresponding Lie algebras are usually introduced in terms of matrix Lie groups and algebras, the most fundamental being the n^2-dimensional Lie group $GL(n, \mathbb{R})$ and its corresponding Lie algebra $gl(n, \mathbb{R})$. In [1, Chp.8], it was shown that the group product of the Lie group $SU(2)$ of the 3-sphere S^3 in $\mathbb{R}^4$, can be represented in terms of the geometric product in the geometric algebra $\mathbb{G}_4$ of the Euclidean space. However, the simplest expression of the group product was missed, leading to more difficult computations and mistakes. One of the main purposes of the present work is to correct this oversight, and to extend these results to the corresponding Lie group of the 3-hyperboloid in the $(2, 2)$-pseudo-Euclidean space $\mathbb{R}^{2,2}$ with the geometric algebra $\mathbb{G}_{2,2}$. We also briefly explore the Lie group $GL(2,\mathbb{C})$, using the complexified spacetime geometric algebra $\mathbb{G}_{1,3}(\mathbb{C})$ of Minkowski spacetime.

Various geometric algebras are used in this work, and the interlocking relationships between these algebras becomes important. The quaternions and spit quaternions make their appearance in a natural way when the *spectral basis* of a matrix is utilized. The spectral basis makes possible the full unification of the matrix framework of linear algebra with the geometric framework provided by geometric algebra. Our approach is based on the fundamental assumption that (Clifford) geometric algebras should be considered to be the natural extension of the real number system $\mathbb{R}$ to include the concept of direction by introducing new *anti-commuting* square roots of ± 1, representing orthonormal unit vectors along orthogonal coordinate axes.

2. Quaternions and split quaternions

The quaternions were discovered by Hamilton over 170 years ago, and the split-quaternions by James Cockle six years later. While undergraduate students are taught nodding acquiantance with quaternions, split quaternions are never mentioned. Allowing 2×2 matrices to have quaternion, or split quaternion entries greatly extends their reach into advanced mathematics and physics. In this section, we use quaternion matrices to gain familiarity with the still less familiar geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_4$ on the one hand, and the split quaternions to gain familiarity with the geometric algebras $\mathbb{G}_{1,2}$ and $\mathbb{G}_{2,2}$ on the other.

The unit *anti-commutative* quaternions, usually denoted by $\mathbf{i},\mathbf{j},\mathbf{k}$, are naturally identified with elements of the even subalgebra $\mathbb{G}_3^+$ of the geometric algebra

$$\mathbb{G}_3 := gen_{\mathbb{R}}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\} = span_{\mathbb{R}}\{1,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_{23},\mathbf{e}_{13},\mathbf{e}_{12},\mathbf{e}_{123}\}, \tag{2.1}$$

known as the *Pauli algebra*, with $\mathbf{i} := \mathbf{e}_{23}$, $\mathbf{j} := \mathbf{e}_{13}$, $\mathbf{k} := \mathbf{e}_{12}$. They satisfy the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ijk} = -1.$$

We also use the special symbol $i := \mathbf{e}_{123}$ for the unit trivector, or *pseudoscalar* of $\mathbb{G}_3$, which is in the center $Z(\mathbb{G}_3)$ of the algebra since it commutes with all elements of $\mathbb{G}_3$.

The geometric algebra $\mathbb{G}_3$ is algebraically isomorphic to the algebra of Pauli matrices $\mathcal{P}$, generated by the famous Pauli matrices

$$[\mathbf{e}_1] := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [\mathbf{e}_2] := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [\mathbf{e}_3] := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

Rather than just "pulling" the Pauli matrices "out of a hat", more geometric insight is gained by introducing the concept of the *spectral basis* of the Pauli algebra over $Z(\mathbb{G}_3)$. Defining the *idempotents* $\mathbf{e}_3^{\pm} := \frac{1}{2}(1 \pm \mathbf{e}_3)$, and noting that $\mathbf{e}_1\mathbf{e}_3^+ = \mathbf{e}_3^-\mathbf{e}_1$, by the *spectral basis* of $\mathbb{G}_3$ we mean

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \mathbf{e}_3^+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_3^+ & \mathbf{e}_1\mathbf{e}_3^- \\ \mathbf{e}_1\mathbf{e}_3^+ & \mathbf{e}_3^- \end{pmatrix}. \tag{2.3}$$

The matrix $[g]$ of any element $g \in \mathbb{G}_3$ then satisfies the basic relationship

$$g = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} \mathbf{e}_3^+ [g] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}. \tag{2.4}$$

For example, plugging in the Pauli matrices for the basis vectors given in (2.2), and carrying out the indicated matrix multiplication over the geometric algebra $\mathbb{G}_3$, gives the identities

$$\mathbf{e}_k = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} \mathbf{e}_3^+ [\mathbf{e}_k] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}$$

for $k = 1, 2, 3$. Refering to the spectral basis (2.3), for the matrix $[g] = [g_{ij}]$, where the formally complex numbers $g_{ij} \in Z(\mathbb{G}_3)$,

$$g = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} \mathbf{e}_3^+ [g] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = g_{11}\mathbf{e}_3^+ + g_{12}\mathbf{e}_1\mathbf{e}_3^- + g_{21}\mathbf{e}_1\mathbf{e}_3^+ + g_{22}\mathbf{e}_3^-.$$

Split or s-quaternions were discovered by James Cockle (1819-1895) in 1849, six years after Hamilton's quaternions. Split quaternions are almost never mentioned, even though algebraically they are closely related to the quaternions. Denoting the anti-commutative unit split quaternions by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, they satisfy the quaternion-like rules

$$\mathbf{i}^2 = -1, \quad \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ijk} = 1.$$

Just as the quaternions are identified with the even sub-algebra $\mathbb{G}_3^+$ of $\mathbb{G}_3$, the split quaternions are naturally identified with the even sub-algebra $\mathbb{G}_{1,2}^+$ of the associative geometric algebra $\mathbb{G}_{1,2}$. The geometric algebra $\mathbb{G}_{1,2}$ is generated by three anticommuting square roots $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of ± 1, satisfying $\mathbf{e}_1^2 = \mathbf{e}_2^2 = -1$, $\mathbf{e}_3^2 = 1$. The standard basis of $\mathbb{G}_{1,2}$ is

$$\mathbb{G}_{1,2} := gen_{\mathbb{R}}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = span_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12}, \mathbf{e}_{123}\}, \tag{2.5}$$

and the s-quaternions are defined by $\mathbf{i} := \mathbf{e}_{23}$, $\mathbf{j} := \mathbf{e}_{13}$, $\mathbf{k} := \mathbf{e}_{12}$.[1] Once again, in the context of the geometric algebra $\mathbb{G}_{1,2}$, we use the special symbol $i := \mathbf{e}_{123}$, for which $i^2 = -1$, to denote the pseudoscalar in the center of the algebra.

The geometric algebra $\mathbb{G}_{1,2}$ has a matrix representation that is similar to the Pauli algebra of matrices for $\mathbb{G}_3$. In this case, we use the spectral basis

$$\begin{pmatrix} 1 \\ \mathbf{j} \end{pmatrix} \mathbf{k}_+ \begin{pmatrix} 1 & \mathbf{j} \end{pmatrix} = \begin{pmatrix} \mathbf{k}_+ & \mathbf{j}\,\mathbf{k}_- \\ \mathbf{j}\,\mathbf{k}_+ & \mathbf{k}_- \end{pmatrix}.$$

[1] In order to maintain the sign conventions of the definition of the s-quaternions, we are forced to interchange the roles of $\mathbf{e}_{23}$ and $\mathbf{e}_{12}$ used in the defintion of the quaternions.

The matrix representations of the basis vectors and bivectors of $\mathbb{G}_{1,2}$ are

$$[\mathbf{e}_1] := i\,[\mathbf{k}] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ [\mathbf{e}_2] := i\,[\mathbf{j}] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ [\mathbf{e}_3] := -i\,[\mathbf{i}] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.6}$$

and

$$[\mathbf{i}] = [\mathbf{e}_{12}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ [\mathbf{j}] = [\mathbf{e}_{31}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ [\mathbf{k}] = [\mathbf{e}_{23}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

Note that all the matrices above represent quite different elements in the Pauli algebra of matrices, the difference being a different geometric interpretation of what is a *vector* or *bivector* in the respective geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_{1,2}$. Never-the-less, in both of these geometric algebras the pseudoscalar element $i = \mathbf{e}_{123}$ has the property that $i^2 = -1$, and is in the center of the respective algebras.

There are two other geometric algebras which we need, $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$, both of which are simply obtained by extending the geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_{1,2}$ to include an additional orthogononal unit vector $\mathbf{e}_0$ with the property that $\mathbf{e}_0^2 = 1$. The associative geometric algebra $\mathbb{G}_4 := \mathbb{G}(\mathbb{R}^4) = gen_{\mathbb{R}}\{\mathbb{G}_3, \mathbf{e}_0\}$ has the standard basis consisting of the following $2^4 = 16$ elements

$$\mathbb{G}_4 = \{1; \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3; \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03}, \mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12}; \mathbf{e}_{012}, \mathbf{e}_{013}, \mathbf{e}_{023}, \mathbf{e}_{123}; \mathbf{e}_{0123}\}, \tag{2.8}$$

called, respectively, *the scalar* 1*, vectors, bivectors, trivectors*, and the 4-*vector*

$$I := \mathbf{e}_{0123} = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$

called the *pseudoscalar* of the algebra. Note also that $I = \mathbf{e}_0 i = -i\mathbf{e}_0$.

The geometric algebra $\mathbb{G}_{2,2} := \mathbb{G}(\mathbb{R}^{2,2}) = gen_{\mathbb{R}}\{\mathbb{G}_{2,2}, \mathbf{e}_0\}$ has the same standard basis (2.8) as the geometric algebra $\mathbb{G}_4$, but in this case the anti-commuting orthonormal basis vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ satisfy the rules

$$\mathbf{e}_0^2 = \mathbf{e}_3^2 = +1, \quad \text{and} \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = -1.$$

Never-the-less, in both $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$, the pseudoscalar element $I = \mathbf{e}_{0123} = \mathbf{e}_0 i$ has the property that $I^2 = 1$, as can be easily verified. Two other important properties that both algebras share are

$$\mathbf{e}_{10}\mathbf{e}_{20}\mathbf{e}_{30} = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = I, \tag{2.9}$$

and for any vector $\mathbf{x}$ in either $\mathbb{G}_4^1$ or $\mathbb{G}_{2,2}^1$, $\mathbf{x}I = -I\mathbf{x}$. Note also that

$$\mathbb{G}_4 \cap \mathbb{G}_{2,2} = gen_{\mathbb{R}}\{\mathbf{e}_0, \mathbf{e}_3\} = \mathbb{G}_2.$$

Formally, a general quaternion $\alpha \in \mathbb{H} \subset \mathbb{G}_3$ has the form

$$\alpha = a_0 + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13} + a_{12}\mathbf{e}_{12}, \quad \text{for} \quad a_0, a_{23}, a_{13}, a_{12} \in \mathbb{R},$$

and a general split quaternion $\alpha \in \mathbb{H}_s \subset \mathbb{G}_{1,2}$ has the form

$$\mathbb{H}_s := \{\alpha |\ \ \alpha = a_0 + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}, \quad \text{for} \quad a_0, a_{12}, a_{13}, a_{23} \in \mathbb{R}\}.$$

For a quaternion $\alpha \in \mathbb{H}$, or an s-quaternion $\alpha \in \mathbb{H}_s$, its *conjugate* is defined by

$$\alpha^\dagger = a_0 - a_{23}\mathbf{e}_{23} - a_{13}\mathbf{e}_{13} - a_{12}\mathbf{e}_{12}.$$

Of course, the quaternions $\mathbb{H}$ and the split quaternions $\mathbb{H}_s$ obey quite different multiplication rules, belonging to the respective different geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_{1,2}$, so care must always be taken not to confuse which algebra we are working in.

The group

$$SU(2) := \{\alpha \in \mathbb{H}| \;\; \alpha\alpha^\dagger = a_0^2 + a_{23}^2 + a_{13}^2 + a_{12}^2 = 1\},$$

with the group product quaternion multiplication. On the other hand, the group

$$SL(2,R) := \{\alpha \in \mathbb{H}_s| \;\; \alpha\alpha^\dagger = a_0^2 + a_{12}^2 - a_{13}^2 - a_{23}^2 = 1\},$$

with the group product s-quaternion multiplication. The group $\mathbb{H}_s^*$, consisting of all non-null s-quaternions, with $\alpha\alpha^\dagger \neq 0$, is also a group with s-quaternion multiplication, and it is isomorphic to the group $GL(2,\mathbb{R})$.

3. Matrices over the quaternions and split quaternions

Because $I^2 = 1$ in both $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$, we can use it to define the *mutually anihilating idempotents* I_+ and I_-, $I_\pm := \frac{1}{2}(1 \pm I)$, which in both algebras satisfy the properties

$$I_\pm^2 = I_\pm^2, \quad I_+I_- = 0, \quad I_+ + I_- = 1, \quad \text{and} \quad I_+ - I_- = I. \tag{3.1}$$

These properties are used in what follows to reduce algebraic properties in $\mathbb{G}_4$ or $\mathbb{G}_{2,2}$, to properties of 2×2 matrices over the quaternions $\mathbb{H} = \mathbb{G}_3^+$, or s-quaternions $\mathbb{H}_s = \mathbb{G}_{1,2}^+$, respectively. An introductory treatment of idempotents is given in [6].

The *spectral basis* of both $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$, over either the respective quaternions $\mathbb{H}$, or split quaternions $\mathbb{H}_s$, is

$$\begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} = \begin{pmatrix} I_+ & \mathbf{e}_0 I_- \\ \mathbf{e}_0 I_+ & I_- \end{pmatrix}. \tag{3.2}$$

Since every element in $\mathbb{G}_3$ can be uniquely represented in the form $\mathcal{A} = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{H}$, and every element in $\mathbb{G}_{1,2}$ can be uniquely represented in the form $\mathcal{A}_s = \alpha_s + i\beta_s$ for $\alpha_s, \beta_s \in H_s$, it follows, since $\mathbf{e}_0 i = I$, that every element in $\mathbb{G}_4$ or $\mathbb{G}_{2,2}$, respectively, can be uniquely represented in the form $\mathcal{A} + I\mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are in $\mathbb{G}_3$ or $\mathbb{G}_{1,2}$, respectively. Given $g = \mathcal{A} + I\mathcal{B}$, we can now find the *quaternion matrix* $[g]_{\mathbb{H}}$, or the split quaternion matrix $[g]_{\mathbb{H}_s}$ that represents g for the general element $g \in \mathbb{G}_4$ or $g \in \mathbb{G}_{2,2}$, respectively.

Noting that

$$\begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix} = I_+ + \mathbf{e}_0 I_+ \mathbf{e}_0 = I_+ + I_- = 1,$$

we calculate

$$g = \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix} (\mathcal{A} + I\mathcal{B}) \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} I_+(\mathcal{A}+I\mathcal{B})I_+ & I_+(\mathcal{A}+I\mathcal{B})I_-\mathbf{e}_0 \\ \mathbf{e}_0 I_-(\mathcal{A}+I\mathcal{B})I_+ & \mathbf{e}_0 I_-(\mathcal{A}+I\mathcal{B})I_-\mathbf{e}_0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix},$$

$$= \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ [g]_{\mathbb{H}} \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix}, \tag{3.3}$$

where $[g]_{\mathbb{H}} := \begin{pmatrix} \alpha_1+\beta_1 & -\alpha_2-\beta_2 \\ \alpha_2-\beta_2 & \alpha_1-\beta_1 \end{pmatrix}$ for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{H} \equiv \mathbb{G}_3^+$, or where $[g]_{\mathbb{H}_s} := \begin{pmatrix} \alpha_1+\beta_1 & -\alpha_2-\beta_2 \\ \alpha_2-\beta_2 & \alpha_1-\beta_1 \end{pmatrix}$ for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{H}_h \equiv \mathbb{G}_{2,2}^+$, respectively.

For a vector $\mathbf{x} = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{x}_3 \in \mathbb{R}^4$, or in $\mathbb{R}^{2,2}$,

$$\mathbf{x}\mathbf{e}_0 = x_0 + x_1\mathbf{e}_{10} + x_2\mathbf{e}_{20} + x_3\mathbf{e}_{30}, \text{ and } I\mathbf{x}\mathbf{e}_0 = x_0 I \pm x_1\mathbf{e}_{23} \mp x_2\mathbf{e}_{13} + x_3\mathbf{e}_{12}, \tag{3.4}$$

respectively, so the quaternion, or s-quaternion, corresponding to $\mathbf{x}$ is defined by $\alpha_1 = 0 = \beta_1$, and

$$\alpha_2 = -I\mathbf{x}\wedge\mathbf{e}_0 = \mp x_1\mathbf{e}_{23} \pm x_2\mathbf{e}_{13} - x_3\mathbf{e}_{12}, \quad \text{and} \quad \beta_2 = -\mathbf{e}_0\cdot\mathbf{x} = -x_0. \tag{3.5}$$

Consequently, the quaternion or s-quaternion matrix $[\mathbf{x}]$ for the vector $\mathbf{x} \in \mathbb{R}^4$, or $\mathbf{x} \in \mathbb{R}^{2,2}$ is given by

$$[\mathbf{x}] = \begin{pmatrix} 0 & x_0 \pm x_1\mathbf{e}_{23} \mp x_2\mathbf{e}_{13} + x_3\mathbf{e}_{12} \\ x_0 \mp x_1\mathbf{e}_{23} \pm x_2\mathbf{e}_{13} - x_3\mathbf{e}_{12} & 0 \end{pmatrix},$$

or by using (3.5),

$$[\mathbf{x}] = \begin{pmatrix} 0 & \mathbf{x}\cdot\mathbf{e}_0 + I\mathbf{x}\wedge\mathbf{e}_0 \\ \mathbf{x}\cdot\mathbf{e}_0 - I\mathbf{x}\wedge\mathbf{e}_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_{\mathbf{x}} \\ \alpha_{\mathbf{x}}^\dagger & 0 \end{pmatrix}, \tag{3.6}$$

for the quaternions, or s-quaternions $\alpha_{\mathbf{x}} = \mathbf{x}\cdot\mathbf{e}_0 + I\mathbf{x}\wedge\mathbf{e}_0$ and $\alpha_{\mathbf{x}}^\dagger = \mathbf{x}\cdot\mathbf{e}_0 - I\mathbf{x}\wedge\mathbf{e}_0$ in $\mathbb{H}$ or $\mathbb{H}_s$, respectively.

Referring to (3.6), and noting that $I_+ I = I_+$, the vector $\mathbf{x} \in \mathbb{R}^4$ or $\mathbf{x} \in \mathbb{R}^{2,2}$ can be expressed in the form

$$\mathbf{x} = \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ \begin{pmatrix} 0 & \mathbf{x}\mathbf{e}_0 \\ \mathbf{e}_0\mathbf{x} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix} = (I_+\alpha_{\mathbf{x}} + I_-\alpha_{\mathbf{x}}^\dagger)\mathbf{e}_0, \tag{3.7}$$

but the quantities $\mathbf{e}_0\mathbf{x}, \mathbf{x}\mathbf{e}_0$ are no longer quaternions in $\mathbb{H}$ or $\mathbb{H}_s$, respectively. The last expression on the right shows that there is a one-one correspondence between points $\mathbf{x} \in \mathbb{R}^4$, or $\mathbf{x} \in \mathbb{R}^{2,2}$ and quaternions $\alpha_{\mathbf{x}} \in \mathbb{H} = \mathbb{G}_3^+$, or s-quaternions $\alpha_{\mathbf{x}} \in \mathbb{H}_s = \mathbb{G}_{2,2}^+$.

From (3.7), for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$, or in $\mathbb{R}^{2,2}$,

$$\mathbf{x} = (I_+\alpha_{\mathbf{x}} + I_-\alpha_{\mathbf{x}}^\dagger)\mathbf{e}_0, \quad \mathbf{y} = (I_+\alpha_{\mathbf{y}} + I_-\alpha_{\mathbf{y}}^\dagger)\mathbf{e}_0 \tag{3.8}$$

for unique quaternions or s-quaternions $\alpha_{\mathbf{x}}, \alpha_{\mathbf{y}}$. It is not hard to show that if $\mathbf{x}\mathbf{y} = -\mathbf{y}\mathbf{x}$, then $(\alpha_{\mathbf{x}}\alpha_{\mathbf{y}})^\dagger = -\alpha_{\mathbf{x}}\alpha_{\mathbf{y}}^\dagger$. Taking the product of three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{G}_4^1$, gives

$$\mathbf{x}\mathbf{y}\mathbf{z} = (I_+\alpha_{\mathbf{x}}\alpha_{\mathbf{y}}^\dagger\alpha_{\mathbf{z}} + I_-\alpha_{\mathbf{x}}^\dagger\alpha_{\mathbf{y}}\alpha_{\mathbf{z}}^\dagger)\mathbf{e}_0.$$

Similarly, if $\mathbf{xyz} = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$, then

$$\alpha_{\mathbf{z}}\alpha_{\mathbf{y}}^{\dagger}\alpha_{\mathbf{x}} = -\alpha_{\mathbf{x}}\alpha_{\mathbf{y}}^{\dagger}\alpha_{\mathbf{z}} = \alpha_{\mathbf{x}}\alpha_{\mathbf{z}}^{\dagger}\alpha_{\mathbf{y}},$$

since quaternion and s-quaternion multiplication is associative.

It follows from (3.6), that any quaternion $\alpha_{\mathbf{x}} \in \mathbb{G}_3^+$, or s-quaternion $\alpha_{\mathbf{x}} \in \mathbb{G}_{1,2}^+$, has the Euler-like canonical form

$$\alpha_{\mathbf{x}} = \rho e^{\theta I\hat{\mathbf{B}}} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^4 \quad \text{or} \quad \mathbb{R}^{2,2} \quad \text{and} \quad \mathbf{e}_0\hat{\mathbf{B}} = -\hat{\mathbf{B}}\mathbf{e}_0, \tag{3.9}$$

for some unit bivector $\hat{\mathbf{B}} \in \mathbb{G}_4^+$ or $\mathbf{B} \in \mathbb{G}_{2,2}^+$, respecitively, which anticommutes with $\mathbf{e}_0$ so that $I\hat{\mathbf{B}} \in \mathbb{G}_3^2$ or in $\mathbb{G}_{1,2}^+$. This important canonical form will be used later.

If $[g_1]$ and $[g_2]$ are the quaternion or s-quaternion matrices of $g_1, g_2 \in \mathbb{G}_4$ or $g_1, g_2 \in \mathbb{G}_{2,2}$, then the matrix $[g_1 g_2] = [g_1][g_2]$, as follows from

$$g_1 g_2 = \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ [g_1 g_2] \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{e}_0 \end{pmatrix} I_+ [g_1][g_2] \begin{pmatrix} 1 \\ \mathbf{e}_0 \end{pmatrix},$$

so multiplication of arbitrary elements in $\mathbb{G}_4$ or $\mathbb{G}_{2,2}$ is reduced to 2×2 matrix multiplication over the quaternions $\mathbb{H}$ or s-quaternions $\mathbb{H}_s$, respectively [9, p.79].

4. Conjugations in $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$

There are three kinds of conjugation on $\mathbb{G}_4$ and $\mathbb{G}_{2,2}$ that we use, and which induce corresponding conjugations on the subalgebras of quaternions $\mathbb{H}$ in $\mathbb{G}_4$ and s-quaternions $\mathbb{H}_s$ in $\mathbb{G}_{2,2}$.

For $g = \mathcal{A} + I\mathcal{B}$ in $\mathbb{G}_4$, or in $\mathbb{G}_{2,2}$, for $\mathcal{A}, \mathcal{B}$ in $\mathbb{G}_3$, or in $\mathbb{G}_{1,2}$, respectively,

$$\overline{g} := \mathbf{e}_0 g \mathbf{e}_0 = \overline{\mathcal{A}} - I\overline{\mathcal{B}}, \quad \text{and} \quad g^{\dagger} := \mathcal{A}^{\dagger} + I\mathcal{B}^{\dagger},$$

where

$$\overline{\mathcal{A}} := \alpha_1 - i\alpha_2 \quad \text{and} \quad \mathcal{A}^{\dagger} := \alpha_1^{\dagger} - i\alpha_2^{\dagger}.$$

The *reverse* $\mathcal{A}^{\dagger}$ reverses the order of all geometric products of vectors in $\mathcal{A}$. The third conjugation, or automorphism, is the composition of the previous two,

$$g^* := \overline{g}^{\dagger} = \overline{\mathcal{A}}^{\dagger} - I\overline{\mathcal{B}}^{\dagger} = \alpha_1^{\dagger} + i\alpha_2^{\dagger} - I(\beta_1^{\dagger} + i\beta_2^{\dagger}).$$

5. Lie group manifolds for $\mathbb{H}^*$ and $\mathbb{H}_s^*$

For the remainder of this paper, we use the notation $x = x_0 e + \mathbf{x}$, where $e := \mathbf{e}_0 \in \mathbb{R}^4 \cap \mathbb{R}^{2,2}$, and $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ in $\mathbb{R}^3$ or $\mathbb{R}^{1,2}$, to clearly distinguish between vectors in the larger $\mathbb{G}_4$ or $\mathbb{G}_{2,2}$, from vectors in the geometric algebras $\mathbb{G}_3$ or $\mathbb{G}1,2$, respectively. We also introduce the special symbol $e = \mathbf{e}_0$; it will represent the group identity in the group manifolds to be defined.

The product of unit quaternions defines the group $SU(2)$, and by (3.8), each quaternion $\alpha_x \in \mathbb{H}$ corresponds to a unique unit vector x on the 3-D sphere S^3 in $\mathbb{R}^4$, defined by

$$S^3 := \{x \in \mathbb{R}^4 | \ \ x^2 = \alpha_x \alpha_x^\dagger = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Similarly, the product of s-quaternions defines the group $SL(2, \mathbb{R})$, and by (3.8), each s-quaternion $\alpha_x \in \mathbb{H}_s$ corresponds to a unique unit vector $x \in \mathbb{R}^{2,2}$ on the 3-D identity component L^3 of the unit hyperboloid in $\mathbb{R}^{2,2}$ defined by

$$L^3 := \{x \in \mathbb{R}^{2,2} | \ \ x^2 = \alpha_x \alpha_x^\dagger = x_0^2 - x_1^2 - x_2^2 + x_3^2 = 1\}.$$

To complete the description of the group manifold of $SU(2)$ on the unit sphere S_3 in $\mathbb{R}^4$, and the group manifold of $SL(2, \mathbb{R})$, we must define a group product $\phi(x, y)$ for all pairs of points $x, y \in S^3$, or in L^3. The natural group product is

$$z = \phi(x, y) = I_+ xey + I_- yex = \Big(I_+ \alpha_x \alpha_y + I_- \alpha_y^\dagger \alpha_x^\dagger\Big) e, \tag{5.1}$$

for $\alpha_x, \alpha_y \in \mathbb{H}$, or in $\mathbb{H}_s$, respectively. It should be noted that the group product $\phi(x, y)$ is well defined for all x, y, and corresponding quaternions α_x, α_y, and not just for points on the unit 3-sphere S^3, or on the unit 3-hyperboloid L^3. Expressing z in terms of the dot and wedge products, gives

$$z = (x \cdot e)y + (x \wedge e) \cdot y + I(x \wedge e \wedge y), \tag{5.2}$$

but it is much more efficient to use (5.1) when making calculations.[2]

The group product $z = \phi(x, y)$ was first introduced in [1, p.291], with a different sign in the last term on the right side of (5.2). Its group properties are easily verified. The identity element of the group is $e := \mathbf{e}_0$. The group's associative property follows directly from the associativity of the geometric product in (5.1). Finally, given any $x \in S^3$, or in L^3, its group inverse y, satisfying $\phi(x, y) = e$, is uniquely defined by

$$y = \overline{x} = exe = x_\parallel - x_\perp,$$

where $x_\parallel$ and $x_\perp$ are components of x parallel and perpendicular to e, respectively. Not surprisingly, the group product can be expressed as a rotation. Using (5.1), we have for $z = \phi(x, y)$

$$\begin{aligned} z &= (zy)y = \sqrt{zy}\ y\sqrt{yz} \\ &= \sqrt{(I_+ xey + I_- yex)y}\, y\sqrt{y(I_+ xey + I_- yex)} \\ &= \sqrt{(I_+ xe + I_- yexy)}\, y\sqrt{(I_+ ex + I_- yxey)} \\ &= (I_+\sqrt{xe} + I_-\sqrt{yexy})\, y(I_+\sqrt{ex} + I_-\sqrt{yxey}). \end{aligned} \tag{5.3}$$

The case when $x = y$ is interesting. In this case, (5.3) simplifies to $x^{(2)} := \phi(x, x) = (xe)x$, and more generally,

$$x^{(k)} := \phi(x^{(k-1)}, x) = (xe)^{k-1} x \quad \text{for} \quad k \in \mathbb{R}. \tag{5.4}$$

Thus, as an element of the group manifold, x can be chosen in such a way as to have any order. For $k = 0$, $x^{(0)} = e$, and for $k = -1$, $x^{(-1)} = \overline{x}$ the group

[2] The idempotent form of the group product (5.1) was missed in [1, p.292].

inverse of x. Of course $x^{(1)} = x$ as expected. More generally, for any non-zero vector $x \in \mathbb{R}^4$, or non null vectors $x \in \mathbb{R}^{2,2}$, the group inverse with respect to the group product $\phi(x,y)$ is $x^{(-1)} = \overline{x}/x^2$ as is easily verified.

Since both S^3 and L^3 are group manifolds, we can apply the methods of geometric calculus in the study of their properties [1], [9]. Introducing the notation $\lambda_y(x) = \phi(y,x)$, by group associativity for the elements x, y, z, we find that

$$\lambda_z(\lambda_y(x)) = \phi(z, \phi(y,x)) = \phi(\phi(z,y), x) = \lambda_{\phi(z,y)}(x). \tag{5.5}$$

The 3-dimensional tangent space $\mathcal{T}_x$ of S^3, or L^3, at any point x is defined by

$$\mathcal{T}_x := \{a| \;\; a \in \mathbb{R}_4 \;\text{ and }\; a \cdot x = 0\}.$$

For the tangent vector $a \in \mathcal{T}_x$, the differential mapping

$$\underline{\lambda}_y(a) := a \cdot \partial_x \lambda_y(x),$$

where ∂_x is the *vector derivative* or *gradient* of $\mathbb{R}^4$ or $\mathbb{R}^{2,2}$ at the point x, satisfies

$$\underline{\lambda}_y(a) = a \cdot \partial_x \phi(y,x) = \phi(y,a),$$

and is a tangent vector in the tangent algebra $\mathcal{T}_y$ at the the point y, [9, p.63]. The definition of the vector derivative on the Euclidean space $\mathbb{R}^4$, or the pseudo-Euclidean space $\mathbb{R}^{2,2}$, is based upon the *Levi-Civita connection* of Riemannian geometry [5, p.235].

A tangent vector $a \in \mathcal{T}_e$ at the identity is simply extended to a vector field on all of S^3 or L^3 by *left translation*

$$a(x) := \underline{\lambda}_x(a(e)) = a \cdot \partial_e \lambda_x(e) = \phi(x, a(e)) = x \cdot (e \wedge a) + Ix \wedge e \wedge a. \tag{5.6}$$

The set of all tangent vectors in the tangent algebra $\mathcal{T}_e$ at the identity, extended to vector fields on S^3 or L^3, make up the *Lie algebras* $su(2)$ or $sl(2,\mathbb{R})$ respectively, of the Lie groups $SU(2)$ or $SL(2,\mathbb{R})$ at the identity e.

Letting $w = \lambda_y(x)$, using the chain rule for differentiation for the mappings defined in (5.5), we calculate

$$\underline{\lambda}_{\phi(z,y)}(a) = \underline{\lambda}_y(a) \cdot \partial_w \lambda_z(w) = \underline{\lambda}_z(\underline{\lambda}_y(a)) = \underline{\lambda}_z \circ \underline{\lambda}_y(a), \tag{5.7}$$

showing that the composition of the linear mappings of the differentials obeys the same group properties as the group manifolds of points on S^3 or L^3. Note also that

$$\underline{\lambda}_z \circ \underline{\lambda}_y(a) = \underline{\lambda}_z \circ \underline{\lambda}_y \circ \underline{\lambda}_a(e) = \phi(z, \phi(y,a)).$$

It follows that the group manifolds of S^3 and L^3 can be largely replaced by the study of the composition of linear mappings in their Lie algebras $su(2)$ or $sl(2,\mathbb{R})$ at the identity e.

Property (5.7) is equivalent to the first of three theorems which make up the *Fundamental Theorem of Lie Group Theory*, first formulated by Sophus Lie (1842-1899). Lie's second theorem is a basic property of an *outermorphism* applied to the *Lie bracket of vector fields*, defined by

$$[a(x), b(x)] := a(x) \cdot \partial_x b(x) - b(x) \cdot \partial_x a(x).$$

For $a(x)=\underline{\lambda}_x(a)$ and $b(x)=\underline{\lambda}_x(b)$ for $a,b\in T_e$,

$$\underline{\lambda}_x([a(e),b(e)])=[a(x),b(x)]=[\underline{\lambda}_x(a),\underline{\lambda}_x(b)], \tag{5.8}$$

as is shown in the steps

$$[\underline{\lambda}_x(a(e)),\underline{\lambda}_x(b(e))]=\phi(x,a)\cdot\partial_x\underline{\lambda}_x(b(e))-\phi(x,b)\cdot\partial_x\underline{\lambda}_x(a(e))$$
$$=\phi(\phi(x,a),b)-\phi(\phi(x,b),a))=\phi(x,\phi(a,b))-\phi(x,\phi(b,a))$$
$$=\underline{\lambda}_x(\phi(a,b)-\phi(b,a))=\underline{\lambda}_x([a(e),b(e)]).$$

Calculating $[a(x),b(x)]$ explicitly at the identity, we find that

$$[a(e),b(e)]=a(e)\cdot\partial_e b(e)-b(e)\cdot\partial_e a(e)=\phi(a(e),b(e))-\phi(b(e),a(e))$$
$$=I_+aeb+I_-bea-I_+bea-I_-aeb=Iaeb-Ibea=2Ia\wedge e\wedge b.$$

Using Lie's second theorem (5.8), we then find that[3]

$$[a(x),b(x)]=\underline{\lambda}_x([a(e),b(e)])=2\phi(x,Ia\wedge e\wedge b)=2\Big((a\wedge b)\cdot x+Ia\wedge x\wedge b\Big). \tag{5.9}$$

When $x=e$, (5.9) reduces to the previous formula for $[a(e),b(e)]$, as it must.

Lie's third theorem, as applied to either of the Lie algebras $su(2)$ or $sl(2,\mathbb{R})$, is that the vector fields satisfy the *Jacobi identity*,

$$[a(x),[b(x),c(x)]]+[c(x),[a(x),b(x)]]+[b(x),[c(x),a(x)]]=0. \tag{5.10}$$

Since

$$\underline{\lambda}_x([[a(e),b(e)],c(e)])=[[a(x),b(x)],c(x)]]$$

by a double application of Lie's second theorem (5.8), we need only show that (5.10) is valid at the identity. But, calculating at the identity,

$$[a,[b,c]]=a\cdot\partial_e b\cdot\partial_e c-a\cdot\partial_e c\cdot\partial_e b-b\cdot\partial_e c\cdot\partial_e a+c\cdot\partial_e b\cdot\partial_e a,$$

and adding two permuted copies to this result gives zero, so we are done.

6. Dirac algebra of spacetime

There is a very interesting relationship between the geometric algebra $\mathbb{G}_4$ of Euclidean space $\mathbb{R}^4$ and the geometric algebra $\mathbb{G}_{1,3}$ of the pseudo-Euclidean space $\mathbb{R}^{1,3}$ known as the *Dirac algebra*, or the *spactime algebra* of *Minkowski spacetime* [2, 7]. Just as the Pauli algebra $\mathbb{G}_3$ can be *factored* into, or identified with the even subalgebra $\mathbb{G}_{1,3}^+$ of $\mathbb{G}_{1,3}$, the geometric algebra $\mathbb{G}_4$ of Euclidean space $\mathbb{R}^4$ can be factored into the spacetime algebra $\mathbb{G}_{1,3}$.

Recall the standard orthonormal basis of the geometric algebra

$$\mathbb{G}_4:=gen_{\mathbb{R}}\{\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

Essentially, we *split* or factor the geometric algebra $\mathbb{G}_4$ into the geometric algebra

$$\mathbb{G}_{1,3}:=gen_{\mathbb{R}}\{\gamma_0,\gamma_1,\gamma_2,\gamma_3\}$$

[3] This corrects two errors that were made in [1, p.294].

by making the identification

$$\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \equiv \{\gamma_{123}, \gamma_{10}, \gamma_{20}, \gamma_{30}\}, \tag{6.1}$$

or, equivalently, solving for the spacetime vectors of $\mathbb{G}_{1,3}$ in terms of the basis vectors of $\mathbb{G}_4$,

$$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \equiv \{-I, \mathbf{e}_{023}, \mathbf{e}_{031}, \mathbf{e}_{012}\}, \tag{6.2}$$

With this identification, the geometric algebras $\mathbb{G}_{1,3}$ and $\mathbb{G}_4$ are algebraically isomorphic [3, p.217]; we have discovered Euclidean 4-space hidden in the geometric algebra $\mathbb{G}_{1,3}$ of Minkowski spacetime. It is interesting to note that in the Euclidean split (6.1), $\mathbf{e}_0 = \gamma_{123}$ is the *dual* $\gamma_0 I$ of the timelike vector $\gamma_0 \in \mathbb{G}_{1,3}$, and is uniquely determined by the frame of spacelike vectors $\{\gamma_1, \gamma_2, \gamma_3\}$.

All of the geometric algebras $\mathbb{G}_4$, $\mathbb{G}_{2,2}$, and $\mathbb{G}_{1,3}$, can be considered to be real subalgebras of the *complex geometric algebra* $\mathbb{G}_4(\mathbb{C})$. By a *complex vector* $z \in \mathbb{G}^1_{1,3}(\mathbb{C})$, we mean $z = x + iy$, where $x, y \in \mathbb{G}^1_{1,3}$ and $i := \sqrt{-1}$. The non-null complex Dirac vectors in $\mathbb{G}^1_{1,3}(\mathbb{C})$ of spacetime can be made into the Lie group $GL(2,\mathbb{C})$ in the larger $\mathbb{G}_{1,3}(\mathbb{C})$. Given two vectors $z_1, z_2 \in \mathbb{G}_{1,3}(\mathbb{C})$, such that $z_1^2 \neq 0$, and $z_2^2 \neq 0$, we define the group product by

$$\phi(z_1, z_2) := \gamma^+ z_1 \gamma_0 z_2 + \gamma^- z_2 \gamma_0 z_1, \tag{6.3}$$

where $\gamma^{\pm} := \frac{1}{2}(1 \pm i\gamma_{0123})$. Clearly, the group identity is γ_0, and given any complex Dirac vector z such that $z^2 \neq 0$, the group inverse

$$z^{(-1)} := \frac{\gamma_0 z \gamma_0}{z^2},$$

as shown by

$$\phi(z, z^{(-1)}) = \left(\gamma^+ z^2 + \gamma^- z^2\right)\frac{\gamma_0}{z^2} = \gamma_0.$$

The complex geometric algebras $\mathbb{G}_4(\mathbb{C}), \mathbb{G}_{2,2}(\mathbb{C})$ and $\mathbb{G}_{1,3}(\mathbb{C})$, are all algebraically isomorphic.

7. Haar measure on S^3

In this section, we will restrict ourselves to considering the *Harr measure* of the group manifold $SU(2)$ of S^3, although the Haar measures on the other group manifolds discussed in this paper can be treated in much the same way.

We have seen in the last section that the group product (5.1) on S^3 is very closely related to the group of unit quaternions in $SU(2)$. This relationship can be expressed in several different ways. Again identifying $e = \mathbf{e}_0$,

$$\phi(x, y) = I_+ xey + I_- yex = (I_+ xeye)e + e(I_+ eyex) = (I_+ \alpha_x \alpha_y + I_- \alpha_y^\dagger \alpha_x^\dagger)e, \tag{7.1}$$

for $\alpha_x, \alpha_y \in \mathbb{H}$. Of course, the group product $\phi(x, y)$ differs from the geometric product xy. For comparison, using (3.7), we find for the geometric product that

$$xy = (I_+ \alpha_x + I_- \alpha_x^\dagger)e(I_+ \alpha_y + I_- \alpha_y^\dagger)e = (I_+ \alpha_x \alpha_y^\dagger + I_- \alpha_x^\dagger \alpha_y).$$

Naturally, the associative group product (5.1), (7.1) can be extended to any number of terms, $\phi(x, y, z, \ldots)$, expressing the group product directly in terms of the corresponding quaternions, $\alpha_x, \alpha_y, \alpha_z, \ldots$, respectively. The group of all unit vectors in $\mathbb{R}^4$, under the geometric product, make up the *Pin group* Pin(4), which is the double covering group of $O(4)$, [3, p.146]. The set of non-zero vectors in $\mathbb{R}^4$, under the group product $\phi(x, y)$, is isomorphic to the group of all non-zero quaternions. Modulo I_+, the unit quaternions α_x and α_y are identified with the elements $xe, ye \in \mathbb{G}_4^+$, generators of the group $Spin(4)$, the double covering of the group $SO(4)$. Thus, the group of unit quaternions $SU(2)$ is identified with the ideal $I_+ Spin(4) = Spin(4) I_+ \subset \mathbb{G}_4^+$, [3, p.270]. There are many applications of groups in differential geometry which are important in mathematics and physics. For example, see [1, p.210] and [10].

Let $a \in T_e$ be a tangent vector at the identity $e \in S^3$, so that $e \cdot a = 0$. Calculating

$$a(x) = \underline{\lambda}_x(a) = a \cdot \partial_e \lambda_x(e) = \phi(x, a) = I_+ xea + I_- aex,$$

it follows that $\underline{\lambda}_x(a) \in T_x$. Checking,

$$a(x) \cdot x = \underline{\lambda}_x(a) \cdot x = x \cdot e\, x \cdot a + e \cdot a - x \cdot a\, x \cdot e = 0.$$

For a second and third vector $b, c \in T_e$,

$$\begin{aligned} a(x) \wedge b(x) = \underline{\lambda}_x(a) \wedge \lambda_x(b) &= \langle (I_+ xea + I_- aex)(I_+ xeb + I_- bex) \rangle_2 \\ &= \langle I_+ xabx + I_- ab \rangle_2 = x\Big(x \wedge a \wedge b + Ix \cdot (a \wedge b)\Big), \end{aligned} \tag{7.2}$$

and

$$\underline{\lambda}_x(a \wedge b \wedge c) = \langle (I_+ xabx + I_- ab)(I_+ xec + I_- cex) \rangle_3 = (a \wedge b \wedge c \wedge e)x,$$

or

$$\underline{\lambda}_x(Ie) = Ix \quad \Longrightarrow \quad \det[\underline{\lambda}_x] = 1. \tag{7.3}$$

This last result shows the relationship between the unit tangent 3-vector Ie at $e \in S^3$ and the tangent unit 3-vector at the point $x \in S^3$, and that the Jacobian of the mapping, $\det[\underline{\lambda}_x] = 1$ for all $x \in S^3$. This reduces the problem of finding the *Haar Measure* on the group vector manifold S^3 to the problem in calculus of calculating the mass of an object with a homogeneous unit density, which is its volume.[4]

Let us consider the "half angle" version of (7.3), which will be useful in explicitly finding the Haar measure on the vector group manifold of $SU(2)$ on S^3. Letting

$$y = \sqrt{xe}e = e \cos\frac{\theta}{2} + \hat{\mathbf{n}} \sin\frac{\theta}{2} = (\cos\frac{\theta}{2} + \hat{\mathbf{n}}\, e \sin\frac{\theta}{2})e = e^{-\frac{\theta}{2} e\, \hat{\mathbf{n}}} e, \tag{7.4}$$

where $\hat{\mathbf{n}} \in \mathbb{R}^3$ and $\hat{\mathbf{n}} \cdot e = 0$, the relationship (7.3) takes the form

$$\underline{\lambda}_y(Ie) = I\sqrt{xe}e = Ie\, e^{\frac{\theta}{2} e\, \hat{\mathbf{n}}} = I\frac{x + e}{|x + e|} = I(e \cos\frac{\theta}{2} + \hat{\mathbf{n}} \sin\frac{\theta}{2}).$$

[4] Our derivation of the Haar measure for $SU(2)$ given here corrects the misleading discussion that was given in [1, p.295-96].

We are now prepared to derive the Haar measure on the group manifold $SU(2)$ of S^3, with the the group product $\phi(x,y) = I_+ xey + I_- yex$. Following [4], the key idea is to calculate, and then normalize, the integral

$$\int_{S^3} f(x) d^3x := \int_{\mathbb{R}^4} f(x)\delta(x^2-1)d^4x, \tag{7.5}$$

where $d^4x = dx_0dx_1dx_2dx_3$, and dx^3 is the corresponding element of the 3-D surface S^3, and $\delta(x^2-1)$ is the Dirac delta function on $\mathbb{R}^4$. As before, we clearly distinguish the identity component x_0 from the other components x_1, x_2, x_3 for the vector $x \in \mathbb{R}^4$.

Writing $x = x_0 e + \mathbf{x}$ for $\mathbf{x}_3 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, the right side of (7.5) becomes

$$\int_{\mathbb{R}^4} f(x)\delta(x^2-1)d^4x = \int_{\mathbb{R}^3}\int_{\mathbb{R}} f(x_0,\mathbf{x})\delta(x_0^2+\mathbf{x}^2-1)dx_0 d^3\mathbf{x}$$

$$= \int_{|\mathbf{x}|\le 1}\Big[\int_0^1 \Big(f(x_0,\mathbf{x}) + f(-x_0,\mathbf{x})\Big)\delta(x_0^2+\mathbf{x}^2-1)dx_0\Big]d^3\mathbf{x}$$

where $d^3\mathbf{x} := dx_1dx_2dx_3$. Making the substitution $w = x_0^2 + \mathbf{x}^2 - 1$ and $dw = 2x_0dx_0$ in the inside integral, and evaluating the delta function, leads to the further simplification of the outside integral to

$$\int_{|\mathbf{x}|\le 1} \frac{1}{x_0(\mathbf{x})}\Big(f(x_0(\mathbf{x}),\mathbf{x}) + f(-x_0(\mathbf{x}),\mathbf{x})\Big)d^3\mathbf{x}. \tag{7.6}$$

Introducing the new parameters θ and $\hat{\mathbf{n}}$ from (7.4),

$$x = \mathbf{e}_0\Big|\cos\frac{\theta}{2}\Big| - \hat{\mathbf{n}}\sin\frac{\theta}{2} \quad\Longleftrightarrow\quad x_0 = \sqrt{1-\mathbf{x}^2} = \Big|\cos\frac{\theta}{2}\Big|,\ \mathbf{x} = -\hat{\mathbf{n}}\sin\frac{\theta}{2},$$

the integral (7.6) is reduced to an integral over S^2, given by

$$\int_0^{2\pi}\int_{\hat{\mathbf{n}}\in S^2} f(\theta,\hat{\mathbf{n}})\sin^2\frac{\theta}{2}d^2\hat{\mathbf{n}}\,d\theta, \tag{7.7}$$

where

$$d^2\hat{\mathbf{n}} = \frac{1}{4\pi}\sin\phi_1\ d\phi_1 d\phi_2 \ \text{ and } \ d^3\mathbf{x} = \frac{1}{2}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}d^2\hat{\mathbf{n}}\,d\theta.$$

Putting together the integrals (7.5), (7.6), (7.7), and normalizing, gives the *Haar measure* on the vector group manifold S^3 of $SU(2)$ with the group product $\phi(x,y)$,

$$\int_{x\in S^3} f(x)d\mu_H(x) = \frac{1}{\pi}\int_0^{2\pi}\int_{\hat{\mathbf{n}}\in S^2} f(\theta,\hat{\mathbf{n}})\sin^2\frac{\theta}{2}d^2\hat{\mathbf{n}}\,d\theta.$$

8. Riemannian geometry of S^3

This final section is devoted to exploring some of the most fundamental properties of the geometry of the Riemannian sphere S^3 induced by its Lie group $SU(2)$ and Lie algebra $su(3)$.

Following [1, p.290], we define the *extensor function* h_x by

$$h_x := \underline{\lambda}_x^{-1} = \underline{\lambda}_{\overline{x}},$$

so that

$$a(e) = h_x(a(x)) = \underline{\lambda}_{\overline{x}}(a(x)) \tag{8.1}$$

for all $a(e) \in T_e$. Then the composition

$$g_x := \overline{h}h = \overline{\lambda}_{\overline{x}}\,\underline{\lambda}_{\overline{x}},$$

where $\overline{h}$ is the linear *adjoint mapping* of h (not to be confused with the conjugation $\overline{x}$ of $x \in \mathbb{R}^4$). The *metric tensor* g_x has the property for the vector fields $a(x) = \underline{\lambda}(a(e)), b(x) = \underline{\lambda}(b(e))$,

$$a(x) \cdot g(b(x)) = a(x) \cdot \overline{h}_x\, \underline{h}_x(b(x)) = \underline{h}_x(a(x)) \cdot \underline{h}_x(b(x)) = a(e) \cdot b(e),$$

so the metric tensor at any point $x \in S^3$ is completely defined by the metric tensor at the identity [1, p.50], [9, p.113].

One of the most important concepts of curvature on a differentiable manifold is the *Riemann curvature tensor*, and it is this concept that is at the heart of Einstein's general theory of relativity. The curvature tensor on S^3 is most simply defined in terms of it's *shape operator* $S_x(a(x))$. The shape operator is a measure of how the tangent trivector (7.3) to S^3 at the point x changes when moved off in the direction $a(x) \in T_x$. It is defined by

$$S_x(a(x)) := Ix\, a(x) \cdot \partial_x\, xI = x\, a(x) = x \wedge a(x). \tag{8.2}$$

The *Riemann curvature bivector* is now easily found,

$$R_x(a(x) \wedge b(x)) := \langle S_x(a(x))S_x(b(x)\rangle_2 = \langle xa(x)xb(x)\rangle_2 = -a(x) \wedge b(x), \tag{8.3}$$

[1, p.190], [9, p.272]. It is interesting to compare this result with (7.2). Note also that these results remain valid for the hyperboloid L^3, as determined by its Lie group $SL(2, \mathbb{R})$.

Acknowledgement

The author thanks the reviewers for comments which led to major revisions of the original paper under the title, "Geometry of the 3-Sphere".

References

[1] D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, D. Reidel Pub. Co., 1984.

[2] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, N.Y. 1966.

[3] P. Lounesto, *Clifford Algebras and Spinors, 2nd Edition.* Cambridge University Press, Cambridge, 2001.

[4] Majr, *Introduction to Lie groups, Lie algebras and their representations*, Lectures 5 and 6: SU(2) and SO(3), University of Bristol, Autumn 2013.
`http://www.maths.bris.ac.uk/~majr/lectures/`
`Lecture%205%20and%206.pdf`

[5] R.W. Sharpe, Differential Geometry: Cartan's Generalization of Klein's Erlangen Program, Springer-Verlag, New York 1997.

[6] G. Sobczyk, Hyperbolic Number Plane, *The College Mathematics Journal*, Vol. 26, No. 4, pp.268-280, September 1995.

`http://www.garretstar.com/`

[7] G. Sobczyk, *Geometry of Spin 1/2 Particles*, Revista Mexicana de Física, **61**, (2015) 211-223.

[8] G. Sobczyk, *Geometric Spinors, Relativity and the Hopf Fibration*, 2015.

`http://www.garretstar.com/geo2hopf26-9-2015.pdf`

[9] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, New York 2013.

[10] G. Sobczyk, *Conformal Mappings in Geometric Algebra*, Notices of the AMS, Volume 59, Number 2, p.264-273, 2012.

Garret SobczykrUniversidad de las Américas-Puebla
Departamento de Físico-Matemáticas
72820 Puebla, Pue., México
URL: http://www.garretstar.com
e-mail: `garret_sobczyk@yahoo.com`