GEOMETRIC MATRIX ALGEBRA

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Abstract

Matrix multiplication was first introduced by Arthur Cayley in 1855 in agreement with the composition of linear transformations. We explore an underlying geometric framework or basis in which matrix multiplication naturally arises as the product of geometric numbers in a geometric (Clifford) algebra. Consequently, all invariants of a linear operator become geometric invariants of the multivectors that they represent. Two different kinds of basis for matrices emerge, a spectral basis of idempotents and nilpotents, and a standard basis of scalars, vectors, bivectors, and higher order k-vectors. We also discuss the geometric basis for the Kronecker product of matrices.

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PART I: Geometric Extension of Number.

1. The real number system $\mathbb{R}$.

2. $i := \sqrt{-1} \notin \mathbb{R}$ gives *complex numbers* $\mathbb{C}$.

a) $z = x + iy \in \mathbb{C}$ in the *standard basis* $\{1, i\}$.

b) $i^2 = -1$ and $\mathbb{C}$ enjoys all the rules of $\mathbb{R}$.

c) $\mathbb{C}$ is a *field.*

d) $\mathbb{C}$ is algebraically closed.

e) Euler formula: $z = x + iy = r \exp i\theta$.

f) Euclidean distance: $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

g) Equation of circle with radius r:

$$z\overline{z} = x^2 + y^2 = r^2.$$

3. Let $u := \sqrt{1} \notin I\!R$ **leads to** *hyperbolic numbers*

$w = x + uy \in I\!H.$

a) $w = x + uy \in I\!H$ in the *standard basis* $\{1, u\}$.

b) $u^2 = 1$ and $I\!H$ enjoys all the rules of $I\!R$, except $I\!H$ has divisors of zero.

c) $w = w_+ u_+ + w_- u_-$ in the *spectral basis* $\{u_+, u_-\}$ where $w_+ = x + y$ and $w_- = x - y$, and

$u_+ = \frac{1}{2}(1 + u)$ and $u_- = \frac{1}{2}(1 - u)$. Note that

$u_+ + u_- = 1,\ u_+^2 = u_+,\ u_-^2 = u_-,\ u_+ u_- = 0.$

d) $I\!H$ is a *ring.*

e) $I\!H$ is NOT algebraically closed.

f) Euler formula: $w = x + uy = \pm\rho \exp u\phi$,

$w = x + uy = \pm\rho u \exp u\phi.$

g) Hyperbolic distance: $|w_1 - w_2| = \sqrt{|(x_1 - x_2)^2 - (y_1 - y_2)^2|}.$

h) Equation of hyperbola (4 branches):

$|ww^-| = |x^2 - y^2| = \rho^2.$

4. *Complex Hyperbolic Numbers*

$\mathbb{H}_{\mathbb{C}} = span_{\mathbb{R}}\{1, u, i, ui = iu\},\ w = z_1 + z_2 u.$

a) $w = z_1 + z_2 u \in \mathbb{H}_{\mathbb{C}}$ in the *standard basis over* $\mathbb{C}$ $\{1, u\}$.

b) $\mathbb{H}_{\mathbb{C}}$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{H}$ has zero divisors.

c) $w = w_+ u_+ + w_- u_-$ in the *spectral basis*$_{\mathbb{C}}$ $= \{u_+, u_-\}$ where $w_+ = z_1 + z_2$ and $w_- = z_1 - z_2$,

$u_+ = \frac{1}{2}(1 + u)$ and $u_- = \frac{1}{2}(1 - u)$. Note that

$u_+ + u_- = 1,\ u_+^2 = u_+,\ u_-^2 = u_-,\ u_+ u_- = 0.$

d) $\mathbb{H}_{\mathbb{C}}$ is a *ring*.

e) $\mathbb{H}_{\mathbb{C}}$ is algebraically closed:

The zeros of any polynomial

$f(w) = f(w_+)u_+ + f(w_-)u_-$, are just the complex zeros of $f(w_+) = 0$ and $f(w_-) = 0$.

f) Euler formula:

$w = z_1 + z_2 u = Z(\frac{z_1}{Z} + \frac{z_1}{Z} u) = Z \exp \Omega u,$

$Z = \sqrt{z_1^2 - z_2^2}$, and Ω is complex hyperbolic angle. (Singular at $z_1 = \pm z_2$.)

g) Complex Hyperbolic distance to origin:

$|w| = \sqrt{Z\overline{Z}}.$

h) Equation of 3-D hypersurface:

$Z\overline{Z} = \rho^2.$

5. Geometric Numbers of the Plane $\mathbb{G}_2$.

$\mathbb{G}_2 = span_{\mathbb{R}}\{1, e_1, e_2, e_1 e_2\},$

where $e_1^2 = e_2^2 = 1$ and $e_{12} := e_1 e_2 = -e_2 e_1$.

$i = e_{12}$ has the geometric interpretation of a *unit bivector* in the plane of e_1 and e_2, and $i^2 = -1$.

a) $g = (\alpha_1 + \alpha_2 e_{12}) + (v_1 e_1 + v_2 e_2)$

in the *standard basis* $\{1, e_1, e_2, e_1 e_2\}$.

b) $\mathbb{G}_2$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_2$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_2$ is

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u_+ & e_1 u_- \\ e_1 u_+ & u_- \end{pmatrix},$$

where $u_\pm = \frac{1}{2}(1 \pm e_2)$.

d) Let $g = (\alpha_1 + \alpha_2 i) + (v_1 e_1 + v_2 e_2)$. Then

$$g = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} \alpha_1 + v_2 & v_1 - \alpha_2 \\ v_1 + \alpha_2 & \alpha_1 - v_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.$$

$$[g] = u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_-$$

$$= \begin{pmatrix} \alpha_1 + v_2 & v_1 - \alpha_2 \\ v_1 + \alpha_2 & \alpha_1 - v_2 \end{pmatrix}$$

is called the (real) *matrix* of g.

e) $\mathbb{G}_2$ is algebraically closed:

Using d) above, the zeros of any polynomial $f(g)$ can be defined in terms of the zeros of the matrix $f([g])$.

6. Geometric Numbers of 3-Space $\mathbb{G}_3$.

$\mathbb{G}_3 = span_{\mathbb{R}}\{1, e_1, e_2, e_3, e_2e_3, e_3e_1, e_1e_2, i = e_{123}\}$,

where $e_1^2 = e_2^2 = e_3^2 = 1$ and $e_{ij} := e_ie_j = -e_je_i$ for $i \neq j$.

$i = e_{123}$ has the geometric interpretation of a *unit trivector* and $i^2 = -1$.

a) $g = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

in the *standard basis* $\{1, e_1, e_2, e_3\}_{\mathbb{C}}$.

b) $\mathbb{G}_3$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_3$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_3$ is

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u_+ & e_1u_- \\ e_1u_+ & u_- \end{pmatrix},$$

where $u_\pm = \frac{1}{2}(1 \pm e_2)$.

d) For $g = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$,

$$g = (\,1 \quad e_1\,)\, u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g\, (\,1 \quad e_1\,)\, u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= (\,1 \quad e_1\,)\, u_+ \begin{pmatrix} g & ge_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$= (\,1 \quad e_1\,)\, u_+ \begin{pmatrix} \alpha_0 + \alpha_2 & \alpha_1 + i\alpha_3 \\ \alpha_1 - i\alpha_3 & \alpha_0 - \alpha_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.$$

$$[g] = u_+ \begin{pmatrix} g & ge_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_-$$

$$= \begin{pmatrix} \alpha_0 + \alpha_2 & \alpha_1 + i\alpha_3 \\ \alpha_1 - i\alpha_3 & \alpha_0 - \alpha_2 \end{pmatrix}.$$

is called the (complex) *matrix* of g.

e) $\mathbb{G}_3$ is algebraically closed:

Using d) above, the zeros of any polynomial $f(g)$ can be difined in terms of the zeros of the matrix $f([g])$.

7. Geometric Numbers of the (4,1)-Space $\mathbb{G}_{4,1}$.

$\mathbb{G}_{4,1} = span_{\mathbb{R}}\{\mathbb{G}_3, e_4, e_5\}$,

where $e_4^2 = 1 = -e_5^2$, and $i := e_{12345}$

has the geometric interpretation of a

unit 5-vector, and $i^2 = -1$.

a) $g = (h_0 + ih_2) + (h_1 + ih_3)e_4$

in the *standard basis*

$\{h_0 + ih_2, (h_1 + ih_3)e_4 | \ h_0, h_1, h_2, h_3 \in \mathbb{G}_3\}_{\mathbb{G}_3}$.

b) $\mathbb{G}_{4,1}$ enjoys all the rules of $\mathbb{R}$, except $\mathbb{G}_{4,1}$ is not universally commutative and has zero divisors.

c) The *spectral basis* of $\mathbb{G}_{4,1}$ over $\mathbb{G}_3$ is

$$\begin{pmatrix} 1 \\ e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 & e_4 \end{pmatrix} = \begin{pmatrix} v_+ & e_4 v_- \\ e_4 v_+ & v_- \end{pmatrix},$$

where $v_\pm = \frac{1}{2}(1 \pm e_{45})$.

d) For $g = (h_0 + ih_2) + (h_1 + ih_3)e_4$,

$$g = (\,1 \quad e_4\,)\, v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix} g\,(\,1 \quad e_4\,)\, v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix}$$

$$= (\,1 \quad e_4\,)\, v_+ \begin{pmatrix} g & ge_4 \\ e_4 g & e_4 g e_4 \end{pmatrix} v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix}$$

$$= (\,1 \quad e_4\,)\, v_+ \begin{pmatrix} h_0 + ih_2 & h_1 + ih_3 \\ \tilde{h}_1 - i\tilde{h}_3 & \tilde{h}_0 - i\tilde{h}_2 \end{pmatrix} \begin{pmatrix} 1 \\ e_4 \end{pmatrix}.$$

$$[g] = v_+ \begin{pmatrix} g & ge_4 \\ e_4 g & e_4 g e_4 \end{pmatrix} v_+ + v_- \begin{pmatrix} e_4 g e_4 & e_4 g \\ ge_4 & g \end{pmatrix} v_-$$

$$= \begin{pmatrix} h_0 + ih_2 & h_1 + ih_3 \\ \tilde{h}_1 - i\tilde{h}_3 & \tilde{h}_0 - i\tilde{h}_2 \end{pmatrix}.$$

is called the (complex $\mathbb{G}_3$) *matrix* of g, where

for $h \in \mathbb{G}_3$, $\tilde{h} := e_4 h e_4$.

e) We can also represent g as a complex scalar 4×4 matrix, by employing the Kronecker product of matrices:

$$(\,1 \quad e_1\,)\,(\,1 \quad e_4\,)\, u_+ v_+ \begin{pmatrix} 1 \\ e_4 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix} =$$

$$(\,1 \quad e_1 \quad e_4 \quad e_{14}\,)\, u_+ v_+ \begin{pmatrix} 1 \\ e_1 \\ e_4 \\ e_{41} \end{pmatrix} = 1.$$

f) Similarly, the geometric algebras $\mathcal{G}_{n,n+1} \widetilde{=} Mat_{\mathbb{C}}(2^n \times 2^n)$.

f) The geometric algebras $\mathcal{G}_{n,n+1}$ are all algebraically closed.

Using d) above, the zeros of any polynomial $f(g)$ can be defined in terms of the zeros of the matrix $f([g])$.

PART II: Conformal Geometric Algebra in $\mathbb{G}_{4,1}$

1. The matrix geometric algebra $G_{4,1}$.

The basis elements of $\mathbb{G}_{4,1} = Mat_{\mathbb{G}_3}(2 \times 2)$ are:

$$[e_k] = \begin{pmatrix} e_k & 0 \\ 0 & -e_k \end{pmatrix}, \; k = 1, 2, 3.$$

$$[\sigma] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$[\gamma] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$[u] = [\sigma\gamma] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[u_+] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$[u_-] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Other elements are calcuated by taking sums and products of the matrix representations of the vector basis elements.

For example, for $e = \frac{1}{2}(\sigma + \gamma)$ and $\overline{e} = \sigma - \gamma$,

$$[e] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [\overline{e}] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

A (real) vector $x \in R^{4,1}$ can be written

$$x = \mathbf{x} + \alpha e + \frac{1}{2}\beta\overline{e},$$

where $\mathbf{x} \in R^3$ and $\alpha, \beta \in R$.

Definition: By a *complex vector* $[x] \in M_{2\times 2}$ we mean any element of the form $[x] = \begin{pmatrix} \mathbf{x} & \beta \\ \alpha & -\mathbf{x} \end{pmatrix}$ for $\mathbf{x} \in G_3^{1+2}$ and $\alpha, \beta \in G_3^{0,3}$.

A general *complex vector* $x \in G_{4,1}^{1+4}$ has the form

$$x = \mathbf{x} + iu\mathbf{y} + (\alpha_1 + iu\alpha_2)e + \frac{1}{2}(\beta_1 + iu\beta_2)\overline{e}$$

The matrix representation of the real x is

$$[x] = \begin{pmatrix} \mathbf{x} & \beta \\ \alpha & -\mathbf{x} \end{pmatrix},$$

and for the complex x,

$$[x] = \begin{pmatrix} \mathbf{x} + i\mathbf{y} & \beta_1 + i\beta_2 \\ \alpha_1 + i\alpha_2 & -\mathbf{x} - i\mathbf{y} \end{pmatrix}.$$

The determinant of $[x]$ for both the real and complex x is

$$\det[x] = \mathbf{x}^4 + 2\alpha\beta\mathbf{x}^2 + \alpha^2\beta^2 = (\mathbf{x}^2 + \alpha\beta)^2.$$

The *pseudodeterminant* of $[x]$ is

$$pdet[x] := -(x^2 + \alpha\beta),$$

and x is invertible iff $pdet[x] \neq 0$.

2. The group $G^*_{4,1}$ of all invertible elements of $G_{4,1}$ is isomorphic to the general linear group $M_{4\times 4}(C)$.

3. The *Lipschitz subgroup* $\Gamma_{4,1}$ of $G^*_{4,1}$ consists of those elements in $G^*_{4,1}$ for which $gx\overline{g} \in R^{4,1}$ for all $x \in R^{4,1}$ and is generated by the product of invertible vectors $x \in R^{4,1}$.

4. The *complex Lipschitz subgroup*

$\Gamma^c_{4,1}$ of $G^*_{4,1}$

consists of those elements of $g \in G_{4,1}^{*}$ for which

$$gxg^{\dagger} \in G_{4,1}^{1+4} \text{ for all } x \in G_{4,1}^{1+4}.$$

Letting $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$[g][x][g^{\dagger}] = \begin{pmatrix} a\mathbf{x}\overline{d} + \alpha b\overline{d} + \beta a\overline{c} - b\mathbf{x}\overline{c} & a\mathbf{x}\overline{b} + \alpha b\overline{b} + \beta a\overline{a} - b\mathbf{x}\overline{a} \\ c\mathbf{x}\overline{d} + \alpha d\overline{d} + \beta c\overline{c} - d\mathbf{x}\overline{c} & c\mathbf{x}\overline{b} + \alpha d\overline{b} + \beta c\overline{a} - d\mathbf{x}\overline{a} \end{pmatrix}.$$

Examining the complex products,

$$< a\mathbf{x}\overline{d} - b\mathbf{x}\overline{c} >_{0+3} = \mathbf{x} \circ [b_0\mathbf{c} - a_0\mathbf{d} + d_0\mathbf{a} - c_0\mathbf{b} + \mathbf{a} \otimes \mathbf{d} - \mathbf{b} \otimes \mathbf{c}] = 0$$

for all $\mathbf{x}$, or equivalently, $< \overline{a}d - \overline{b}c >_{1+2} = 0$.

Also

$$< \alpha b\overline{d} + \beta a\overline{c} >_{0+3} = 0$$

for all $\alpha, \beta \in G_3^{0+3}$, or equivalently, $b\overline{d} = -d\overline{b}$ and $a\overline{c} = -c\overline{a}$.

5. The pseudodeterminant function is related to the ordinary determinant function for elements $g \in \Gamma_{4,1}^{c}$.

$$\det[g] = \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = (\overline{d}a\overline{a} - \overline{b}a\overline{c})\frac{1}{a\overline{a}}(a\overline{a}d - c\overline{a}b)$$

$$= (\overline{d}a\overline{a} + \overline{b}c\overline{a})\frac{1}{a\overline{a}}(a\overline{a}d + a\overline{c}b)$$

$$= (\overline{d}a + \overline{b}c)(\overline{a}d + \overline{c}b)$$

$$= (\overline{a}d + \overline{c}b)^2 = (\mathrm{pdet}[g])^2,$$

since $\overline{a}d + \overline{c}b =< \overline{a}d + \overline{c}b >_{0+3}$.

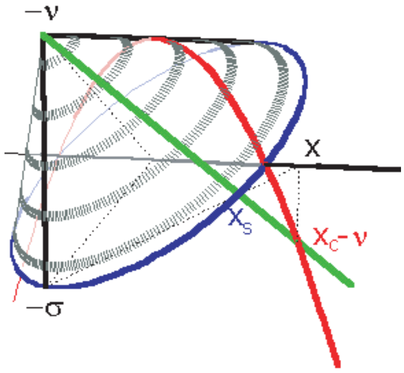

Figure 1: The Horosphere in $I\!R^3$.

6. The 3-Affine space $\mathcal{A}_e(R^3)$.

DEFINITION: The *affine space* $\mathcal{A}_e(R^3) := \{x_h = \mathbf{x} + e|\ \mathbf{x} \in R^3\}$.

Note that $x_h \cdot \overline{e} = 1$ for all $x_h \in \mathcal{A}_e(R^3)$.

7. The 3-Dimensional Horosphere.

DEFINITION: The *horosphere* $H(R^3)$, is defined by the condition that $x_c := x_h + \beta\overline{e}$ is a null vector for all $x_h = \mathbf{x} + e \in \mathcal{A}_e(R^3)$.

Calculating

$$x_c^2 = x_h^2 + 2\beta\overline{e} \cdot x_h = \mathbf{x}^2 + 2\beta = 0$$

or $\beta = -\frac{1}{2}\mathbf{x}^2$. Thus,

$$H(I\!R^3) := \{x_c = \mathbf{x} - \frac{\mathbf{x}^2}{2}\overline{e} + e| \ \ \mathbf{x} \in R^3\}.$$

The horosphere consists of homogeneous points, since

$x_c = \frac{\alpha x_c}{\overline{e}\cdot(\alpha x_c)}$ for all $x_c \in H(R^3)$

and $\alpha \in R^*$.

Calculating $[x_c]$,

$$[x_c] = [x_h] - [\frac{\mathbf{x}^2}{2}\overline{e}] = \begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & -\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{x} \end{pmatrix}.$$

By the *column h-twistor* $[x_c]_t$, we mean

$$[x_c]_t = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}.$$

By the space of *column h-twistor*

$\mathcal{T}_{G_3}$ of G_3

we mean

$$\mathcal{T}_{G_3} := \{[w]_t = \begin{pmatrix} a \\ b \end{pmatrix} | \ \ a, b \in G_3\}.$$

For the column h-twistor

$$[w]_t = \begin{pmatrix} a \\ b \end{pmatrix},$$

we define a *conjugate row h-twistor* by

$$[w]_t^\dagger = (\,\overline{b} \quad \overline{a}\,).$$

The *h-twistor inner product* is

$$< [w_1]_t, [w_2] >_t := [w_1]_t^\dagger [w_2]_t = \overline{b_1} a_2 + \overline{a_1} b_2 \in G_3^{1+3},$$

where $[w_1]_t = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $[w_2]_t = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ are h-twistors.

8. We can now express any point x_c on the horosphere by

$$[x_c] = [x_c]_t [x_c]_t^\dagger.$$

Actually, we can do much more.

DEFINITION: (*Equivalence of h-twistors*)

$[w_1]_t \equiv [w_2]_t$ iff $[w_1]_t [w_1]_t^\dagger = [w_2]_t [w_2]_t^\dagger$,

and that they are *projectively equivalent* iff

$[w_1]_t [w_1]_t^\dagger = \alpha [w_2]_t [w_2]_t^\dagger$ for $\alpha \in R^*$.

It follows that $[x_c]_t$ and $[x_c h]_t$ are projectively equivalent for all $h \in G_3$ such that $h\overline{h} \in R^*$.

Thus points on the horosphere need only be defined up to a invertible multivector $h \in G_3$.

The concept of an h-twistor cuts calculations on the horosphere in half. For example, for any $g \in G_{4,1}$ with $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$[g x_c g^\dagger] = [g][x_c][g]^\dagger = [g][x_c]_t([g][x_c]_t)^\dagger.$$

Reflections on the horosphere have the form

$$S_{\mathbf{a}}(x_c) = \mathbf{a}u x_c(\mathbf{a}u)^\dagger = -\mathbf{a}u x_c \mathbf{a}u.$$

In terms of the h-twistor representation, we have

$$[S_{\mathbf{a}}(x_c)] = ([\mathbf{a}u][x_c]_t)([\mathbf{a}u][x_c]_t)^\dagger$$

$$= \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} \mathbf{ax} \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} -\mathbf{a} & \mathbf{xa} \end{pmatrix} = \begin{pmatrix} -\mathbf{axa} & \mathbf{a}^2\mathbf{x}^2 \\ -\mathbf{a}^2 & \mathbf{axa} \end{pmatrix}.$$

Rotations are the composition of two reflections. We find that

$$S_{\mathbf{b}}S_{\mathbf{a}}(x_c) = \mathbf{ba}x_c(\mathbf{ba})^{\dagger} = \mathbf{ba}x_c\mathbf{ab}.$$

In terms of the h-twistor construction, we find

$$[S_{\mathbf{c}}S_{\mathbf{a}}(x_c)] = ([\mathbf{ba}][x_c]_t)([\mathbf{ba}][x_c]_t)^{\dagger}$$

$$= \begin{pmatrix} \mathbf{ba} & 0 \\ 0 & \mathbf{ba} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} \mathbf{ba} & 0 \\ 0 & \mathbf{ba} \end{pmatrix}^{\dagger}$$

$$= \begin{pmatrix} \mathbf{bax} \\ \mathbf{ba} \end{pmatrix} \begin{pmatrix} \mathbf{ab} & -\mathbf{xab} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{baxab} & -\mathbf{a}^2\mathbf{b}^2\mathbf{x}^2 \\ \mathbf{a}^2\mathbf{b}^2 & -\mathbf{baxab} \end{pmatrix}.$$

We can also represent *translations* in the horosphere. For $\mathbf{a} \in R^3$,

$$T_{\mathbf{a}}(x_c) := (1 + \frac{\mathbf{a}\overline{e}}{2})x_c(1 - \frac{\mathbf{a}\overline{e}}{2}).$$

The expression for translations on the horosphere is simpler than translations in affine space, since the last term in the expression for translation in affine space is no longer necessary.

In terms of the h-twistor construction,

$$[T_{\mathbf{a}}(x_c)] = [1 + \frac{\mathbf{a}\overline{e}}{2}][x_c]_t([1 + \frac{\mathbf{a}\overline{e}}{2}][x_c]_t)^\dagger$$

$$= \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}^\dagger =$$

$$\begin{pmatrix} \mathbf{x} + \mathbf{a} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{x} - \mathbf{a} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{x} + \mathbf{a} & -(\mathbf{x} + \mathbf{a})^2 \\ 1 & -\mathbf{x} - \mathbf{a} \end{pmatrix}.$$

For a general element $g \in G^*_{4,1}$, with $[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the h-twistor transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} a\mathbf{x} + b \\ c\mathbf{x} + d \end{pmatrix},$$

leads to the general linear fraction *Möbius transformation* or conformal transformation

$$f(\mathbf{x}) = (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1},$$

because of the projective equivalence of the h-twistors

$$\begin{pmatrix} a\mathbf{x} + b \\ c\mathbf{x} + d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1} \\ 1 \end{pmatrix}$$

at all points for which $(c\mathbf{x} + d)^{-1}$ is defined.

The linear fractional transformation $f(\mathbf{x})$ defines a conformal transformation:

Letting $f = f(\mathbf{x})$ we calculate the differential of

$f(\mathbf{x})(c\beta x + d) = (a\mathbf{x} + b)$, getting

$df(c\mathbf{x} + d) + fcd\mathbf{x} = ad\mathbf{x}$ or

$df = (a - fc)d\mathbf{x}(c\mathbf{x} + d)^{-1}$. Continuing the calculation,

$$df = \frac{[a(c\mathbf{x} + b)(\overline{d} - \mathbf{x}\overline{c}) - (a\mathbf{x} + b)(\overline{d} - \mathbf{x}\overline{c})c]d\mathbf{x}(\overline{d} - \mathbf{x}\overline{c})}{(c\mathbf{x} + b)^2(\overline{d} - \mathbf{x}\overline{c})^2}.$$

Simplifying the first part of the numerator,

$$a(c\mathbf{x} + b)(\overline{d} - \mathbf{x}\overline{c}) - (a\mathbf{x} + b)(\overline{d} - \mathbf{x}\overline{c})c$$

$$= ac\mathbf{x}\overline{d} - ad\mathbf{x}\overline{c} + b\mathbf{x}\overline{c}c - a\mathbf{x}\overline{d}c + ad\overline{d} - ac\overline{c}\mathbf{x}^2 - b\overline{d}c + a\mathbf{x}^2 c\overline{c}$$

$$= ac\mathbf{x}\overline{d} - ad\mathbf{x}\overline{c} + b\mathbf{x}\overline{c}c - a\mathbf{x}\overline{d}c + d(\overline{d}a + \overline{b}c)$$

$$= ac\mathbf{x}\overline{d} + (ad\mathbf{x} + b\mathbf{x}c)\overline{c} - a\mathbf{x}\overline{d}c + d\ \mathrm{pdet}(g)$$

$$= \cdots = -a\overline{c}d\mathbf{x} + b\overline{c}c\mathbf{x} + d\,\mathrm{pdet}(g) = \mathrm{pdet}(g)(c\mathbf{x}+d).$$

Finally, we get

$$df = \frac{\mathrm{pdet}(g)(c\mathbf{x}+d)d\mathbf{x}(\overline{d} - \mathbf{x}\overline{c})}{(c\mathbf{x}+d)^2(\overline{d} - \mathbf{x}\overline{c})^2}$$

$$= \mathrm{pdet}(g)(\overline{d} - \mathbf{x}\overline{c})^{-1} d\mathbf{x}(c\mathbf{x}+d)^{-1}.$$

Squaring both sides gives

$$(df)^2 = \frac{\det g}{(c\mathbf{x}+d)^2(\mathbf{x}\overline{c} - \overline{d})^2}(d\mathbf{x})^2,$$

showing that $\beta y = f(\mathbf{x})$ is conformal at all points at which it is defined.