Hyperbolic Numbers Revisited

Garret Sobczyk
Universidad de las Américas-Puebla
Departamento de Físico-Matemáticas
72820 Puebla, Pue., México

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Abstract

In 1995, my article "The Hyperbolic Number plane" was published in the *College Mathematics Journal*. The unit four branched hyperbola in the hyperbolic number plane plays the same role for the hyperbolic trigonometric functions as the unit circle in the complex number plane plays for the trigonometric functions. Here we unify these different number planes into a much more powerful number system that is easy to learn, and which provides a fun stepping stone to higher mathematics.

0 Introduction

The development of the real and complex number systems was truly a multimillennial effort undertaken by key players from many civilizations with different pressing needs. Sheep had to be accounted for, records needed to be kept for increasingly complicated trading relationships, legs to reach the lofty dreams of understanding the motion of the planets and stars had to be invented, and the more sinister demands for ever more sophisticated weapons of mass killing and destruction had to be met. A beautiful accounting of this story is related by Tobias Danzig in his book, "Number the Language of Science" [1].

The reader may have the impression that with the development of the real and complex number systems, the job had been completed and that these number systems form the basis of all higher mathematics. On the contrary, we show that there is a very simple extension of the real and complex number systems that is easy to learn, and gives insight into higher mathematics that is usually considered in the domain of the experts and out of reach of the beginning student. Indeed, there are many geometric extensions of the number system that should be incorporated into the concept of number itself [2, 3]. In another paper [4], meant to demonstrate the power and naturalness of the *geometric extension of number*, the authors show how the famous Pythagorean theorem for a right-triangle serves as an elementary model for constructing the geometric multiplication of vectors.

1 Real new numbers

The real number system $\mathbb{R}$ is generally represented on the *real number line*. What makes the real number system so powerful is the ability to both *add* and *multiply* real numbers to get other real numbers. We assume that the reader is familiar with the addition and multiplication of real numbers, and all of the basic properties of these operations. We review only the properties that are most pertinent to what we wish to do here. For $r, s, t \in \mathbb{R}$,

R1) $rs = sr$ *Commutative law of multiplication.*

R2) $r(s+t) = rs + rt$ *Distributive law of multiplication over addition.*

R3) $(rs)t = r(st)$ *Associative law of multiplication.*

R4) $rs = 0 \iff r = 0$ and/or $s = 0$.

We now introduce two *new numbers* $\mathbf{a}$ and $\mathbf{b}$ not in $\mathbb{R}$. To emphasize that $\mathbf{a}$ and $\mathbf{b}$ are *not* real numbers, we write $\mathbf{a} \notin \mathbb{R}$ and $\mathbf{b} \notin \mathbb{R}$. To make the extended number system $\mathcal{N} := \mathbb{R}(\mathbf{a}, \mathbf{b})$ fully functional, and compatible with $\mathbb{R}$, we extend the operations of addition and multiplication to include the new numbers $\mathbf{a}, \mathbf{b}$. This is accomplished by assuming that the extended numbers in $\mathcal{N}$ obey exactly the same rules of addition and multiplication as do the numbers in $\mathbb{R}$, with the *exception* that $\mathbf{ab} \neq \mathbf{ba}$. Regarding the the new numbers $\mathbf{a}$ and $\mathbf{b}$, they satisfy the following two special properties:

N1) $\mathbf{a}^2 = 0 = \mathbf{b}^2$ *The new numbers* $\mathbf{a}$ *and* $\mathbf{b}$ *are nilpotents.*

N2) $\mathbf{ab} + \mathbf{ba} = 1$ *The sum of* $\mathbf{ab}$ *and* $\mathbf{ba}$ *is* 1.

In addition, we assume that numbers in $\mathcal{N}$ *commute* with real numbers in $\mathbb{R}$ and that the associative and distributive properties R2) and R3) above, remain valid for our new numbers. Because of the geometric interpretation that we give these new numbers, we call them g-numbers. The g-numbers in the table

$$\begin{pmatrix} \mathbf{ab} & \mathbf{a} \\ \mathbf{b} & \mathbf{ba} \end{pmatrix} \tag{1}$$

are called the *standard canonical basis* of $\mathcal{N} = \mathbb{R}(\mathbf{a}, \mathbf{b})$ over the real numbers.

Clearly the non-zero g-numbers $\mathbf{a}$ and $\mathbf{b}$ cannot be real numbers because $\mathbf{a}^2 = 0 = \mathbf{b}^2$, since there are no non-zero real numbers with this property. Any g-number g such that $g^2 = 0$ is said to be a *nilpotent*. Also, the products $\mathbf{ab}$ and $\mathbf{ba}$ cannot be real numbers since $\mathbf{ab} \neq \mathbf{ba}$. Never-the-less the property N2) tells us that the *sum* $\mathbf{ab} + \mathbf{ba} = 1 \in \mathbb{R}$, providing a direct relationship between the extended g-numbers in $\mathcal{N}$ and the real numbers, and showing that $\mathbb{R} \subset \mathcal{N} = \mathbb{R}(\mathbf{a}, \mathbf{b})$.

Each g-number $g \in \mathcal{N}$ is uniquely specified by four real numbers, which we write in tabular form $[g] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. Thus,

$$g = g_{11}\mathbf{ab} + g_{12}\mathbf{a} + g_{21}\mathbf{b} + g_{22}\mathbf{ba} \quad \leftrightarrow \quad [g] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \tag{2}$$

where $g_{11}, g_{12}, g_{21}, g_{22} \in \mathbb{R}$.[1] The *Multiplication Table* 1 for g-numbers is easily derived from the assumed properties N1) and N2), and the associative law. Since half of its entries are zeros, it is easily remembered.

Table 1: Multiplication table.

	a	**b**	**ab**	**ba**
a	0	**ab**	0	**a**
b	**ba**	0	**b**	0
ab	**a**	0	**ab**	0
ba	0	**b**	0	**ba**

To show that $\mathbf{aba} = \mathbf{a}$, we use both properties N1) and N2), and in particular N2) to substitute in $1 - \mathbf{ba}$ for $\mathbf{ab}$, getting

$$\mathbf{aba} = (\mathbf{ab})\mathbf{a} = (1 - \mathbf{ba})\mathbf{a} = \mathbf{a} - \mathbf{ba}^2 = \mathbf{a},$$

and similarly, $\mathbf{bab} = \mathbf{b}$. The same substitution works for showing that

$$(\mathbf{ab})^2 = (1 - \mathbf{ba})(\mathbf{ab}) = \mathbf{ab} + \mathbf{ba}^2\mathbf{b} = \mathbf{ab},$$

and similarly that $(\mathbf{ba})^2 = \mathbf{ba}$. Any non-zero g-number g with the property that $g^2 = g$ is said to be an *idempotent*, so $\mathbf{ab}$ and $\mathbf{ba}$ are idempotents, and since

$$\mathbf{ab} + \mathbf{ba} = 1, \quad \text{and} \quad (\mathbf{ab})(\mathbf{ba}) = 0 = (\mathbf{ba})(\mathbf{ab}),$$

they are said to *partition unity* and to be *mutually annihilating.*

We can now easily derive the general rule for the addition and multiplication of two g-numbers $f, g \in \mathcal{N}$. In addition to g, already defined, let

$$f = f_{11}\mathbf{ab} + f_{12}\mathbf{a} + f_{21}\mathbf{b} + f_{22}\mathbf{ba} \quad \leftrightarrow \quad [f] := \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Calculating $f + g$ and fg, we find that

$$f + g = (f_{11} + g_{11})\mathbf{ab} + (f_{12} + g_{12})\mathbf{a} + (f_{21} + g_{21})\mathbf{b} + (f_{22} + g_{22})\mathbf{ba}$$

$$\longleftrightarrow \quad [f + g] = \begin{pmatrix} f_{11} + g_{11} & f_{12} + g_{12} \\ f_{21} + g_{21} & f_{22} + g_{22} \end{pmatrix}$$

for addition, and

$$fg = (f_{11}g_{11} + f_{12}g_{21})\mathbf{ab} + (f_{11}g_{12} + f_{12}g_{22})\mathbf{a}$$

$$+(f_{21}g_{11} + f_{22}g_{21})\mathbf{b} + (f_{21}g_{12} + f_{22}g_{22})\mathbf{ba}$$

[1] We include an Appendix on the relationship of our new number system $\mathcal{N}$ to real 2×2 matrices.

$$\longleftrightarrow \quad [fg] = \begin{pmatrix} f_{11}g_{11} + f_{12}g_{21} & f_{11}g_{12} + f_{12}g_{22} \\ f_{21}g_{11} + f_{22}g_{21} & f_{21}g_{12} + f_{22}g_{22} \end{pmatrix},$$

for multiplication. Readers who know how to add and multiply matrices will be surprised to find that

$$[f+g] = [f] + [g] \quad \text{and} \quad [fg] = [f][g].$$

We leave further discussion of this relationship to the Appendix.

2 Conjugations and inverses

To complete our new number system, we define three powerful *conjugation operators* on $\mathcal{N}$. First note that each g-number $g \in \mathcal{N}$, with respect to the canonical basis (1), is the sum of two parts, $g = g_o + g_e$, an *odd part* g_o and an *even part* g_e, where

$$g_o := g_{12}\mathbf{a} + g_{21}\mathbf{b}, \quad \text{and} \quad g_e := g_{11}\mathbf{ab} + g_{22}\mathbf{ba}, \tag{3}$$

respectively.

We define the *reverse* $g^\dagger$ of g, with respect to the standard canonical basis (1), by

$$g^\dagger := (g_o + g_e)^\dagger = g_o^\dagger + g_e^\dagger = g_o + g_e^\dagger, \tag{4}$$

where

$$g_o^\dagger := g_o \quad \text{and} \quad g_e^\dagger = g_{11}\mathbf{ba} + g_{22}\mathbf{ab}.$$

The reverse operation reverses the order of the multiplication of $\mathbf{a}$ and $\mathbf{b}$, *i.e.*, $(\mathbf{ab})^\dagger = \mathbf{ba}$, leaving the odd part g_o unaffected. It follows that for $f, g \in \mathcal{N}$,

$$(f+g)^\dagger = f^\dagger + g^\dagger \quad \text{and} \quad (fg)^\dagger = g^\dagger f^\dagger.$$

The *inversion* g^- of g, with respect to the standard canonical basis (1), is defined by

$$g^- := (g_o + g_e)^- = g_o^- + g_e^- = -g_o + g_e, \tag{5}$$

where

$$g_o^- := -g_{12}\mathbf{a} - g_{21}\mathbf{b} = -g_o \quad \text{and} \quad g_e^- = g_e.$$

The inverse operation changes the *sign* of both $\mathbf{a}$ and $\mathbf{b}$, *i.e.*, $\mathbf{a}^- = -\mathbf{a}$ and $\mathbf{b}^- = -\mathbf{b}$, leaving g_e unaffected. Clearly

$$g_o = \frac{1}{2}(g - g^-), \quad \text{and} \quad g_e = \frac{1}{2}(g + g^-),$$

and for $f, g \in \mathcal{N}$,

$$(f+g)^- = f^- + g^- \quad \text{and} \quad (fg)^- = f^- g^-.$$

Combining the operations of reverse and inversion gives the third *mixed conjugation*. For $g \in \mathcal{N}$, the mixed conjugation g^* of g, with respect to the standard canonical basis (1), is defined by

$$g^* := (g^\dagger)^- = (g_o + g_e^\dagger)^- = -g_o + g_e^\dagger. \tag{6}$$

The mixed conjugation of the sum and product of $f, g \in \mathcal{N}$, satisfies

$$(f+g)^* = f^* + g^* = -(f_o + g_o) + (f_e^\dagger + g_e^\dagger),$$

and

$$(fg)^* = g^* f^* = (-g_o + g_e^\dagger)(-f_o + f_e^\dagger) = (fg)_o^* + (fg)_e^*,$$

where

$$(fg)_o^* = -(g_o f_e^\dagger + g_e^\dagger f_o)$$

and

$$(fg)_e^* = g_o f_o + g_e^\dagger f_e^\dagger.$$

Using the mixed conjugation, we find that for $g \in \mathcal{N}$,

$$tr(g) := g + g^* = g_e + g_e^\dagger = g_{11} + g_{22} \in \mathbb{R}, \tag{7}$$

called the *trace* of g. Also, using that an even g-number times an odd g-number is odd, we calculate

$$\begin{aligned} \det g := gg^* &= (g_o + g_e)(-g_o + g_e^\dagger) \\ &= g_o g_e^\dagger - (g_o g_e^\dagger)^\dagger + g_e g_e^\dagger - g_o g_o \\ &= g_e g_e^\dagger - g_o g_o = g_{11} g_{22} - g_{12} g_{21} \in \mathbb{R}, \end{aligned} \tag{8}$$

since

$$g_e g_e^\dagger = (g_{11}\mathbf{ab} + g_{22}\mathbf{ba})(g_{11}\mathbf{ba} + g_{22}\mathbf{ab}) = g_{11} g_{22}$$

and

$$g_o g_o = (g_{12}\mathbf{a} + g_{21}\mathbf{b})^2 = g_{12} g_{21}(\mathbf{ab} + \mathbf{ba}) = g_{12} g_{21}.$$

For $g \in \mathcal{N}$, $\det g$ is called its *determinant.*

Given a g-number $g \in \mathcal{N}$, when is there a $f \in \mathcal{N}$ such that $gf = fg = 1$? When such an f exists, we say that $g^{-1} := f$ is the *multiplicative inverse* of g. Since

$$\frac{g^* g}{g^* g} = \frac{g g^*}{g g^*} = 1,$$

it immediately follows that

$$g^{-1} := \frac{g^*}{gg^*} = \frac{g^*}{g^* g} = \frac{-g_o + g_e^\dagger}{gg^*}$$

provided that $gg^* \neq 0$. Whenever a g-number has the property that $\det g \neq 0$, we say that g is *non-sigular.* If $gg^* = 0$, we say that g is *singular.*

Given g-numbers $f, g \in \mathcal{N}$, the product fg can be decomposed into even and odd parts. We have

$$fg = (fg)_o + (fg)_e$$

where

$$(fg)_o = f_o g_e + f_e g_o \quad \text{and} \quad (fg)_e = f_o g_o + f_e g_e$$

The product of two g-numbers fg can also be decomposed into the sum of a *symmetric part* $f \circ g$ and a *skew-symmetric part* $f \otimes g$. We have

$$fg = \frac{1}{2}(fg + gf) + \frac{1}{2}(fg - gf) = f \circ g + f \otimes g, \tag{9}$$

where $f \circ g := \frac{1}{2}(fg + gf)$ and $f \otimes g := \frac{1}{2}(fg - gf)$. We give here a couple of useful vector analysis-like identities satisfied by the symmetric and skew-symmetric products of three odd g-numbers $f_o, g_o, h_o \in \mathcal{N}$.

$$f_o \otimes (g_o \otimes h_o) = (f_o \circ g_o)h_o - (f_o \circ h_o)g_o, \tag{10}$$

and

$$f_o \circ (g_o \otimes h) = 0 = (f \otimes g) \circ h. \tag{11}$$

For the odd g-numbers, $f = \mathbf{a}$ and $g = \mathbf{b}$, we find that

$$\mathbf{a} \circ \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \frac{1}{2} \quad \text{and} \quad \mathbf{a} \otimes \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}).$$

Squaring $\mathbf{a} \otimes \mathbf{b}$, gives

$$(\mathbf{a} \otimes \mathbf{b})^2 = \frac{1}{4}(\mathbf{ab} - \mathbf{ba})^2 = \frac{1}{4}\Big((\mathbf{ab})^2 + (\mathbf{ba})^2\Big) = \frac{1}{4},$$

so that $(\mathbf{a} \otimes \mathbf{b})^2 = (\mathbf{a} \circ \mathbf{b})^2$.

3 Geometry of $\mathcal{N}$

Much conceptual clarity is gained when it is possible to pictorially represent fundamental concepts. To pictorial represent properties of g-numbers

$$g = g_{11}\mathbf{ab} + g_{12}\mathbf{a} + g_{21}\mathbf{b} + g_{22}\mathbf{ba} = g_o + g_e \in \mathcal{N},$$

we picture the odd and even parts of g, separately, in the *odd g-number plane* $\mathcal{N}_o$, and in the *even g-number plane* $\mathcal{N}_e$, pictured in Figure 1. Because $\mathbf{a}$ and $\mathbf{b}$ are nilpotents, they are pictured on the 2-dimensional *light-cone* in $\mathcal{N}_o$, [5]. Similarly, since $\mathbf{ab}$ and $\mathbf{ba}$ are *singular idempotents*, they are also pictured on the light-cone in $\mathcal{N}_e$.

The even g-number plane $\mathcal{N}_e$ has been studied under different guises. In [6], it is referred to as the "hyperbolic number plane" because of the prominence of the four branched hyperbola. Whereas here we have placed the idempotents $\mathbf{ab}$ and $\mathbf{ba}$ along the asymtotes, in [6] the coordinate x-axis is the real line, and the coordinate y-axis is $u = 2(\mathbf{a} \otimes \mathbf{b})$, where $u^2 = 1$. This difference is no more than a change of basis, see Figure 1.

There are only two idempotents 0 and 1 in the real number system $\mathbb{R}$. Idempotents in $\mathcal{N}$ have a much richer structure. If $P^2 = P$ is an idempotent, then $(P^\dagger)^2 = P^\dagger$, $(P^-)^2 = P^-$ and $(P^*)^2 = P^*$ are also idempotents, so once a

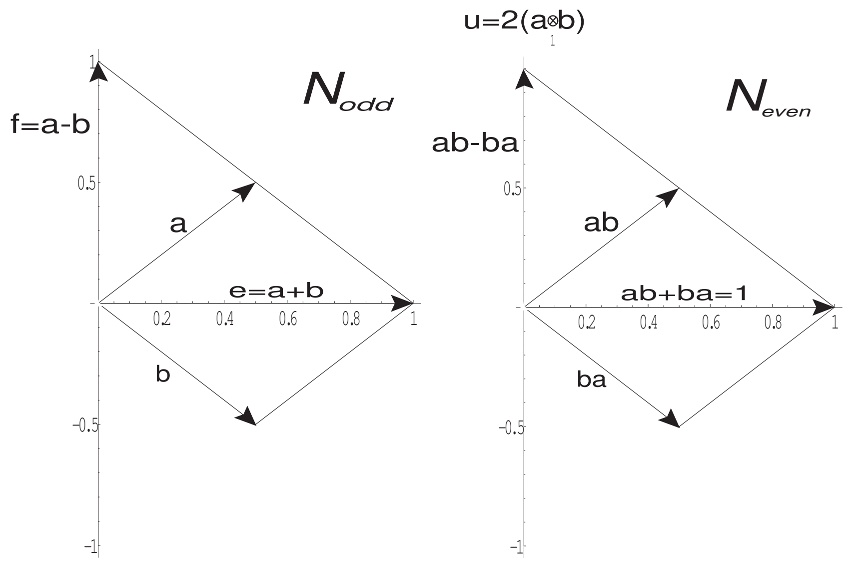

Figure 1: The odd number plane $\mathcal{N}_o$. The even number plane $\mathcal{N}_e$.

canonical form is found for P we have also found canonical forms for $P^\dagger, P^-$ and P^*. We characterize the idempotent

$$P = p_{11}\mathbf{ab} + p_{12}\mathbf{a} + p_{21}\mathbf{b} + p_{22}\mathbf{ba} \quad \longleftrightarrow \quad \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

in the following

Lemma 1 *Suppose the matrix* $[P]$ *of* $P \in \mathcal{N}$ *satisfies*

$$[P] := \begin{pmatrix} 1 - p_{22} & \frac{p_{22}(1-p_{22})}{p_{21}} \\ p_{21} & p_{22} \end{pmatrix}, \tag{12}$$

then P *is an idempotent with two degrees of freedom* $p_{21}, p_{22} \in \mathbb{R}$.

Proof: Since the matrix of P is given, it can be directly verified that $P^2 = P$. Instead, we will show that the assumption that $P^2 = P$ leads to the given matrix $[P]$ of P as a solution.

For $P = P_o + P_e$, we calculate

$$P^2 = (P_o + P_e)^2 = (P_oP_e + P_eP_o) + (P_o^2 + P_e^2) = P_o + P_e = P,$$

where

$$P_o = P_oP_e + P_eP_o \quad \text{and} \quad P_e = P_o^2 + P_e^2.$$

For $P_o = p_{12}\mathbf{a} + p_{21}\mathbf{b}$ and $P_e = p_{11}\mathbf{ab} + p_{22}\mathbf{ba}$, we calculate

$$P_o^2 = (p_{12}\mathbf{a} + p_{21}\mathbf{b})^2 = p_{12}p_{21}(\mathbf{ab} + \mathbf{ba}) = p_{12}p_{21},$$

$$P_e^2 = (p_{11}\mathbf{ab} + p_{22}\mathbf{ba})^2 = p_{11}^2\mathbf{ab} + p_{22}^2\mathbf{ba},$$

and similarly,

$$P_oP_e = p_{12}p_{22}\mathbf{a} + p_{11}p_{21}\mathbf{b} \;\text{ and }\; P_eP_o = p_{11}p_{12}\mathbf{a} + p_{21}p_{22}\mathbf{b}.$$

Substituting these values into the equations $P_oP_e + P_eP_o - P_o = 0$ and $P_o^2 + P_r^2 - P_e = 0$ leads to the respective equations

$$p_{12}(p_{11} + p_{22} - 1)\mathbf{a} + p_{21}(p_{11} + p_{22} - 1)\mathbf{b} = 0$$

and

$$(p_{11}^2 - p_{11} + p_{12}p_{21})\mathbf{ab} + (p_{22}^2 - p_{22} + p_{12}p_{21})\mathbf{ba} = 0.$$

Equating the coefficients of the canonical basis elements to zero gives a system of four linear equations in four unknowns, one of whose solutions is the matrix $[P]$ given in the Lemma. Other solutions are obtained by noting that $P^\dagger$, P^-, and P^* are all idempotents.

□

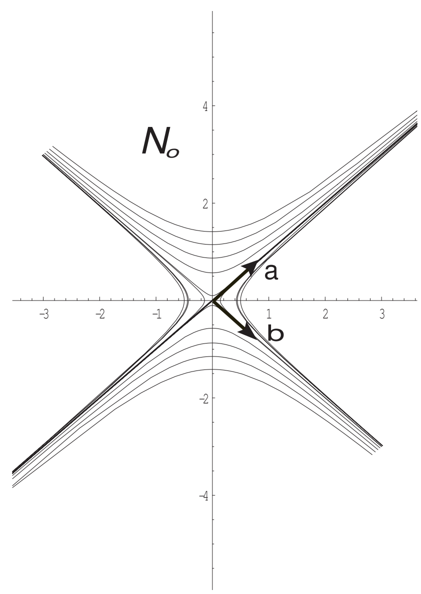

Figure 2: Each constant p_{22} defines a point in the even number plane $\mathcal{N}_e$ (not shown) and a hyperbola in the odd number plane $\mathcal{N}_o$.

If for $[P]$, given in (12), we consider p_{22} to be a constant, the resulting equation is a family of points in $\mathcal{N}_e$, defined by $p_{11} = 1 - p_{22}$, and a family of hyperbolas in $\mathcal{N}_o$, defined by $p_{12} = \frac{p_{22}(1-p_{22})}{p_{21}}$ shown in Figure 2.

Theorem 1 *Given a non-zero singular idempotent $P \in \mathcal{N}$.*

i) There exists a canonical basis

$$\mathcal{B} := \begin{pmatrix} \mathbf{AB} & \mathbf{A} \\ \mathbf{B} & \mathbf{BA} \end{pmatrix} \tag{13}$$

with the properties that $P = \mathbf{AB}$, $\mathbf{A}^2 = 0 = \mathbf{B}^2$ *and* $\mathbf{A} \circ \mathbf{B} = \frac{1}{2}$.

ii) For each canonical basis (13), there exists a $g \in \mathcal{N}$ *such that*

$$g^{-1} \begin{pmatrix} \mathbf{ab} & \mathbf{a} \\ \mathbf{b} & \mathbf{ba} \end{pmatrix} g = \begin{pmatrix} \mathbf{AB} & \mathbf{A} \\ \mathbf{B} & \mathbf{BA} \end{pmatrix}.$$

For the idempotent P. *with matrix the* $[P]$ *given in Lemma 1, the matrix of* g *is*

$$[g] := \begin{pmatrix} \frac{g_{12}p_{21}}{p_{22}} & g_{12} \\ g_{21} & -\frac{g_{21}(1-p_{22})}{p_{21}} \end{pmatrix},$$

with $\det[g] = -\frac{g_{12}g_{21}}{p_{22}}$.

Proof: *i)* For the non-zero singular idempotent $P = P_o + P_e$, with matrix (12) given in Lemma 1, we find a non-singular g-number $g = g_o + g_e$ such that $g^{-1}\mathbf{ab}g = P$, or equivalently,

$$\mathbf{ab}g_o = g_oP_e + g_eP_o \quad \text{and} \quad \mathbf{ab}g_e = g_eP_e + g_oP_o.$$

Calculating,

$$\mathbf{ab}g_o = \mathbf{ab}(g_{12}\mathbf{a} + g_{21}\mathbf{b}) = g_{12}\mathbf{a},$$

and

$$g_oP_e + g_eP_o = (g_{12}\mathbf{a} + g_{21}\mathbf{b})(p_{11}\mathbf{ab} + p_{22}\mathbf{ba}) + (g_{11}\mathbf{ab} + g_{22}\mathbf{ba})(p_{12}\mathbf{a} + p_{21}\mathbf{b})$$

$$= (g_{12}p_{22}\mathbf{a} + g_{21}p_{11}\mathbf{b}) + (g_{11}p_{12}\mathbf{a} + g_{22}p_{21}\mathbf{b})$$

$$= (g_{12}p_{22} + g_{11}p_{12})\mathbf{a} + (g_{21}p_{11} + g_{22}p_{21})\mathbf{b}.$$

Similarly, we calculate both sides of $\mathbf{ab}g_e = g_eP_e + g_oP_o$, getting

$$\mathbf{ab}g_e = \mathbf{ab}(g_{11}\mathbf{ab} + g_{22}\mathbf{ba}) = g_{11}\mathbf{ab},$$

and

$$g_eP_e + g_oP_o = (g_{11}\mathbf{ab} + g_{22}\mathbf{ba})(p_{11}\mathbf{ab} + p_{22}\mathbf{ba}) + (g_{12}\mathbf{a} + g_{21}\mathbf{b})(p_{12}\mathbf{a} + p_{21}\mathbf{b})$$

$$= (g_{11}p_{11}\mathbf{ab} + g_{22}p_{22}\mathbf{ba}) + (g_{12}p_{21}\mathbf{ab} + g_{21}p_{12}\mathbf{ba})$$

$$= (g_{11}p_{11} + g_{12}p_{21})\mathbf{ab} + (g_{22}p_{22} + g_{21}p_{12})\mathbf{ba}.$$

It follows that $g_oP_e + g_eP_o - \mathbf{ab}g_o = 0$, and $g_eP_e + g_oP_o - \mathbf{ab}g_e = 0$, giving the relationships

$$(g_{12}p_{22} + g_{11}p_{12} - g_{12})\mathbf{a} + (g_{21}p_{11} + g_{22}p_{21})\mathbf{b} = 0$$

and

$$(g_{11}p_{11} + g_{12}p_{21} - g_{11})\mathbf{ab} + (g_{22}p_{22} + g_{21}p_{12})\mathbf{ba} = 0.$$

Setting the coefficients of the canonical basis $= 0$, and imposing the conditions from Lemma 1 for the idempotent P, gives the solution

$$[g] := \begin{pmatrix} \frac{g_{12}p_{21}}{p_{22}} & g_{12} \\ g_{21} & -\frac{g_{21}(1-p_{22})}{p_{21}} \end{pmatrix}, \tag{14}$$

with $\det[g] = -\frac{g_{12}g_{21}}{p_{22}}$.

The canonical basis (1), with the required properties, is then easily constructed from g, getting

$$g^{-1}\begin{pmatrix} \mathbf{ab} & \mathbf{a} \\ \mathbf{b} & \mathbf{ba} \end{pmatrix} g = \begin{pmatrix} \mathbf{AB} & \mathbf{A} \\ \mathbf{B} & \mathbf{BA} \end{pmatrix},$$

where $\mathbf{A} = g^{-1}\mathbf{a}g$, $\mathbf{B} = g^{-1}\mathbf{b}g$.

ii) Notice in the solution for $g \in \mathcal{N}$ in part *i)*, the two parameters g_{12} and g_{21} are free. These extra parameters can be used to adjust the scale of the nilpotents $\mathbf{A}$ and $\mathbf{B}$.

□

4 Structure of a geometric number

The fact that the standard canonical basis (1) consists only of g-numbers which are idempotents or nilpotents, suggests that these g-numbers are of fundamental importance. With this in mind, we make the following

Definition 1 *Given a geometric number $f \in \mathcal{N}$. The characteristic polynomial of f is*

$$\varphi_f(x) := x^2 - tr(f)x + \det f = (x - \lambda_1)(x - \lambda_2). \tag{15}$$

The roots λ_1, λ_2 of this polynomial are said to be the eigenvalues of f.

The structure of a geometric number f is completely determined by its characteristic polynomial $\varphi_f(x)$, [7, 8, 9]. The eigenvalues of f can be either real or complex numbers. Discussion of g-numbers f with complex eigenvalues is deferred to the Appendix.

We have the following

Theorem 2 *Given a g-number $f \in \mathcal{N}$ with real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. Then f is either type i) or type ii) given below.*

i) When $\lambda_1 \neq \lambda_2$, then $f = \lambda_1 s_1 + \lambda_2 s_2$, where $s_1, s_2 \in \mathcal{N}$ are idempotents satisfying the conditions

$$s_1 + s_2 = 1, \quad \text{and} \quad s_1 s_2 = s_2 s_1 = 0.$$

When $\varphi_f(x) = (x - \lambda)^2$ and $f - \lambda = 0$, then $f = \lambda$.

ii) When $\varphi_f(x) = (x-\lambda)^2$ but $(f-\lambda) \neq 0$, then $f = \lambda + m$, where $m \in \mathcal{N}$ is a non-zero nilpotent.

Proof: We first note that every $f \in \mathcal{N}$ trivially satisfies its characteristic polynomial, that is

$$\varphi_f(f) = f^2 - (f^* + f)f + f^* f = 0. \tag{16}$$

i) For type *i)*, when $\lambda_1 \neq \lambda_2$. Define

$$s_1 := \frac{f-\lambda_2}{\lambda_1 - \lambda_2}, \quad \text{and} \quad s_2 := \frac{f-\lambda_1}{\lambda_2-\lambda_1}.$$

To see that s_1 and s_2 are idempotents, we calculate for s_1

$$s_1^2 = \Big(\frac{f-\lambda_2}{\lambda_1-\lambda_2}\Big)^2 = \Big(\frac{(f-\lambda_1)+(\lambda_1-\lambda_2)}{\lambda_1-\lambda_2}\Big)\frac{f-\lambda_2}{\lambda_1-\lambda_2}$$

$$= \Big(\frac{(f-\lambda_1)(f-\lambda_2)+(\lambda_1-\lambda_2)(f-\lambda_2)}{\lambda_1-\lambda_2}\Big)\frac{1}{\lambda_1-\lambda_2} = s_1,$$

since by (16),

$$\varphi_f(f) = (f-\lambda_1)(f-\lambda_2) = 0.$$

That s_2 is an idempotent is similarly established. It also trivially follows that $s_1 s_2 = 0 = s_2 s_1$, $s_1 + s_2 = 1$, and the case when $f - \lambda = 0$.

ii) This is the case when the characteristic polynomial $\varphi_f(x) = (x-\lambda)^2$, and $f - \lambda \neq 0$. Since by (16), $\varphi_f(f) = (f-\lambda)^2 = 0$, we simply let the nilpotent $m = f - \lambda$, and we are done.

□

Given that f is type *i)*, so that $f = \lambda_1 s_1 + \lambda_2 s_2$, by multiplying both sides of this equation on the right by s_1 and s_2, successively, we get

$$f s_1 = \lambda_1 s_1, \quad \text{and} \quad f s_2 = \lambda_2 s_2, \tag{17}$$

respectively. We say that s_1 and s_2 are *eigenpotents* for the respective eigenvalues λ_1 and λ_2. When $f = \lambda + m$ for type *ii)*, multiplying on the right by m gives $fm = \lambda m$. In this case, we say that m is an *eigen-nilpotent* of f. It is interesting that for type *i*) $f \in \mathcal{N}$, that the eigenpotents are idempotents, whereas for type *ii*) f, the eigen-nilpotent is a nilpotent. We explore this situation further as follows.

Applying part *i*) of Theorem 1, to the nonzero singular idempotent $P = s_1$, we can find a non-singular $g \in \mathcal{N}$ and construct a canonical basis (1) such that

$$\mathcal{B} = g^{-1}\begin{pmatrix} \mathbf{ab} & \mathbf{a} \\ \mathbf{b} & \mathbf{ba} \end{pmatrix} g = \begin{pmatrix} \mathbf{AB} & \mathbf{A} \\ \mathbf{B} & \mathbf{BA} \end{pmatrix},$$

and where $s_1 = \mathbf{AB}$, $s_2 = \mathbf{BA}$ and $\mathbf{A}^2 = 0 = \mathbf{B}^2$, so that

$$f = \lambda_1 \mathbf{AB} + \lambda_2 \mathbf{BA}.$$

Multiplying both sides this equation on the right by $\mathbf{A}$, and then by $\mathbf{B}$, we get

$$f\mathbf{A} = \lambda_1 \mathbf{ABA} = \lambda_1 \mathbf{A} \quad \text{and} \quad f\mathbf{B} = \lambda_2 \mathbf{BAB} = \lambda_2 \mathbf{B}, \tag{18}$$

respectively. Equations (17) and (18) are equivalent, however, since we can easily get back the first equations from the second. In the canonical basis $\mathcal{B}$, the matrix of f is

$$[f] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Let us also find a canonical basis for a type *ii)* $f = \lambda + m$, where $[f]$ takes its simplest form. In this case, we find a non-singular $g \in \mathcal{N}$ such that

$$\mathbf{a}g = gm. \tag{19}$$

Aside from the standard canonical nilpotents like $\mathbf{a}$ and $\mathbf{b}$, a class of quite general nilpotents m have the form

$$[m] = \begin{pmatrix} -s & -s^2/t \\ t & s \end{pmatrix}, \quad \text{for} \quad s, t \in \mathbb{R}.$$

For a nilpotent m of this form, g defined by

$$[g] := \begin{pmatrix} -\frac{\sqrt{t}}{s} & 0 \\ \sqrt{t} & \frac{s}{\sqrt{t}} \end{pmatrix}$$

satisfies the property (19).

Now define the nilpotent n by specifying

$$[n] := \begin{pmatrix} 0 & 0 \\ -t/s^2 & 0 \end{pmatrix}.$$

The nilpotent n is said to be *compatible* with m, because together they define the canonical basis of $\mathcal{N}$, given by

$$\mathcal{B}_{m,n} := \begin{pmatrix} AB & A \\ B & BA \end{pmatrix} = g^{-1} \begin{pmatrix} \mathbf{ab} & \mathbf{a} \\ \mathbf{b} & \mathbf{ba} \end{pmatrix} g,$$

where

$$A := g^{-1}\mathbf{a}g = m \quad \text{and} \quad B := g^{-1}\mathbf{b}g = n.$$

With respect to $\mathcal{B}_{m,n}$, f is specified by

$$[f] := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

5 Appendix: Geometric algebras and matrices

The purpose of this Appendix is to give readers familiar with matrix algebra insight into how matrix and geometric algebras compliment each other, [10, 11]. Complex eigenvalues are dealt with by extending the real g-numbers $\mathcal{N}$ to the complex g-numbers $\mathcal{N}_{\mathbb{C}}$. The real and complex g-numbers $\mathcal{N}$ and $\mathcal{N}_{\mathbb{C}}$ are algebraically isomorphic to the Clifford geometric algebras $\mathbb{G}_{1,1}$ and $\mathbb{G}_{1,2}$, respectively.

In terms of its matrix $[f]$, since $(ab)(ab) = ab$ and $ab[f] = [f]ab$,

$$f = (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} = (\,\mathbf{ba} \quad \mathbf{a}\,)\,[f^\dagger]^T \begin{pmatrix} \mathbf{ba} \\ \mathbf{b} \end{pmatrix}$$

$$= (\,\mathbf{ab} \quad \mathbf{b}\,) \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} = f_{11}\mathbf{ab} + f_{12}\mathbf{a} + f_{21}\mathbf{b} + f_{22}\mathbf{ba}. \qquad (20)$$

Furthermore,

$$f^\dagger = (\,\mathbf{ba} \quad \mathbf{a}\,)\,[f]^T \begin{pmatrix} \mathbf{ba} \\ \mathbf{b} \end{pmatrix}, \quad \text{and} \quad f^* = (\,\mathbf{ba} \quad -\mathbf{a}\,)\,[f]^T \begin{pmatrix} \mathbf{ba} \\ -\mathbf{b} \end{pmatrix},$$

where $[f]^T$ is the *transpose* of the matrix $[f]$. If $g = (\,\mathbf{ab} \quad \mathbf{a}\,)\,[g] \begin{pmatrix} \mathbf{ab} \\ \mathbf{b} \end{pmatrix}$, then

$$fg = (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} (\,\mathbf{ab} \quad \mathbf{b}\,)\,[g] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix}$$

$$= (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f] \begin{pmatrix} \mathbf{ab} & 0 \\ 0 & \mathbf{ab} \end{pmatrix} [g] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} = (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f][g] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix},$$

showing that matrix multiplication of g-numbers is preserved.

The equation (20) can be directly solved for the matrix $[f]$ of f. Multiplying equation (20) on the left and right by $\begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix}$ and $(\,\mathbf{ab} \quad \mathbf{b}\,)$, respectively, gives the equation

$$\begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} f\,(\,\mathbf{ab} \quad \mathbf{b}\,) = \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} (\,\mathbf{ab} \quad \mathbf{b}\,)$$

$$= \begin{pmatrix} \mathbf{ab} & 0 \\ 0 & \mathbf{ab} \end{pmatrix} [f] \begin{pmatrix} \mathbf{ab} & 0 \\ 0 & \mathbf{ab} \end{pmatrix} = \mathbf{ab}[f].$$

Similarly, multiplying equation (20) on the left and right by $\begin{pmatrix} \mathbf{b} \\ \mathbf{ba} \end{pmatrix}$ and $(\,\mathbf{a} \quad \mathbf{ba}\,)$, respectively, gives

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{ba} \end{pmatrix} f\,(\,\mathbf{a} \quad \mathbf{ba}\,) = \begin{pmatrix} \mathbf{b} \\ \mathbf{ba} \end{pmatrix} (\,\mathbf{ab} \quad \mathbf{b}\,)\,[f] \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} (\,\mathbf{a} \quad \mathbf{ba}\,)$$

$$= \begin{pmatrix} \mathbf{b} & 0 \\ 0 & \mathbf{b} \end{pmatrix} [f] \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} = \mathbf{ba}[f].$$

Adding these two equations together give the desired result

$$[f] = \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix} f \begin{pmatrix} \mathbf{ab} & \mathbf{b} \end{pmatrix} + \begin{pmatrix} \mathbf{b} \\ \mathbf{ba} \end{pmatrix} f \begin{pmatrix} \mathbf{a} & \mathbf{ba} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{ab}f\mathbf{ab} + \mathbf{b}f\mathbf{a} & \mathbf{ab}f\mathbf{b} + \mathbf{b}f\mathbf{ba} \\ \mathbf{a}f\mathbf{ab} + \mathbf{ba}f\mathbf{a} & \mathbf{a}f\mathbf{b} + \mathbf{ba}f\mathbf{ba} \end{pmatrix}. \tag{21}$$

The geometric algebra $\mathbb{G}_{1,1}$ is defined by

$$\mathbb{G}_{1,1} := \mathbb{R}(\mathbf{e}, \mathbf{f})$$

where $\mathbf{e}^2 = 1 = -\mathbf{f}^2$, and $\mathbf{ef} = -\mathbf{fe}$. The geometric algebra $\mathbb{G}_{1,1}$ is the real number system $\mathbb{R}$ extended to include the new anticommuting square roots $\mathbf{e}, \mathbf{f}$ of ± 1, respectively. Since

$$\begin{pmatrix} \mathbf{e} & \mathbf{f} \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{e} & \mathbf{f} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

it follows that

$$\mathcal{N} = \mathbb{R}(\mathbf{a}, \mathbf{b}) = \mathbb{R}(\mathbf{e}, \mathbf{f}) = \mathbb{G}_{1,1},$$

are different names for the same thing.

We have defined $\mathcal{N} = \mathbb{R}(\mathbf{a}, \mathbf{b})$ to be the real number system extended to include the new elements $\mathbf{a}, \mathbf{b}$. Because of the problem of *complex* eigenvalues, we extend $\mathcal{N}$ to $\mathcal{N}_{\mathbb{C}}$ by allowing the matrix $[g]_c$ to consists of *complex numbers.* Thus

$$g = \begin{pmatrix} \mathbf{ab} & \mathbf{b} \end{pmatrix} [g]_c \begin{pmatrix} \mathbf{ab} \\ \mathbf{a} \end{pmatrix},$$

where $[g]_c$ is 2×2 matrix over the complex numbers $\mathbb{C}$. Of course, we make the additional adhoc assumption that complex numbers commute with $\mathbf{a}$ and $\mathbf{b}$, and therefore with all the g-numbers in $\mathcal{N}$.

On the level of geometric algebras, we can give the complex g-numbers a comprehensive geometric interpretation. In either of the geometric algebras $\mathbb{G}_{1,2}$ or $\mathbb{G}_3$,

$$\mathcal{N}_{\mathbb{C}} = \mathbb{G}_{1,2} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2) = \mathbb{R}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{G}_3,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and $\{\mathbf{f}_1, \mathbf{f}_2\}$, are new *anti-commuting* square roots of ± 1, respectively. A complex g-number $f \in \mathbb{G}_{1,2}$ has the form

$$f = f_{11}\mathbf{ab} + f_{12}\mathbf{a} + f_{21}\mathbf{b} + f_{22}\mathbf{ba},$$

where $f_{jk} \in \mathbb{C} \cong \mathbb{G}_{1,2}^{0+3}$. This means that the complex scalars f_{jk} are of the form

$$f_{jk} = x_{jk} + i y_{jk}, \quad \text{where} \quad i := \mathbf{e}_1 \mathbf{f}_1 \mathbf{f}_2$$

for $x_{jk}, y_{jk} \in \mathbb{G}_{1,2}^{0+3}$. In $\mathcal{N}_{\mathbb{C}}$, the definitions (3) of the even an odd parts of a complex g-number must be carefully re-examined [3].

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