Vector Analysis

Of Spinors

by

Garret Sobczyk

Universidad de Las Americas-P

Cholula, Mexico

October 2014

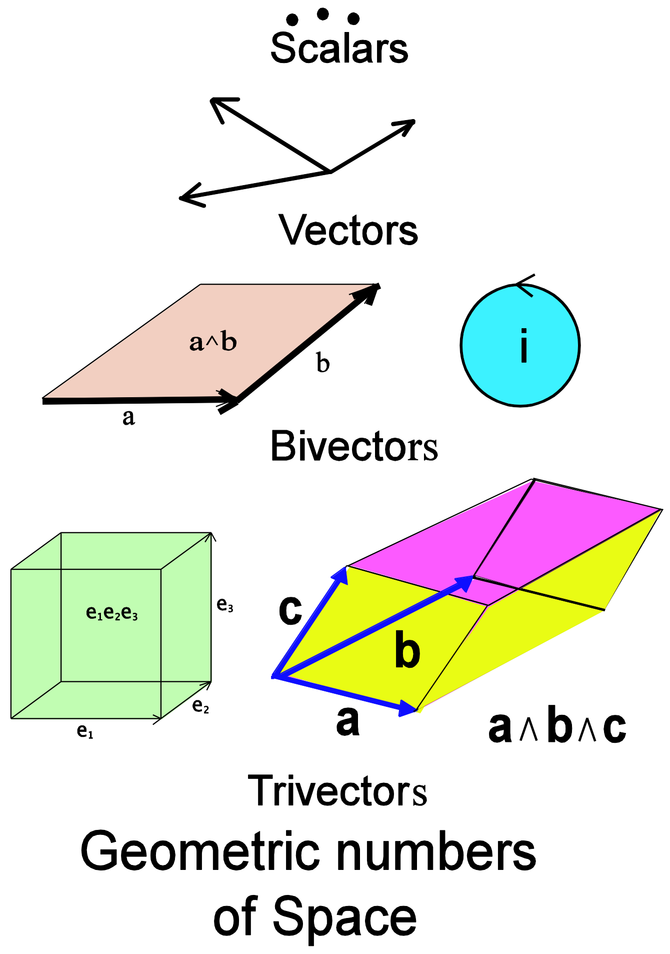
Contents

Contents

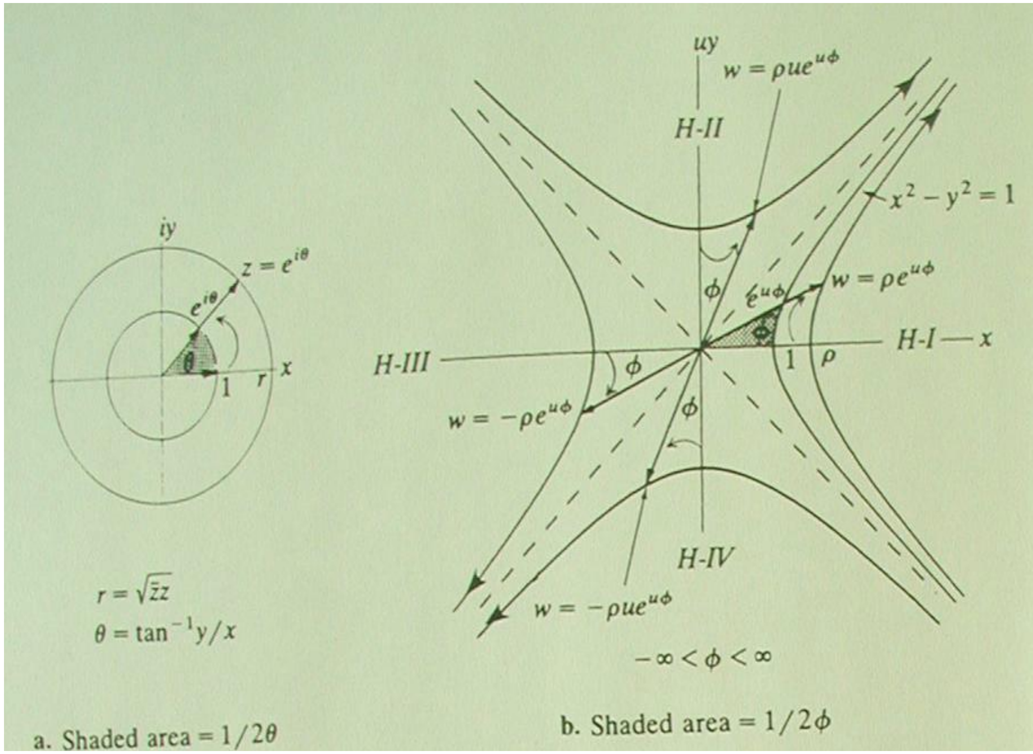
Contents

What is Geometric Algebra?

Geometric algebra is the completion of the real number system to include new anticommuting square roots of plus and minus one, each such root representing an orthogonal direction in successively higher dimensions.



Complex and Hyperpolic Numbers



$$u \notin R$$

$$u^2=1$$

$$z = x + iy = re^{i\theta}, \quad w = x + uy = \rho e^{u\phi}$$

$$i = \sqrt{-1},\ i^2 = -1,\ u = \sqrt{+1},\ u^2 = 1$$

Hyperbolic Numbers

$$w_1 = x_1 + uy_1, \ w_2 = x_2 + uy_2$$

$$w_1 w_2 = x_1 x_2 + y_1 y_2 + (x_1 y_2 + x_2 y_1)u$$

$$\text{Spectral Basis :}$$

$$w = x + yu = (x+y)u_+ + (x-y)u_-$$

$$\text{where } u_+ = \frac{1}{2}(1+u), \ u_- = \frac{1}{2}(1-u),$$

$$u_+ + u_- = 1, \ u_+ - u_- = u$$

$$u_+^2 = u_+, \ u_-^2 = u_-, \ u_+ u_- = 0.$$

:

Geometric Numbers G_2 of the Plane

Standard Basis of $G_2=\{1, e_1, e_2, e_{12}\}$.

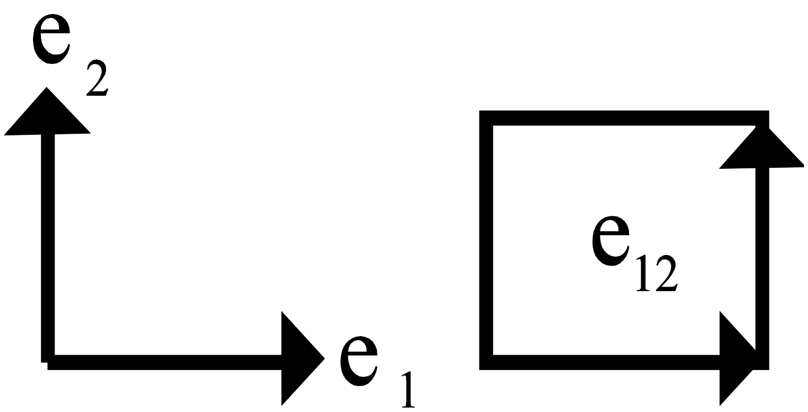
where $i=e_{12}$ is a unit bivector.

$$e_1^2 = e_2^2 = 1$$

$$e_{12} = e_1 e_2 = -e_2 e_1$$

$$i^2 = (e_1 e_2)^2 = e_1 e_2 e_1 e_2$$

$$= -e_1 e_1 e_2 e_2 = -1$$



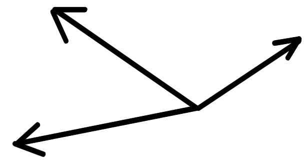
Basic Identities

$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$

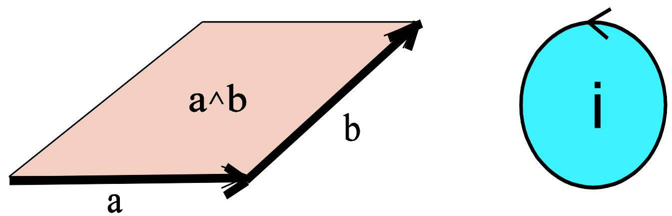
$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$

$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$

$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$



Vectors



Bivectors

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}, \;\; \mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

$$\mathbf{a} \wedge \mathbf{b} = (a_1 b_2 - a_2 b_1)\mathbf{e_{12}}$$

$$\mathbf{ab} = |a||b|e^{i\theta}, \;\; i = \mathbf{e_{12}}$$

Geometric Numbers of 3-Space

$$G_3 = span\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$$

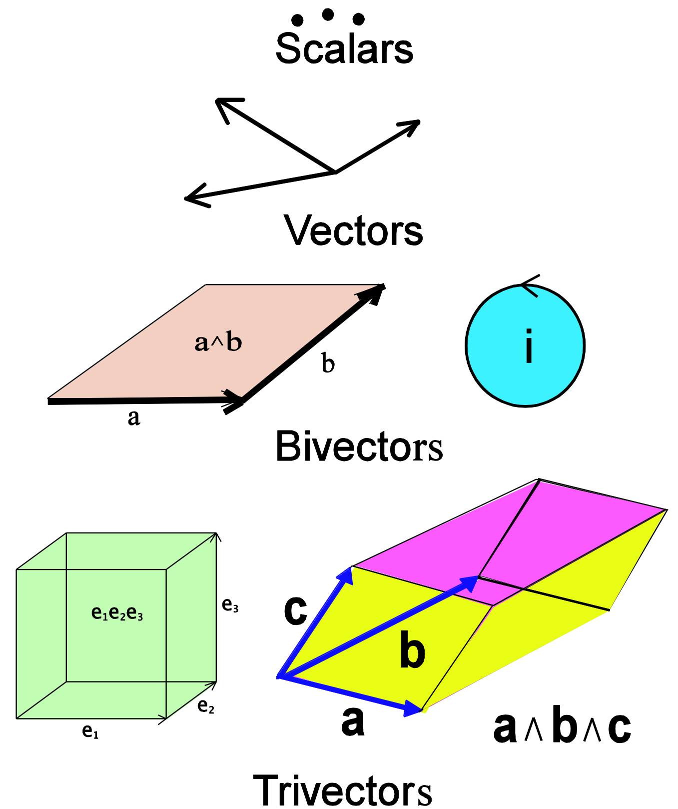
$$a \wedge b = i\, a \times b$$

$$a \wedge b \wedge c = [a \cdot (b \times c)]i$$

where $i = e_1 e_2 e_3 = e_{123}$

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$$

$$= -\, a \times (b \times c).$$



Geometric numbers of Space

Cancellation Property

Let a, b, c be vectors in G_3, then

$$\mathbf{ab} = \mathbf{ac} \iff \mathbf{a}^2\mathbf{b} = \mathbf{a}^2\mathbf{c} \iff \mathbf{b} = \mathbf{c},$$

provided $\mathrm{a}^2 = |\mathrm{a}|^2 \neq 0.$

This is equivalent to:

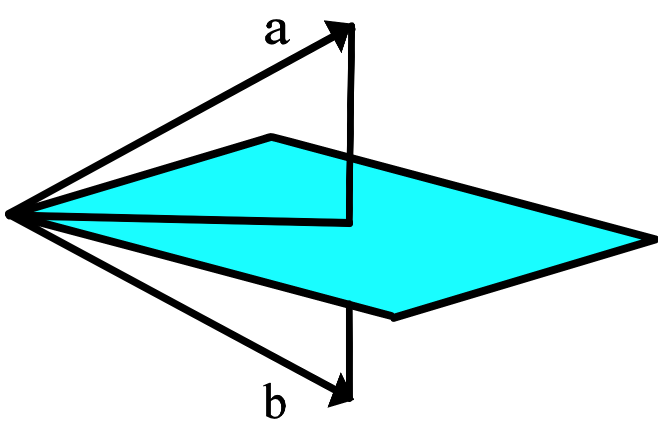
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$$

implies $\mathbf{a} = \mathbf{c}$

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$$

Reflections L(x) and Rotations R(x)

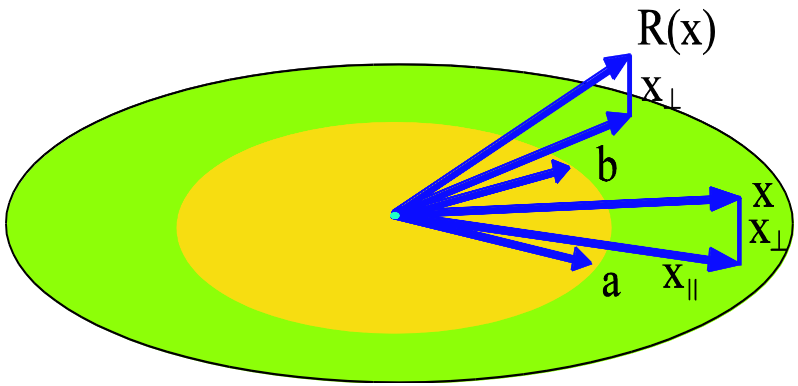
$$L(\mathrm{x}) = -(\mathrm{a}-\mathrm{b})\mathrm{x}(\mathrm{a}-\mathrm{b})^{-1}$$



$$R(\mathbf{x}) = \sqrt{\mathbf{ba}}\ \mathbf{x}\ \sqrt{\mathbf{ab}}$$

where |a|=|b|=1 and

$$\sqrt{\mathbf{ab}} = \pm\mathbf{a}\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$$



$$L(\mathrm{a}) = \mathrm{b}, \quad R(\mathrm{a}) = \mathrm{b}$$

The Spectral Basis of the Geometric Algebra G_3

$G_3 = \text{span}\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$.

By the spectral basis of G_3 we mean

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}$$

where $u_\pm = \frac{1}{2}(1 \pm \mathbf{e}_3)$

are mutually annihiliating idempotents.

Note that $e_1 u_+ = u_- e_1$.

For example, if $[\mathbf{g}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in R$

then the element g Ɛ G_3 is

$$\mathbf{g} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [\mathbf{g}] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = a u_+ + b\mathbf{e}_1 u_- + c\mathbf{e}_1 u_+ + d u_-$$

Pauli Matrices:

$$[1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [\mathbf{e}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and}$$

$$[\mathbf{e}_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, [\mathbf{e}_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Idempotents and the Riemann sphere

An element $s \in G_3$ is an idempotent if

$s^2 = s$

This implies that $s = \frac{1}{2}(1 + \mathrm{m} + i\mathrm{n})$

where $(\mathrm{m} + i\mathrm{n})^2 = \mathrm{m}^2 - \mathrm{n}^2 + 2\mathrm{m} \cdot \mathrm{n} = 1,$

and m and n are orthogonal vectors.

We find that

$$s = \mathbf{m}\left(\frac{1}{2}\left(1 + \frac{\hat{\mathbf{m}} + i\,\hat{\mathbf{m}}\mathbf{n}}{|\mathbf{m}|}\right)\right) = \mathbf{m}\hat{\mathbf{b}}_+$$

where $\hat{\mathbf{b}}_+ = \frac{1}{2}\left(1 + \frac{\hat{\mathbf{m}} + \mathbf{i}\,\hat{\mathbf{m}}\mathbf{n}}{|\mathbf{m}|}\right)$

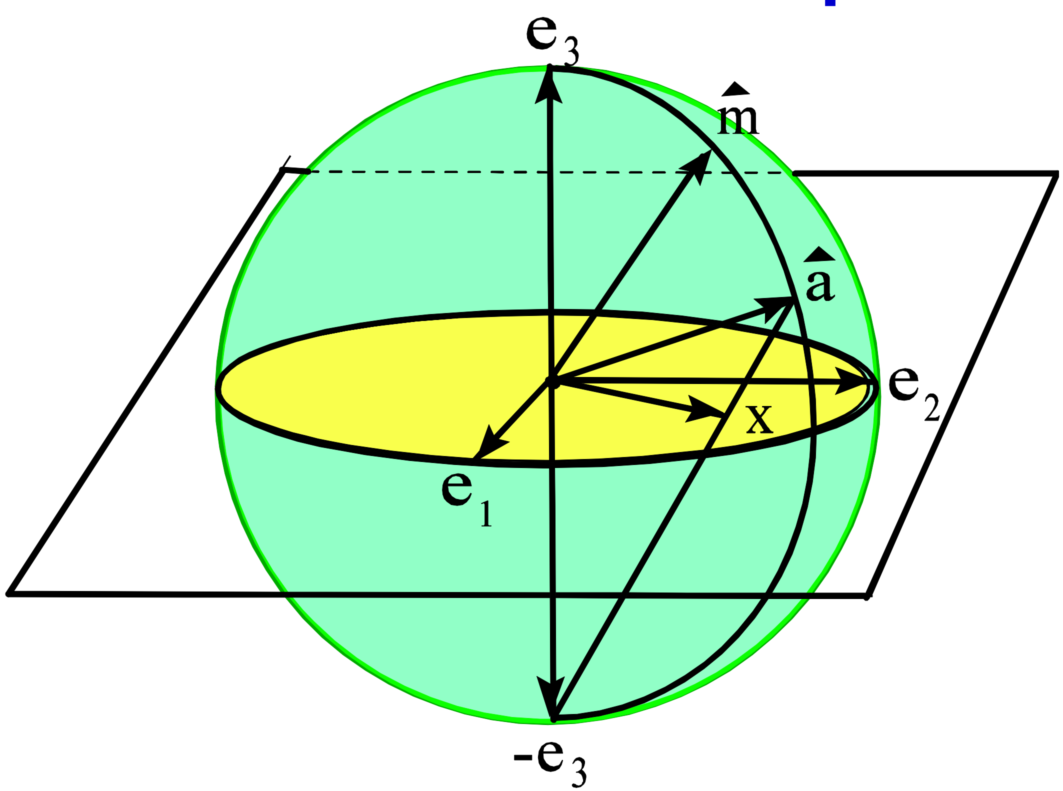
is an idempotent. We also find $s = s^2 = \mathbf{m}^2\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+,$

for the idempotent $\hat{\mathbf{a}}_+ := \hat{\mathbf{m}}\mathbf{b}_+\hat{\mathbf{m}}$

Important property of idempotents:

$$\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\hat{\mathbf{a}}_+ = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{a}}_+.$$

The Riemann Sphere



$$\mathbf{z} = \frac{\alpha_1}{\alpha_0}, \quad \mathbf{x} = \mathbf{P_{xy}}(\mathbf{m}) = \frac{\mathbf{z} + \overline{\mathbf{z}}}{\mathbf{2}}\mathbf{e_1} + \frac{\mathbf{z} - \overline{\mathbf{z}}}{\mathbf{2i}}\mathbf{e_2}$$

$$\mathbf{m} = \frac{\mathbf{z} + \overline{\mathbf{z}}}{\mathbf{2}}\mathbf{e_1} + \frac{\mathbf{z} - \overline{\mathbf{z}}}{\mathbf{2i}}\mathbf{e_2} + \mathbf{e_3}$$

What is a Spinor?

Pauli Spinor: $$|\alpha\rangle_p := \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

Also called a ket-vector.

Geometric Spinor: $$|\alpha\rangle := \sqrt{2}(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+,$$

with matrix $$[|\alpha\rangle] := \begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix}$$

Norm of a Spinor

Sesquilinear inner product:

$$\langle \alpha | \beta \rangle := \alpha_0 \overline{\beta}_0 + \alpha_1 \overline{\beta}_1.$$

A spinor is normalized if

$$\langle \alpha | \alpha \rangle := \alpha_0 \overline{\alpha}_0 + \alpha_1 \overline{\alpha}_1 = 1.$$

where $\langle \alpha | := | \alpha \rangle^\dagger = \sqrt{2} u_+ (\overline{\alpha}_0 + \overline{\alpha}_1 \mathbf{e_1})$

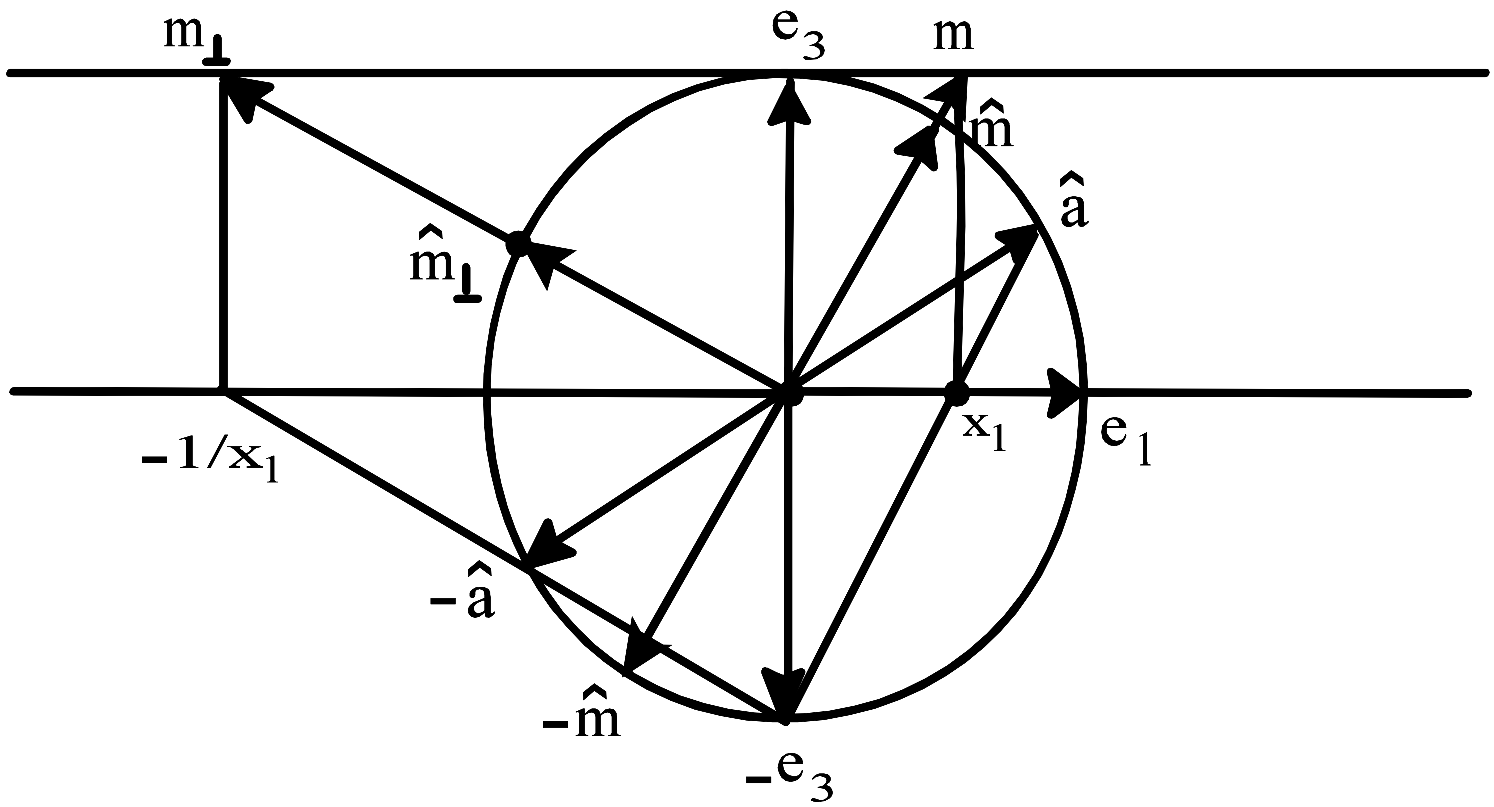
Properties of Spinors

$$\frac{1}{2}|\alpha\rangle\langle\alpha| = \hat{\mathbf{m}}\mathbf{u}_+\hat{\mathbf{m}} = \hat{\mathbf{a}}_+.$$

$$|\alpha\rangle = \sqrt{2}\rho e^{i\theta}\hat{\mathbf{m}}\mathbf{u}_+ = \sqrt{2}\rho \mathbf{e}^{\mathbf{i}(\theta+\hat{\mathbf{v}}\phi)}\mathbf{u}_+$$

$$= \sqrt{2}\rho \mathbf{e}^{\mathbf{i}\hat{\mathbf{v}}\phi}\mathbf{e}^{\mathbf{i}\mathbf{e}_3\theta}\mathbf{u}_+ = \sqrt{2}\rho \mathbf{e}^{\mathbf{i}\hat{\mathbf{c}}\omega}\mathbf{u}_+,$$

Great Circle of Riemann Sphere



$$|\alpha\rangle = \sqrt{2}(1 + x_1\,\mathbf{e_1})u_+, \;\; |\alpha\rangle_\perp = \sqrt{2}(1 - \tfrac{1}{x_1}\mathbf{e_1})u_+$$

Magic of Quantum Mechanics

Observable: $S = s_0 + \mathbf{s} \in \mathbf{G_3}$

where $\mathbf{s} = s_1\mathbf{e_1} + s_2\mathbf{e_2} + s_3\mathbf{e_3}.$

Note that $$[S] = \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix}$$

is an Hermitian matrix.

Expected Value: $\langle S\rangle := \langle\alpha|S|\alpha\rangle = s_0 + \mathbf{s}\cdot\hat{\mathbf{a}},$

Standard deviation:

$$\sigma_S^2 := \langle\alpha|\big(S - \langle S\rangle\big)^2|\alpha\rangle = ||^2\langle\alpha|\big(\hat{s} - \hat{s}\cdot\hat{a}\big)^2|\alpha\rangle = (s\times\hat{a})^2$$

Uncertainty Principle for observables S and T:

$$(s\times\hat{a})^2(t\times\hat{a})^2 = |(s\times\hat{a})\cdot(t\times\hat{a})|^2 + |(s\times t)\cdot\hat{a}|^2,$$

or

$$\sigma_S^2\sigma_T^2 \geq \langle\mathbf{s}\times\mathbf{t}\rangle^2.$$

Schrodinger Equation

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)|\alpha\rangle = H|\alpha\rangle$$

If the Hamiltonian is time independent, then H = S = s_0+**s;** with solution:

$$|\alpha\rangle = \sqrt{2}e^{-\frac{iS}{\hbar}t}u_+ = \sqrt{2}e^{-i\frac{s_0}{\hbar}t}\Big(\cos\frac{|\mathbf{s}|t}{\hbar} + i\,\hat{\mathbf{s}}\sin\frac{|\mathbf{s}|t}{\hbar}\Big)u_+.$$

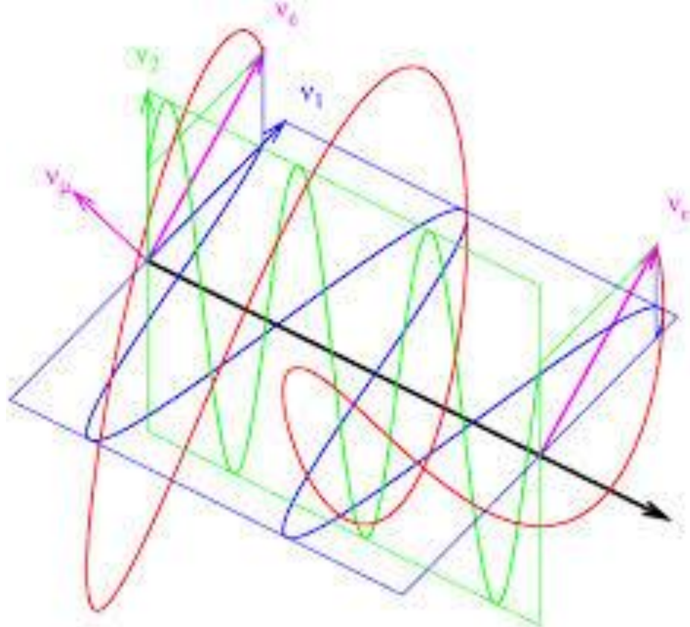
Neutrino Oscillation (two states)

Electron neutrino state: $|\upsilon_\epsilon\rangle := |0\rangle := \sqrt{2}u_+$

Muon neutrino state: $|\upsilon_\mu\rangle := |i\,\hat{\mathbf{s}}\,\mathbf{e_3}\rangle$

Oscillation:

$$\left(\cos\frac{|\mathrm{s}|t}{\hbar} + i\,\hat{\mathrm{s}}\sin\frac{|\mathrm{s}|t}{\hbar}\right)u_+ = \left(\mathbf{e_3}\cos\frac{|\mathrm{s}|t}{\hbar} + i\,\hat{\mathbf{s}}\mathbf{e_3}\sin\frac{|\mathrm{s}|t}{\hbar}\right)u_+,$$



The neutrino in the evolving state $|\alpha(t)\rangle$

will be observed in the state $|\beta\rangle$

with the probability

$$|\langle\alpha|\beta\rangle|^2 = 2\langle\beta|\hat{\mathbf{a}_+}|\beta\rangle_{0+3} = \frac{1}{2}(1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}).$$

Spacetime Algebra $G_{1,3}$

We start with

$$\mathbf{G_3} = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}$$

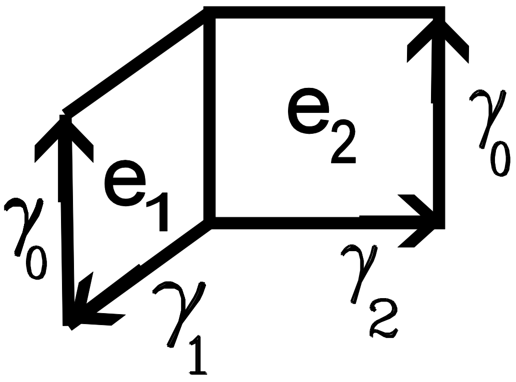
We factor e_1, e_2, e_3 into Dirac bivectors,

$$\mathbf{e}_1 = \gamma_1\gamma_0, \mathbf{e}_2 = \gamma_2\gamma_0, \mathbf{e}_3 = \gamma_3\gamma_0,$$

where

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \gamma_0^2 = 1$$

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \text{ for } \mu \neq \nu.$$



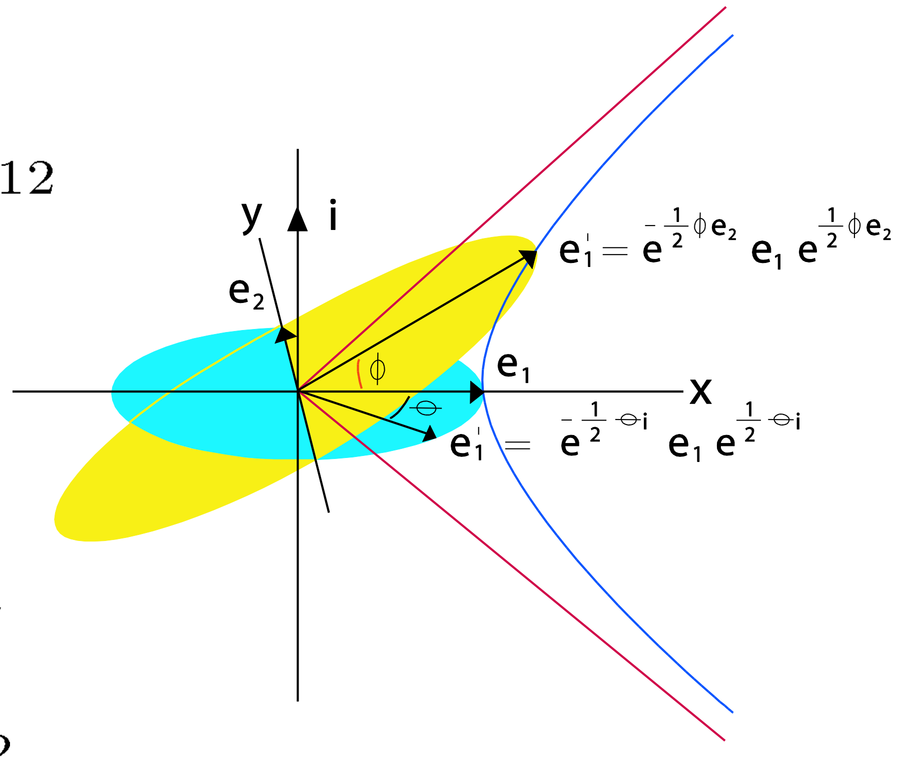
$$\mathbf{G}_{1,3} = span\{1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\omega}, \gamma_{0123}\}$$

Splitting Space and Time

The ordinary rotation

$$R(\mathbf{x}) = e^{-\frac{1}{2}\theta \mathbf{e}_{12}} \mathbf{x} e^{\frac{1}{2}\theta \mathbf{e}_{12}}$$

is in the blue plane of the bivector $i = e_{12}$. The blue plane is boosted into the yellow plane by



$$B(\mathbf{x}) = e^{-\frac{1}{2}\phi \mathbf{e}_{2}} \mathbf{x} e^{\frac{1}{2}\phi \mathbf{e}_{2}}$$

with the velocity $v/c = \tanh \phi$.
The light cone is shown in red.

Selected References

R. Ablamowicz, G. Sobczyk, *Lectures on Clifford (Geometric) Algebras and Application*, Birkhauser, Boston 2004.

T.F. Havel, *GEOMETRIC ALGEBRA: Parallel Processing for the Mind* (Nuclear Engineering) 2002. http://www.garretstar.com/secciones/

D. Hestenes, *New Foundations for Classical Mechanics, 2nd Ed.,* Kluwer 1999.

D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, 2nd edition, Kluwer 1992.

P. Lounesto, *Clifford Algebras and Spinors, 2nd Edition*. Cambridge University Press, Cambridge, 2001.

P. Lounesto, *CLICAL software packet and user manual*. Helsinki University of Technology of Mathematics, Research, Report A248, 1994.

G. Sobczyk, *The missing spectral basis in algebra and number theory*, The American Mathematical Monthly 108 April 2001, pp. 336-346.

G. Sobczyk, *Geometric Matrix Algebra,* Linear Algebra and its Applications, 429 (2008) 1163-1173.

G. Sobczyk, *Hyperbolic Number Plane*, The College Mathematics Journal, Vol. 26, No. 4, pp.269-280, September 1995.

G. Sobczyk, Spacetime Vector Analysis, Physics Letters, 84A, 45-49, 1981.

G. Sobczyk, *Noncommutative extensions of Number*: *An Introduction to Clifford's Geometric Algebra*, Aportaciones Matematicas Comunicaciones}, 11 (1992) 207-218.

G. Sobczyk, *Hyperbolic Number Plane*, The College Mathematics
Journal, 26:4 (1995) 268-280.

G. Sobczyk, *The Generalized Spectral Decomposition of a Linear Operator*, The College Mathematics Journal, 28:1 (1997) 27-38.

G. Sobczyk, *Spectral integral domains in the classroom*,
APORTACIONES MATEMATICAS, Serie Comunicaciones Vol. 20,
(1997) 169-188.

G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number, Springer-Birkhauser* 2012.

Note: Copies of many of my papers can be found on my website:

http://www.garretstar.com/