MAPPINGS OF SURFACES IN EUCLIDEAN SPACE USING GEOMETRIC ALGEBRA

by

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ABSTRACT

A coordinate-free formulation of mappings between surfaces is achieved by utilizing the Geometric Calculus developed by D. Hestenes. Greatly simplifying concepts introduced in this formulation are: (i) differentiation with respect to an r-vector variable; (ii) generalized invariants of a mapping; and (iii) a generalized Lie bracket.

Basic ideas of linear algebra, advanced calculus, differential forms, and differential geometry are then efficiently reformulated in terms of this approach.

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SECTION PAGE

0. Summary

This summary serves several purposes:

(i) It is a listing of the symbols used in this paper, with a brief description of their meanings, and the page numbers on which they first occur.

(ii) It lists some of the basic identities of geometric algebra that will be used repeatedly. (Proofs of most of these identities can be found in [11].)

(iii) It groups properties proved in this paper according to subject area. This serves to bring together related properties that are otherwise apart in the logical exposition of this paper.

An index to the listings by subject headings is found on the next page.

Index to Summary

Symbols Used in This Paper

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.0	XXXX	means "end of proof"	
0.1	$\mathcal{E}_n$	Euclidean n-space	4
0.2	$\mathcal{G}$	geometric algebra of $\mathcal{E}_n$	4
0.3	$\mathcal{X}_m$, $\mathcal{Y}_k$	surfaces in $\mathcal{E}_n$	5
0.4	$\mathcal{G}_{\mathbf{x}}$	geometric algebra of $\mathcal{X}_m$	5
0.5	$\{F(\mathbf{x})\}$	set of multivector fields on $\mathcal{X}_m$	5
0.6	$\{F(\mathbf{x})\}_{\mathbf{x}}$	set of tangent multivector fields on $\mathcal{X}_m$	5
0.7	$\nabla_{\mathbf{x}}$	tangential gradient operator on $\mathcal{X}_m$	6
0.8	$\mathbf{v}\cdot\nabla_{\mathbf{x}}$	directional derivative on $\mathcal{X}_m$	6
0.9	$y: \mathcal{X}_m \to \mathcal{Y}_k$, $\mathbf{y} = y(\mathbf{x})$	mapping of $\mathcal{X}_m$ into $\mathcal{Y}_k$	6
0.10	$i_{\mathbf{x}} = i(\mathbf{x})$	pseudoscalar field on $\mathcal{X}_m$	6
	[illegible]	[illegible] on $\mathcal{X}$ [illegible]	8

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.13	$J_{\bar{y}_r}$	characteristic multivector of $y(\underset{\sim}{x})$	8
0.14	$y_\dagger$	the differential or "push forward" mapping induced by $y(\underset{\sim}{x})$	12
0.15	$y^\dagger$	the adjoint or "pull back" mapping induced by $y(\underset{\sim}{x})$	13
0.16	$z \circ y$	composition of mappings	17
0.17	$(\quad)_{\parallel}$	tangential component to surface	29, 89
	$(\quad)_{\perp}$	normal component to surface	
0.18	$\binom{m}{k}$	binomial coefficient	30
0.19	$[\ ,\ ]$	Lie bracket	40
0.20	$\{\underset{\sim}{e}_i\}$	frame on $\mathcal{X}_m$	50
0.21	$\psi^i(\underset{\sim}{x})$	scalar field on $\mathcal{X}_m$	51
0.22	$g_{\underset{\sim}{x}}$	volume element of $\mathcal{X}_m$	55
0.23	$p_{\underset{\sim}{x}} = p(\underset{\sim}{x})$	unit pseudoscalar field on $\mathcal{X}_m$	62
0.24	$S(\underset{\sim}{a})$	shape operator of $\mathcal{X}_m$	62

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.27	$\int_{\mathcal{A}_r}$	integral over r-surface $\mathcal{A}_r$	84
0.28	$f^r_{\underset{\sim}{x}} = f^r(\underset{\sim}{x})$	differential r-form on $\mathcal{X}_m$	93
0.29	$f^r_{\underset{\sim}{x}} \wedge g^s_{\underset{\sim}{x}}$	exterior product of forms	96
0.30	d	exterior derivative operator	98
0.31	$C_{\underset{\sim}{v}}$	contraction operator w.r.t. $\underset{\sim}{v}$	101
0.32	$D_{\underset{\sim}{v}}$	covariant derivative of forms	103
0.33	$L_{\underset{\sim}{v}}$	Lie derivative of forms	104
0.34	y^*	the pull back mapping	106
0.35	$\nabla_{\underset{\sim}{x}}$	intrinsic gradient operator on $\mathcal{X}_m$	110
0.36	$[\ /\ ]$	intrinsic Lie bracket	111

Algebraic Identities

0.37 $\underset{\sim}{a}\, A_r = \underset{\sim}{a} \cdot A_r + \underset{\sim}{a} \wedge A_r$

0.38 $\underset{\sim}{a} \cdot (A_r \wedge B_s) = (\underset{\sim}{a} \cdot A_r) \wedge B_s + (-1)^r A_r \wedge (\underset{\sim}{a} \cdot B_s)$

0.39 $a \cdot (b \ \wedge b \ ) = a \cdot b \ \ b \ - a \cdot b \ \ b$

0.40 $$a\cdot(b_1 \wedge \ldots \wedge b_s) = \sum_{i=1}^{s} (-1)^{i+1} a\cdot b_i \, b_1 \wedge \ldots \wedge \check{b}_i \wedge \ldots \wedge b_s \,,$$

where $\check{b}_i$ means omit b_i from product.

0.41 $$(a_s \wedge \ldots \wedge a_1)\cdot(b_1 \wedge \ldots \wedge b_s) = \begin{vmatrix} a_1\cdot b_1 , & \ldots , & a_1\cdot b_s \\ \vdots & & \vdots \\ a_s\cdot b_1 , & \ldots , & a_s\cdot b_s \end{vmatrix}$$

0.42 $$(A_r \wedge B_s)\cdot C_t = A_r\cdot(B_s\cdot C_t) \quad \text{for} \quad r + s \leq t$$

0.43 $$(A_r \wedge B_s)\, i_x = A_r\cdot(B_s\, i_x) \,, \text{ where } i_x \text{ is a pseudoscalar}$$

0.44 $$a = i_x^{-1}\, i_x\cdot a + i_x^{-1}\, i_x \wedge a \equiv a_{\parallel} + a_{\perp} \,.$$

0.45 $$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$$

0.46 $$A_r^{\dagger} = (-1)^{\frac{r(r-1)}{2}} A_r \,.$$

Properties of $y_{\dagger}$ and $y^{\dagger}$

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0.47 $$y_{\dagger}A_r \equiv A_r\cdot\nabla_{\bar{x}_r}\, \bar{y}_r \,, \quad y^{\dagger}B^r \equiv \nabla_{\bar{x}_r}\, \bar{y}_r\cdot B^r$$ 12, 13

0.48 $$y_{\dagger}(A \wedge B) = y_{\dagger}A \wedge y_{\dagger}B$$ 13

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0.49 $$\left\{\begin{array}{l} A = y^{\dagger}B \quad \text{iff} \quad i_{\underset{\sim}{y}} B = y_{\dagger} i_{\underset{\sim}{x}} A \\ B = y_{\dagger}A \quad \text{iff} \quad i_{\underset{\sim}{x}}^{-1} A = y^{\dagger} i_{\underset{\sim}{y}}^{-1} B \end{array}\right\} \quad \text{if} \quad J_{\bar{y}_m} \neq 0$$ 20

0.50 Let $\{\underset{\sim}{e}^{i}(\underset{\sim}{x})\}$ and $\{\underset{\sim}{f}^{i}(\underset{\sim}{y})\}$ be frames on $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively, then:

$$y^{\dagger}\underset{\sim}{f}^{i}(\underset{\sim}{y}) = \underset{\sim}{e}^{i}(\underset{\sim}{x}) \quad \text{iff} \quad y_{\dagger}\underset{\sim}{e}_{i} = \underset{\sim}{f}_{i}$$ 53

0.51 $$A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_{\dagger}A_r \;, \quad r \leq i \leq m$$ 15

$$\nabla_{\bar{x}_i} \bar{y}_i \cdot B^s = (y^{\dagger}B^s) \wedge \nabla_{\bar{x}_{i-s}} \bar{y}_{i-s} \;, \quad s \leq i \leq m$$

0.52 $$A_r \cdot y^{\dagger} B^r = (y_{\dagger}A_r) \cdot B^r$$ 16

0.53 $$(y_{\dagger}A_r) \cdot B^s = y_{\dagger}(A_r \cdot y^{\dagger} B^s) \;, \quad r \geq s$$ 15

$$A_r \cdot y^{\dagger} B^s = y^{\dagger}[(y_{\dagger}A_r) \cdot B^s] \;, \quad r \leq s$$

0.54 $$y_{\dagger}A = y^{\dagger}A \quad \text{if} \quad \nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$$ 22

0.55 $$\left\{\begin{array}{l} A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_i} \bar{y}_i \cdot A_r \\ A_r \wedge \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_i} \bar{y}_i \wedge A_r \end{array}\right\} , \quad \text{if} \quad \nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$$ 24

0.56 $$y_{\dagger}A = A = y^{\dagger}A \;, \qquad \text{if} \quad y(\underset{\sim}{x}) \equiv \underset{\sim}{x}$$ 29

0.57 (cont.)

$$A_r \wedge \nabla_{\bar{x}_i} \bar{x}_i = \begin{Bmatrix} \binom{m-r}{i} A_r & \text{if } r+i \leq m \\ 0 & \text{if } r+i > m \end{Bmatrix} = \nabla_{\bar{x}_i} \bar{x}_i \wedge A_r$$

0.58 $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} = m \,, \quad \nabla_{\bar{x}_i} \bar{x}_i = \binom{m}{i}$ 28, 32

<u>Properties of ∇_x</u>

0.59 $\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}} = 0$ 7

0.60 $y^{\dagger} \nabla_{\underset{\sim}{y}} = \nabla_{\underset{\sim}{x}}$ (chain rule) 7,36

0.61 $y_{\dagger}(A_r \cdot \nabla_{\underset{\sim}{x}}) = (y_{\dagger} A_r) \cdot \nabla_{\underset{\sim}{y}}$ 38

0.62 $y_{\dagger} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{y}} \nabla_{\underset{\sim}{y}}$ (dual chain rule) 39

0.63 $\nabla_{\underset{\sim}{x}} = \sum_i \underset{\sim}{e}^i \, \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}$ 52

<u>Derivatives of Fields Under Mappings</u>

0.64 $y^{\dagger}(\nabla_{\underset{\sim}{y}} \wedge B_{\underset{\sim}{y}}) = \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} B_{y(\underset{\sim}{x})}$ 37

0.65 $(y_{\dagger} A_r) \cdot \nabla_{\underset{\sim}{y}} B_{\underset{\sim}{y}} = y_{\dagger}(A_r \cdot \nabla_{\underset{\sim}{x}}) B_{y(\underset{\sim}{x})}$ 38

0.66 $y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger} \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} |J_{\bar{y}_m}(\underset{\sim}{x})|$ 57

0.67 $y_{\dagger} \nabla_x \cdot A_r = \nabla_y \cdot y_{\dagger} A_r$ iff $|J_{\bar{y}_m}(x)|$ is constant. 58

Properties of Lie Brackets

(The formulas below hold only for tangent multivector fields.)

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0.69 $[\underset{\sim}{a},\underset{\sim}{b}] \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}(\underset{\sim}{x}) - \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{a}(\underset{\sim}{x})$ 40

$[\underset{\sim}{a},B_s] \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, B_s(\underset{\sim}{x}) - \underset{\sim}{a}(\underset{\sim}{x})\wedge[\nabla_{\underset{\sim}{x}}^{\dagger}\cdot B_s]$

$[A_r,B_s] \equiv (A_r\cdot\nabla_{\underset{\sim}{x}})\wedge B_s(\underset{\sim}{x}) - A_r(\underset{\sim}{x})\wedge[\nabla_{\underset{\sim}{x}}^{\dagger}\cdot B_s]$

0.70 $[A_r+B_s,\ C_t] = [A_r,C_t] + [B_s,C_t]$ 41

$[A_r,\ B_s+C_t] = [A_r,B_s] + [A_r,C_t]$

0.71 $[A_r\wedge\underset{\sim}{a},\ B_s] = A_r\wedge[\underset{\sim}{a},B_s] + (-1)^r\underset{\sim}{a}\wedge[A_r,B_s]$ 41

$[A_r,\ \underset{\sim}{b}\wedge B_s] = [A_r,\underset{\sim}{b}]\wedge B_s + (-1)^s[A_r,B_s]\wedge\underset{\sim}{b}$

0.72 $[\underset{\sim}{a},\ \underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s] = \sum_{i=1}^{s} \underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_{i-1}\wedge[\underset{\sim}{a},\underset{\sim}{b}_i]\wedge\underset{\sim}{b}_{i+1}\wedge\ldots\wedge\underset{\sim}{b}_s$

45

$[A_r,\underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s] = \sum_{i=1}^{s} (-1)^{i+1}[A_r\wedge\underset{\sim}{b}_i]\wedge[\underset{\sim}{b}_1\wedge\ldots\wedge\check{\underset{\sim}{b}}_i\wedge\ldots\wedge\underset{\sim}{b}_s]$

0.73 $[\underset{\sim}{a},B_s] = -[B_s,\underset{\sim}{a}]$ 41

$[A_r,B_s] = -(-1)^{(r-1)(s-1)}[B_s,A_r]$

$[A\ ,B\ ] = -[B^{\dagger},A^{\dagger}]^{\dagger}$

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0.88 $y^{\dagger}(y_{\dagger}\underset{\sim}{x}) = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in \dot{\mathcal{E}}_n$ iff $y(\underset{\sim}{x})$ is orthogonal . 71

0.89 $\Psi(\lambda) \equiv \sum_{i=0}^{n} (-1)^i \lambda^i [J_{\bar{x}_{n-i}}]_0$ is the characteristic polynomial of $y(\underset{\sim}{x})$. 72

0.90 $\Psi[y(\underset{\sim}{x})] = 0$ 73

0.91 If I_r is a proper invariant r-vector, then $\mathcal{Q}(I_r)$ is an invariant linear subspace. 76

Jacobians and Integral Transformations

0.92
$$J_y(\underset{\sim}{x}) \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & , \ldots , & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & , \ldots , & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = J_{\bar{y}_n}(\underset{\sim}{x})$$
81

0.93 $J_{\bar{y}_m}(\underset{\sim}{x}) = i_{\underset{\sim}{x}}^{-1}\ i_{\underset{\sim}{y}}\ ,\ |J_{\bar{y}_m}(\underset{\sim}{x})| = \frac{\sqrt{g_y}}{\sqrt{g_{\underset{\sim}{x}}}}$ 82

0.94 $\int dY_r\, F(\underset{\sim}{y}) = \int dX_r \cdot \nabla_{\bar{x}_r} \bar{y}_r\, F[y(\underset{\sim}{x})]$ 84

0.94 cont. $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} |dY_r|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} |dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_r|\, F[y(\underset{\sim}{x})]$$

$$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} |dY_m|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} |dX_m|\; |J_{\bar{y}_m}|\; F[y(\underset{\sim}{x})]$$

0.95 $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} dY_r \cdot \nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_{r-1}\, F[y(\underset{\sim}{x}_r)]$$ 84

$$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} dY_m\, \nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} dX_m\, \nabla_{\bar{x}_m}\, \bar{y}_{m-1}\, F[y(\underset{\sim}{x}_m)]$$ 85

<u>Examples of Mappings</u>

0.96 If a mapping is of the kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\, \underset{\sim}{x}$, then

(i) $y_{\dagger} A_r = \psi^{r-1}[\psi A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi) \wedge \underset{\sim}{x}]$ 86

(ii) $y^{\dagger} B^r = \psi^{r-1}[\psi B^r + (\nabla_{\underset{\sim}{x}} \psi) \wedge (\underset{\sim}{x} \cdot B^r)]$

(iii) $J_{\bar{y}_m} = \psi^{m-1}[\psi + (\nabla_{\underset{\sim}{x}} \psi) \cdot \underset{\sim}{x}]$

(iv) $\nabla_{\underset{\sim}{y}} = \psi^{m-1} J_{\bar{y}_m}^{-1} (\psi \nabla_{\underset{\sim}{x}} + \underset{\sim}{x} \cdot [(\nabla_{\underset{\sim}{x}} \psi) \wedge \nabla_{\underset{\sim}{x}}]$.

0.97 If a mapping is of the kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\, \underset{\sim}{p}$, then

(ii) $y^{\dagger}B^{r} = B^{r} + (\nabla_{\underset{\sim}{x}}\psi)\Lambda(\underset{\sim}{p}\cdot B^{r})$ 88

(iii) $J_{\bar{y}_m} = 1 - \underset{\sim}{p}_{\perp}\nabla_{\underset{\sim}{x}}\psi + \underset{\sim}{p}_{\parallel}\cdot\nabla_{\underset{\sim}{x}}\psi$

(iv) $\nabla_{\underset{\sim}{y}} = J_{\bar{y}_m}^{-1}\{\nabla_{\underset{\sim}{x}} - \underset{\sim}{p}_{\perp}(\nabla_{\underset{\sim}{x}}\psi)\Lambda\nabla_{\underset{\sim}{x}} + \underset{\sim}{p}_{\parallel}\cdot[(\nabla_{\underset{\sim}{x}}\psi)\Lambda\nabla_{\underset{\sim}{x}}]\}$

Differential Forms

0.98 $f^{r}_{\underset{\sim}{x}}$ is an r-form iff there is an r-vector field $F^{r}_{\underset{\sim}{x}}$ with the property that $f^{r}_{\underset{\sim}{x}}(\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_r) = F^{r}_{\underset{\sim}{x}}\cdot V^{\dagger}_{r}$, where $V_r = \underset{\sim}{v}_1 \Lambda\cdots\Lambda \underset{\sim}{v}_r$. 98

The following table gives the corresponding operations on differential forms and their respective multivector fields.

	FORMS	MULTIVECTOR FIELDS	
0.99	$f^{r}_{\underset{\sim}{x}}, g^{s}_{\underset{\sim}{x}}$	$F^{r}_{\underset{\sim}{x}}, G^{s}_{\underset{\sim}{x}}$	94
0.100	$f^{r}_{\underset{\sim}{x}} \Lambda g^{s}_{\underset{\sim}{x}}$	$F^{r}_{\underset{\sim}{x}} \Lambda G^{s}_{\underset{\sim}{x}}$	96
0.101	$df^{r}_{\underset{\sim}{x}}$	$\nabla_{\underset{\sim}{x}} \Lambda F^{r}_{\underset{\sim}{x}}$	99
0.102	$C_{\underset{\sim}{v}}f^{r}_{\underset{\sim}{x}}$	$\underset{\sim}{v}\cdot F^{r}_{\underset{\sim}{x}}$	101

The Intrinsic Gradient and Curvature

1. Introduction

In references [9] and [10] D. Hestenes sets down the fundamentals of differential and integral calculus in terms of geometric algebra. Two greatly simplifying features of the resulting "geometric calculus" are that it is coordinate-free and uses only one differential operator.

The purpose of this paper is to apply geometric calculus to the study of smooth mappings between smooth surfaces in Euclidean space. A great simplification of this theory is made possible by the introduction of the following important concepts:

(i) The concept of an r-vector variable, and of differentiating with respect to an r-vector variable.

(ii) The concept of "characteristic multivectors" of a mapping as a generalization of well-known invariants of a mapping, such as the Jacobian, divergence, and curl.

(iii) The concept of the Lie bracket of multivector fields as a generalization of the Lie bracket of vector fields.

This paper is divided into two parts and a series of appendices.

Part I is a study of the differential and adjoint mappings. These linear mappings are induced between the tangent spaces of two

In Part II the "field" properties of the differential and adjoint mappings are studied by considering them as mappings of tangent multivector fields on the two surfaces.

The appendices make up an important part of this paper. They complement the material in Parts I and II, and at the same time relate it to more usual formulations found in the literature.

In Appendix A the methods of Part I are used in the study of linear mappings on Euclidean n-space. By looking at the characteristic equation of a linear mapping, further insight is gained into the nature of the characteristic multivectors of a mapping.

Appendix B discusses the Jacobian, and shows how integral transformation formulas can be easily derived from properties of the differential and adjoint mappings.

Appendix C provides explicit calculations for two kinds of mappings which occur frequently in applications.

Appendix D shows that a one-to-one correspondence exists between differential r-forms and r-vector fields. This correspondence is then exploited to show how all the properties of forms, and operators on forms, follow easily and elegantly from algebraic properties of geometric algebra and the gradient operator.

Appendix E introduces the intrinsic gradient operator on a surface and relates it to the tangential gradient. In addition, the Gauss curvature equation for a surface is formulated in a new and differential geometry. References [9], [10], [11] have already been mentioned in connection with Hestenes. Reference [18] is Whitney's _Geometric Integration Theory_. In Part I of this book, Whitney uses a geometric approach which is the closest to the one adopted here (with the exception of [9], [10] and [11]). However, in most cases, references to Whitney have been avoided since his approach is not as familiar to most readers as some of the others.

2. Preliminaries

This paper makes extensive use of the geometric algebra and calculus as developed by Hestenes in [9], [10], and [11]. A partial list of the algebraic identities that will be used repeatedly is included in the summary.

Let $\mathcal{E}_n$ denote Euclidean n-space. Points in $\mathcal{E}_n$ are named by vectors. These vectors, under the operations of geometric addition and multiplication, generate the geometric algebra $\mathcal{D}$ of 2^n-dimensions. At each point $\underset{\sim}{p} \in \mathcal{E}_n$ there is associated a geometric algebra $\mathcal{D}_{\underset{\sim}{p}}$, called the tangent algebra to $\mathcal{E}_n$ at $\underset{\sim}{p}$. Since $\mathcal{E}_n$ is flat, $\mathcal{D}_{\underset{\sim}{p}} = \mathcal{D}$, i.e., $\mathcal{D}_{\underset{\sim}{p}}$ is a copy of $\mathcal{D}$ at each point $\underset{\sim}{p} \in \mathcal{E}_n$.

Let $\mathcal{X}_m$ denote an m-surface in $\mathcal{E}_n$. At each point $\underset{\sim}{x} \in \mathcal{X}_m$ there is associated a geometric algebra $\mathcal{D}_{\underset{\sim}{x}}$, called the tangent algebra to $\mathcal{X}_m$ at $\underset{\sim}{x}$. Note that $\mathcal{D}_{\underset{\sim}{x}}$ is of 2^m-dimensions and that $\mathcal{D}_{\underset{\sim}{x}} \subset \mathcal{D}$, i.e., the tangent algebra of the m-surface $\mathcal{X}_m$ at each point $\underset{\sim}{x}$ is a 2^m-dimensional subalgebra of $\mathcal{D}$.

Formal definitions are now given.

Definition 2.1 Euclidian n-space is denoted by $\mathcal{E}_n$. The geometric algebra of $\mathcal{E}_n$ is denoted by $\mathcal{D}$. By $\mathcal{D}^r$ is meant the set of r-vectors $A_r \in \mathcal{D}$, where $0 \leq r \leq n$.

<u>Definition 2.2</u> An m-surface in $\mathcal{E}_n$ is denoted by $\mathcal{X}_m$. The tangent algebra of $\mathcal{X}_m$ at a point $\underset{\sim}{x}$ is denoted by $\mathcal{D}_{\underset{\sim}{x}}$. By $\mathcal{D}^r_{\underset{\sim}{x}}$ is meant the set of tangent r-vectors $A_r \in \mathcal{D}_{\underset{\sim}{x}}$, where $0 \leq r \leq m$.

Note that 1-vectors will always be distinguished from other directed quantities by small underlined letters, such as $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{x}$, $\underset{\sim}{v}$, etc.

The vector $\underset{\sim}{x}$ is always used for the name of a point on the surface $\mathcal{X}_m$. Similarly, $\mathcal{D}_{\underset{\sim}{x}}$ always denotes the tangent algebra <u>of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$</u>. The general rule is: Anything subscripted with an $\underset{\sim}{x}$ refers to the surface $\mathcal{X}_m$.

<u>Definition 2.3</u> A surface $\mathcal{X}_m$ is said to be flat, or a tangent m-plane if for any two points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, $\mathcal{D}_{\underset{\sim}{x}_1} = \mathcal{D}_{\underset{\sim}{x}_2}$.

<u>Definition 2.4</u> A function $F(\underset{\sim}{x})$ is said to be a multivector field on $\mathcal{X}_m$ if $F(\underset{\sim}{x}) \in \mathcal{D}$ for each $\underset{\sim}{x} \in \mathcal{X}_m$. If $F(\underset{\sim}{x}) \in \mathcal{D}_{\underset{\sim}{x}}$ for each $\underset{\sim}{x} \in \mathcal{X}_m$, then $F(\underset{\sim}{x})$ is said to be a <u>tangent</u> multivector field on $\mathcal{X}_m$.

Often $F_{\underset{\sim}{x}}$, where $F_{\underset{\sim}{x}} \equiv F(\underset{\sim}{x})$, is used to denote the value of the function $F(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$.

<u>Definition 2.5</u> The set of all multivector fields on $\mathcal{X}_m$ is denoted by $\{F(\underset{\sim}{x})\}$. The set of all tangent multivector fields

continuous.

<u>Definition 2.6</u> The symbol $\nabla_{\underset{\sim}{x}}$ is called the gradient or tangential derivative operator on the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$.

The tangential derivative $\nabla_{\underset{\sim}{x}}$ differentiates multivector fields on $\mathcal{X}_m$, and behaves algebraically like a vector of $\mathcal{D}^1_{\underset{\sim}{x}}$. For a further discussion of $\nabla_{\underset{\sim}{x}}$, see [9] and [10].

"Dotting" the gradient $\nabla_{\underset{\sim}{x}}$ with a tangent vector $\underset{\sim}{v} \in \mathcal{D}^1_{\underset{\sim}{x}}$ gives $\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}$, the directional derivative operator. This can be shown to be equivalent to the following more usual definition:

<u>Definition 2.7</u> $\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x}) \equiv |\underset{\sim}{v}| \lim_{\Delta\underset{\sim}{x}\to 0} \dfrac{F(\underset{\sim}{x}+\Delta\underset{\sim}{x}) - F(\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$, where $F(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}$, and $\Delta\underset{\sim}{x} \to 0$ in such a way that:

(i) $\underset{\sim}{x} + \Delta\underset{\sim}{x}$ is always a point on $\mathcal{X}_m$

(ii) $\lim_{\Delta\underset{\sim}{x}\to 0} \dfrac{\Delta\underset{\sim}{x}}{|\Delta\underset{\sim}{x}|} = \hat{\underset{\sim}{v}}$, where $\hat{\underset{\sim}{v}} \equiv \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$.

<u>Definition 2.8</u> $y: \mathcal{X}_m \to \mathcal{Y}_k$ is said to be a mapping from the m-surface $\mathcal{X}_m$ to the k-surface $\mathcal{Y}_k$, if $\underset{\sim}{y} = y(\underset{\sim}{x}) \in \mathcal{Y}_k$ for each $\underset{\sim}{x} \in \mathcal{X}_m$.

The smooth surfaces and mappings considered in this paper have the following properties:

Property 2.9 There exists a smooth pseudoscaler field

<u>Property 2.10</u> If $A_{r+1}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}^{r+1}$ then there are multivector fields $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(x)\}_{\underset{\sim}{x}}^{1}$, and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}^{r}$ such that $A_{r+1} = \underset{\sim}{a} \wedge A_r$.

<u>Property 2.11</u> $\nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}}$. Property 2.11 guarantees that $\nabla_{\underset{\sim}{x}}$ behaves algebraically like a vector in $\mathcal{D}_{\underset{\sim}{x}}^{1}$.

<u>Property 2.12</u> If $y: \mathcal{X}_m \to \mathcal{Y}_k$ is a mapping and $F(\underset{\sim}{y}) \in \{F(\underset{\sim}{y})\}$, then for each $\underset{\sim}{v} \in \mathcal{D}_{\underset{\sim}{x}}^{1}$,

$$\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = [\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})] \cdot \nabla_{\underset{\sim}{y}} F(\underset{\sim}{y}) \ .$$

(This is a statement of the chain rule for partial differentiation.)

<u>Property 2.13</u> For any smooth multivector field $F(\underset{\sim}{x})$ on $\mathcal{X}_m$, $\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}} F(\underset{\sim}{x}) = 0$.

(This is equivalent to the property that partial derivatives commute in a flat space. For a further discussion of the significance of this property see Appendix E.)

A "chain rule" for the gradient operator is derived from properties 2.11 and 2.12 in the following theorem.

<u>Theorem 2.14</u> $\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \cdot \nabla_{\underset{\sim}{y}} F(\underset{\sim}{y})$.

$\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})]$ property 2.11

Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be points on $\mathcal{X}_m$.

Definition 2.15 Call $\bar{x}_r \equiv \frac{1}{r!}(\mathbf{x}_1-\mathbf{x})\wedge \ldots \wedge(\mathbf{x}_r-\mathbf{x})$ the r-vector variable of the surface $\mathcal{X}_m$ at the point $\mathbf{x} \in \mathcal{X}_m$.

The r-vector variable $\bar{x}_r$ is an oriented measure of the r-simplex with vertices at the points $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_r$. Note that $|\bar{x}_r|$ is volume of the simplex. See [18, p. 80].

Definition 2.16 Call $\bar{y}_r(\mathbf{x}) \equiv \frac{1}{r!}[y(\mathbf{x}_1)-y(\mathbf{x})]\wedge \ldots \wedge [y(\mathbf{x}_r)-y(\mathbf{x})]$ the r-vector variable of the mapping $\mathbf{y} = y(\mathbf{x})$ at the point $\mathbf{y} = y(\mathbf{x}) \in \mathcal{Y}_k$.

Definition 2.17 Call $\nabla_{\bar{x}_r} \equiv \nabla_{\mathbf{x}_r} \wedge \ldots \wedge \nabla_{\mathbf{x}_1}$ the gradient operator with respect to the r-vector variable $\bar{x}_r$ at the point $\mathbf{x} \in \mathcal{X}_m$. It is understood that $\nabla_{\mathbf{x}_i}$ differentiates only with respect to $\mathbf{x}_i$ and is to be evaluated at $\mathbf{x}_i = \mathbf{x}$.

Certain multivectors $J_{\bar{y}_r} = J_{\bar{y}_r}(\mathbf{x})$, called the characteristic multivectors of the mapping $\mathbf{y} = y(\mathbf{x})$ at the point $\mathbf{x}$, are now defined.

Definition 2.18 $J_{\bar{y}_r}(\mathbf{x}) \equiv \nabla_{\bar{x}_r}\bar{y}_r$, for $r = 1, \ldots, m$.

The usual Jacobian $J_y(\mathbf{x})$ of the mapping $\mathbf{y} = y(\mathbf{x})$ is related to $J_{\bar{y}_m}(\mathbf{x})$ by the following equation:

Definition 2.20 A mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be non-singular if $J_{\bar{y}_m}(\underset{\sim}{x}) \neq 0$ for each $\underset{\sim}{x} \in \mathcal{X}_m$.

The relationship of the Jacobian of a mapping to $J_{\bar{y}_m}$ is further discussed in Appendix B.

This section ends with the lemma given below. It is useful in the proofs of theorems in later sections.

Lemma 2.21

(i) $\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = 0 = \nabla_{\bar{x}_2} \underset{\sim}{y}_1 \cdot \underset{\sim}{y}_2$.

(ii) $\frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 \ldots \underset{\sim}{y}_r = \nabla_{\bar{x}_r} \bar{y}_r = \nabla_{\underset{\sim}{x}_r} \nabla_{\underset{\sim}{x}_{r-1}} \ldots \nabla_{\underset{\sim}{x}_1} \bar{y}_r$.

Proof

(i)
$$\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = \frac{1}{2} \nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} [y(\underset{\sim}{x}_1) - y(\underset{\sim}{x})] \wedge [y(\underset{\sim}{x}_2) - y(\underset{\sim}{x})]$$
$$= \frac{-1}{2} \nabla_{\underset{\sim}{x}_1} \cdot \nabla_{\underset{\sim}{x}_2} [y(\underset{\sim}{x}_2) - y(\underset{\sim}{x})] \wedge [y(\underset{\sim}{x}_1) - y(\underset{\sim}{x})]$$
$$= -\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 .$$

Hence $\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = 0$.

Similarly $\nabla_{\bar{x}_2} \underset{\sim}{y}_1 \cdot \underset{\sim}{y}_2 = 0$.

(ii) The proof of (ii), follows by repeated use of (i).

$$\nabla_{\bar{x}} \bar{y} = \frac{1}{\ldots} \nabla_{\bar{x}} (y - y) \wedge \ldots \wedge (y - y)$$

using (i) and 0.40 $= \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 (\underset{\sim}{y}_3 \wedge \ldots \wedge \underset{\sim}{y}_r) - \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 \cdot (\underset{\sim}{y}_3 \wedge \ldots \wedge \underset{\sim}{y}_r)$

$\vdots$

$= \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 \cdots \underset{\sim}{y}_r$.

Similarly $\nabla_{\bar{x}_r} \bar{y}_r = \nabla_{\underset{\sim}{x}_r} \nabla_{\underset{\sim}{x}_{r-1}} \cdots \nabla_{\underset{\sim}{x}_1} \bar{y}_r$.

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PART I

THE DIFFERENTIAL AND ADJOINT MAPPINGS

3. Definitions and Basic Properties

For each point $\underset{\sim}{x} \in \mathcal{X}_m$ the mapping $y: \mathcal{X}_m \to \mathcal{Y}_k$ induces two linear mappings: (i) The differential mapping $y_\dagger$ from the geometric algebra $\mathcal{D}$ of $\mathcal{E}_n$ at the point $\underset{\sim}{x}$, to the tangent algebra $\mathcal{D}_{\underset{\sim}{y}}$ of $\mathcal{Y}_k$ at the point $\underset{\sim}{y} \doteq y(\underset{\sim}{x})$. (ii) The adjoint mapping $y^\dagger$ from the geometric algebra $\mathcal{D}$ of $\mathcal{E}_n$ at the point $\underset{\sim}{y}$, to the tangent algebra $\mathcal{D}_{\underset{\sim}{x}}$ of $\mathcal{X}_m$ at the point $\underset{\sim}{x}$. These mappings are now defined.

Definition 3.1 $y_\dagger: \mathcal{D} \to \mathcal{D}_y$ is given by:

(i) $y_\dagger A_0 \equiv A_0$, for $A_0 \in \mathcal{D}^0$.

(ii) $y_\dagger A_r = A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r$, for $A_r \in \mathcal{D}^r$ and $1 \leq r \leq n$

(iii) $y_\dagger A = \sum_{i=0}^{n} y_\dagger A_i$, where $A = \sum_{i=0}^{n} A_i \in \mathcal{D}$.

Note that the domain of $y_\dagger$ is not restricted to $\mathcal{D}_{\underset{\sim}{x}}$ as might be expected, but is all of $\mathcal{D}$ the geometric algebra of $\mathcal{E}_n$.

The mapping $y_\dagger$ is sometimes called the "push forward" mapping because it maps tangent vectors in the same "direction" as $y(\underset{\sim}{x})$ maps points.

<u>Definition 3.2</u> $y^{\dagger}: \mathfrak{D} \to \mathfrak{D}_{\underset{\sim}{x}}$ is given by:

(i) $y^{\dagger}A^{0} = A^{0}$, for $A^{0} \in \mathfrak{D}^{0}$.

(ii) $y^{\dagger}A^{r} = \nabla_{\bar{x}_r} \bar{y}_r \cdot A^{r}$, for $A^{r} \in \mathfrak{D}^{r}$ and $1 \leq r \leq n$.

(iii) $y^{\dagger}A = \sum_{i=0}^{n} y^{\dagger}A^{i}$, where $A = \sum_{i=0}^{n} A_i \in \mathfrak{D}$.

Just as for $y_{\dagger}$, the domain of $y^{\dagger}$ is not restricted to $\mathfrak{D}_{\underset{\sim}{y}}$ the tangent algebra of the surface $\mathcal{V}_m$ at the point $\underset{\sim}{y}$, but is all of $\mathfrak{D}$ the geometric algebra of $\mathcal{E}_n$.

The mapping $y^{\dagger}$ is sometimes called the "pull back" mapping because it maps tangent vectors in the opposite "direction" to the "direction" that $y(\underset{\sim}{x})$ maps points.

Finally note that upper and lower indices are used to distinguish between what is being "pushed forward" (lower indices), and what is being "pulled back" (upper indices).

Basic properties of the mappings $y_{\dagger}$ and $y^{\dagger}$ are now studied.

<u>Theorem 3.3</u> (i) $y_{\dagger}(A \wedge B) = y_{\dagger}A \wedge y_{\dagger}B$, for $A, B \in \mathfrak{D}$.

(ii) $y^{\dagger}(A \wedge B) = y^{\dagger}A \wedge y^{\dagger}B$, for $A, B \in \mathfrak{D}$.

<u>Proof</u> Since $y_{\dagger}$ and $y^{\dagger}$ are linear, it is sufficient to show the theorem for r and s vectors A and $B \in \mathfrak{D}$

identity 0.40 $\quad = (r+1)\,(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}_{r+1}}\nabla_{\bar{x}_r})\bar{y}_{r+1}$

$$= (A_r\cdot\nabla_{\bar{x}_r}\ \bar{y}_r)\ \Lambda\ (\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}_{r+1}}\underset{\sim}{y}_{r+1})$$

$$= y_{\dagger}A_r\ \Lambda\ y_{\dagger}\underset{\sim}{b}\ .$$

The proof of (i) is completed by induction on s .

(ii) $y^{\dagger}(\underset{\sim}{a}\Lambda B^s) = \nabla_{\bar{x}_{s+1}}\bar{y}_{s+1}\cdot(\underset{\sim}{a}\Lambda B^s)$

identity 0.42 $\quad = \nabla_{\bar{x}_{s+1}}(\bar{y}_{s+1}\cdot\underset{\sim}{a})\cdot B^s$

identity 0.40 $\quad = \nabla_{\bar{x}_{s+1}}(\bar{y}_s\ \underset{\sim}{y}_{s+1}\cdot\underset{\sim}{a})\cdot B^s$

$$= (\nabla_{\underset{\sim}{x}_{s+1}}\underset{\sim}{y}_{s+1}\cdot\underset{\sim}{a})\ \Lambda\ (\nabla_{\bar{x}_s}\ \bar{y}_s\cdot B^s)$$

$$= y^{\dagger}\underset{\sim}{a}\ \Lambda\ y^{\dagger}B^s\ .$$

The proof of (ii) is completed by induction on r .

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The statement of theorem 3.3 with "dots" replacing "wedges" does not hold, i.e.: $y_{\dagger}(A\cdot B) \neq y_{\dagger}A\cdot y_{\dagger}B$ and $y^{\dagger}(A\cdot B) \neq y^{\dagger}A\cdot y^{\dagger}B$, for an arbitrary mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. If $\underset{\sim}{y} = y(\underset{\sim}{x})$ is a linear mapping, the condition that $y_{\dagger}(\underset{\sim}{a}\cdot\underset{\sim}{b}) = y_{\dagger}\underset{\sim}{a}\cdot y_{\dagger}\underset{\sim}{b}$ for all $\underset{\sim}{a}$, $\underset{\sim}{b} \in \mathfrak{X}^1$ is equivalent to saying $\underset{\sim}{y} = y(\underset{\sim}{x})$ is an orthogonal

<u>Theorem 3.4</u> (i) $A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_{\dagger} A_r$, where $A_r \in \mathcal{D}$, and $r \leq i \leq m$.

(ii) $\nabla_{\bar{x}_i} \bar{y}_i \cdot B^s = (y^{\dagger} B^s) \wedge \nabla_{\bar{x}_{i-s}} \bar{y}_{i-s}$, where $B^s \in \mathcal{D}$, and $s \leq i \leq m$.

<u>Proof</u>

(i) $$A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \frac{(i-r)!r!}{i!} A_r \cdot (\nabla_{\bar{x}_r} \wedge \nabla_{\bar{x}_{i-r}}) \bar{y}_{i-r} \wedge \bar{y}_r$$

[11, p. 13, 3.12] $$= \frac{(i-r)!r!}{i!} \binom{i}{r} A_r \cdot \nabla_{\bar{x}_r} \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge \bar{y}_r$$

$$= \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_{\dagger} A_r .$$

(ii) The proof of (ii) is similar to (i) and is omitted.

XXXX

The following theorem relates the differential and adjoint mappings through the inner product.

<u>Theorem 3.5</u> (i) $(y_{\dagger} A_r) \cdot B^s = y_{\dagger}(A_r \cdot y^{\dagger} B^s)$, where $r \geq s$, and A_r , $B^s \in \mathcal{D}$.

(ii)

Proof

(i) $(y_+ A_r)\cdot B^s = A_r \cdot \nabla_{\bar{x}_r} \dot{\bar{y}}_r \cdot B^s$

theorem 3.4(ii) $= A_r \cdot [(y^\dagger B^s) \wedge \nabla_{\bar{x}_{r-s}}]\, \bar{y}_{r-s}$

identity 0.42 $= [A_r \cdot (y^\dagger B^s)] \cdot \nabla_{\bar{x}_{r-s}}\, \bar{y}_{r-s}$

$= y_+[A_r \cdot (y^\dagger B^s)]$.

(ii) $A_r \cdot y^\dagger B^s = A_r \cdot \nabla_{\bar{x}_s}\, \bar{y}_s \cdot B^s$

theorem 3.4(i) $= \nabla_{\bar{x}_{s-r}} (\bar{y}_{s-r} \wedge y_+ A_r)\cdot B^s$

summary 0.42 $= \nabla_{\bar{x}_{s-r}}\, \bar{y}_{s-r} \cdot [(y_+ A_r)\cdot B^s]$

$= y^\dagger[(y_+ A_r)\cdot B^s]$.

XXXX

Corollary 3.6 $A_r \cdot y^\dagger B^r = (y_+ A_r)\cdot B^r$.

Proof Set $r = s$ in part (i) or (ii) of theorem 3.5

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In Appendix B theorems 3.2 and 3.5 are used to prove

4. Composed Mappings

Let $\mathcal{X}_m$, $\mathcal{Y}_k$, and $\mathcal{Z}_l$ be surfaces in $\mathcal{E}_n$, and suppose $y: \mathcal{X}_m \to \mathcal{Y}_k$, and $z: \mathcal{Y}_k \to \mathcal{Z}_l$. Then the composed mapping $z\circ y: \mathcal{X}_m \to \mathcal{Z}_l$.

Lemma 4.1 $\nabla_{\bar{x}_r} \overline{z\circ y}_r = \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r$

Proof: $$\nabla_{\bar{x}_r} \overline{z\circ y}_r = \nabla_{\bar{x}_r} \frac{1}{r!} z[y(\underset{\sim}{x}_1)] \wedge \ldots \wedge z[y(\underset{\sim}{x}_r)]$$

theorem 2.14 $$= \nabla_{\bar{x}_r} \frac{1}{r!} [y(\underset{\sim}{x}_1)\cdot\nabla_{\underset{\sim}{y}_1} z(\underset{\sim}{y}_1)] \wedge \ldots \wedge [y(\underset{\sim}{x}_r)\cdot\nabla_{\underset{\sim}{y}_r} z(\underset{\sim}{y}_r)]$$

theorem 3.3(i) $$= \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r \; .$$

Theorem 4.2 (i) $(z\circ y)_\dagger A = z_\dagger(y_\dagger A)$, where $A \in \mathcal{D}$.
(ii) $(z\circ y)^\dagger A = y^\dagger(z^\dagger A)$, where $A \in \mathcal{D}$.

Proof Since $y_\dagger$ and $y^\dagger$ are linear, it is sufficient to show the theorem for r-vectors $A_r \in \mathcal{D}$.

$$= z_{\dagger}(y_{\dagger}A_r) \quad .$$

(ii) $(z \circ y)^{\dagger} A^r \quad = \nabla_{\bar{x}_r} \overline{z \circ y}_r \cdot A^r$

Lemma 4.1 $\quad = \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r \cdot A^r$

$$= y^{\dagger}(z^{\dagger}A^r) \quad .$$

The following theorem gives the characteristic multivectors of mappings composed by addition or multiplication.

<u>Theorem 4.3</u> (i) $J_{\overline{z \circ y}_r} = (y^{\dagger} \nabla_{\bar{y}_r}) \bar{z}_r$.

(ii) If $y(\underset{\sim}{x}) = g(\underset{\sim}{x}) + h(\underset{\sim}{x})$, where $g: \mathcal{X}_m \to \mathcal{E}_n$ and $h: \mathcal{X}_m \to \mathcal{E}_n$, then $J_{\bar{y}_r} = \sum_{i=o}^{r} \nabla_{\bar{x}_i} \wedge \nabla_{\bar{x}_{r-i}} \bar{g}_{r-i} \wedge \bar{h}_i$.

<u>Proof</u>:

(i) is a restatement of lemma 4.1 .

(ii) $J_{\bar{y}_r} = \nabla_{\bar{x}_r} \bar{y}_r(\underset{\sim}{x})$

$$= \frac{1}{r!} \nabla_{\bar{x}_r} [g(\underset{\sim}{x}_1) + h(\underset{\sim}{x}_1)] \wedge \ldots \wedge [g(\underset{\sim}{x}_r) + h(\underset{\sim}{x}_r)]$$

In Appendix A, theorem 3.9(ii) is used in calculating the characteristic polynomial of a linear mapping.

5. Non-singular mappings

When $y: \mathcal{X}_m \to \mathcal{Y}_m$ is an invertible mapping (non-singular one-to-one) between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$, the differential and adjoint mappings are also invertible, provided their domains are restricted to $\mathcal{D}_{\underline{x}}$ and $\mathcal{D}_{\underline{y}}$ respectively. This is now shown.

Let $i_{\underline{x}} \in \mathcal{D}^{m}_{\underline{x}}$ be a non-zero pseudoscaler on $\mathcal{X}_m$ at the point $\underline{x}$, and let $i_{\underline{y}} = y_{\dagger} i_{\underline{x}}$ be the corresponding pseudoscaler on $\mathcal{Y}_m$ at the point $\underline{y} = y(\underline{x})$. (Note that $J_{\bar{y}_m} \neq 0$ implies $i_{\underline{y}} \neq 0$, since $i_{\underline{y}} = y_{\dagger} i_{\underline{x}} = i_{\underline{x}} \cdot \nabla_{\bar{x}_m} \bar{y}_m = i_{\underline{x}} J_{\bar{y}_m}$.)

<u>Theorem 5.1</u> If $A \in \mathcal{D}_{\underline{x}}$ and $B \in \mathcal{D}_{\underline{y}}$, and $J_{\bar{y}_m}(\underline{x}) \neq 0$,

then: (i) $A = y^{\dagger} B$ iff $i_{\underline{y}} B = y_{\dagger}(i_{\underline{x}} A)$.

(ii) $B = y_{\dagger} A$ iff $i_{\underline{x}}^{-1} A = y^{\dagger}(i_{\underline{y}}^{-1} B)$.

<u>Proof</u> Since $y_{\dagger}$ and $y^{\dagger}$ are linear, it is sufficient to

show the theorem for all r-vectors $A^r \in \mathcal{D}_{\underline{x}}$ and $B^r \in \mathcal{D}_{\underline{y}}$.

$$y_\dagger i_{\underset{\sim}{x}} A^r = y_\dagger (i_{\underset{\sim}{x}} y^\dagger B^r)$$

$$= (i_{\underset{\sim}{x}} \nabla_{\bar{x}_r} \bar{y}_r \cdot B^r) \cdot \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$$

identity 0.42 $$= i_{\underset{\sim}{x}} (\nabla_{\bar{x}_r} \bar{y}_r \cdot B^r) \wedge \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$$

$$= i_{\underset{\sim}{x}} (y^\dagger B^r) \wedge \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$$

theorem 3.4(ii) $$= i_{\underset{\sim}{x}} \nabla_{\bar{x}_m} \bar{y}_m \cdot B^r$$

$$= i_{\underset{\sim}{x}} \cdot \nabla_{\bar{x}_m} \bar{y}_m B^r = i_{\underset{\sim}{y}} B^r .$$

(ii) Let $A^r = i_{\underset{\sim}{x}}^{-1} A_r$ and $B^r = i_{\underset{\sim}{y}}^{-1} B_r$ in part (i) which has just been proved. Then $i_{\underset{\sim}{x}}^{-1} A_r = y^\dagger i_{\underset{\sim}{y}}^{-1} B_r$ iff $i_{\underset{\sim}{y}} i_{\underset{\sim}{y}}^{-1} B_r = y_\dagger i_{\underset{\sim}{x}} i_{\underset{\sim}{x}}^{-1} A_r$, or $i_{\underset{\sim}{x}}^{-1} A_r = y^\dagger i_{\underset{\sim}{y}}^{-1} B_r$ iff $B_r = y_\dagger A_r$, and the proof is complete.

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6. Curl Free Mappings

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the surface $\mathcal{X}_m$ into the surface $\mathcal{Y}_k$.

Definition 6.1 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be curl free at the point $\underset{\sim}{x} \in \mathcal{X}_m$, if $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$. The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be curl free, if it is curl free at each point $\underset{\sim}{x} \in \mathcal{X}_m$.

The following theorem shows that at points $\underset{\sim}{x} \in \mathcal{X}_m$ where the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is curl free, the differential and adjoint mappings are identical.

Theorem 6.2 If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, then $y_{\dagger}A = y^{\dagger}A$ for each $A \in \mathcal{D}$.

Proof It is sufficient to show the theorem is true for r-vectors $A \in \mathcal{D}$.

or $y_\dagger \underset{\sim}{a} - y^\dagger \underset{\sim}{a} = 0$. For $r = i$ suppose $y_\dagger A_i = y^\dagger A_i$, and for $r = i + 1$ write $A_{i+1} = \underset{\sim}{a} \wedge A_i$. Then

$$y_\dagger A_{i+1} = y_\dagger \underset{\sim}{a} \wedge A_i$$

theorem 3.3(i) $\qquad = y_\dagger \underset{\sim}{a} \wedge y_\dagger A_i$

$\qquad = y^\dagger \underset{\sim}{a} \wedge y^\dagger A_i$

theorem 3.3(ii) $\qquad = y^\dagger \underset{\sim}{a} \wedge A_i = y^\dagger A_{i+1}$.

Hence the theorem is proved.

XXXX

<u>Corollary 6.3</u> $\quad y_\dagger \equiv y^\dagger : \mathcal{D} \to \mathcal{D}_{\underset{\sim}{x}} \cap \mathcal{D}_{\underset{\sim}{y}}$.

<u>Proof</u> $\quad$ The proof follows immediately from theorem 6.2 and the facts that $y_\dagger : \mathcal{D} \to \mathcal{D}_{\underset{\sim}{y}}$, and $y^\dagger : \mathcal{D} \to \mathcal{D}_{\underset{\sim}{x}}$.

XXXX

<u>Corollary 6.4</u> $\quad \underset{\sim}{a} \wedge \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) \wedge \underset{\sim}{a}$, for $\underset{\sim}{a} \in \mathcal{D}'$.

<u>Proof</u> $\quad \underset{\sim}{a} \wedge \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) = \underset{\sim}{a}\, \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) - \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})$ $\qquad$ identity 0.37

theorem 6.2 $\qquad = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \underset{\sim}{a} - \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \cdot \underset{\sim}{a}$

identity 0.37 $\qquad = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \wedge \underset{\sim}{a}$

XXXX

Theorem 6.5 If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ and $A_r \in \mathcal{Y}$, then

(i) $A_r \cdot \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \cdot A_r$.

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \wedge A_r$.

Proof (i) $A_r \cdot \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = A_r \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) - A_r \wedge \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

identity 0.37 $\quad = A_r \nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x}) - (\dot{A}_r \wedge \nabla_{\underset{\sim}{x}}) \cdot y(\underset{\sim}{x})$

identity 0.38 $\quad = [A_r \cdot y(\underset{\sim}{x})] \wedge \nabla_{\underset{\sim}{x}}^{\dagger} = \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) \cdot A_r$.

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = A_r \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y} - A_r \cdot \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y}$

using part (i) $\quad = \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y}\, A_r - \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y} \cdot A_r$

identity 0.37 $\quad = \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) \wedge A_r$.

XXXX

Theorem 6.5 is generalized by the next theorem.

Theorem 6.6 If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, then for all i, $r \leq n$, and $A_r \in \mathcal{Y}$, (i) $A_r \cdot \nabla_{\bar{x}_i}\, \bar{y}_i = \nabla_{\bar{x}_i}\, \bar{y}_i \cdot A_r$.

(ii) $A_r \wedge \nabla_{\bar{x}_i}\, \bar{y}_i = \nabla_{\bar{x}_i}\, \bar{y}_i \wedge A_r$.

Proof Part (ii) is proved first, since it is used in the proof of (i) .

theorem 6.5(ii) $\quad = \frac{1}{i!}\nabla_{x_1}[y_1 \wedge A_r \wedge \nabla_{x_i} \wedge \ldots \wedge \nabla_{x_2}\, y_2] y_3 \ldots y_i$

$\vdots$

theorem 6.5(ii) $\quad = \frac{1}{i!}\nabla_{x_1} \ldots \nabla_{x_i}\, y_i \wedge \ldots \wedge y_1 \wedge A_r$

lemma 2.21(ii) $\quad = \nabla_{\bar{x}_i}\, \bar{y}_i \wedge A_r$.

(i) Let $I \in \mathfrak{D}^n$ be a pseudoscaler of $\mathfrak{D}$. Then for all i , $r \leq n$,

$$A_r \cdot \nabla_{\bar{x}_i}\, \bar{y}_i = I^{-1}(I\, A_r) \wedge \nabla_{\bar{x}_i}\, \bar{y}_i \qquad \text{identity 0.43}$$

using part (ii) $\quad = I^{-1}\, \nabla_{\bar{x}_i}\, \bar{y}_i \wedge (I\, A_r)$

identity 0.43 $\quad = \nabla_{\bar{x}_i}\, \bar{y}_i \cdot A_r$.

The proof of the theorem is complete.

XXXX

The characteristic multivectors of a curl free mapping are particularly simple, as is shown by the final theorem of this section.

Theorem 6.7 If $\nabla_x \wedge y(x) = 0$, then $J_{\bar{y}_r} = \nabla_{\bar{x}_r} \cdot \bar{y}_r$, for $1 \leq r \leq m$.

Proof. The proof is by induction on $r \leq m$. Corollary to simplify expressions.

For $r = 1$ there is nothing to prove. Suppose now for all $r < i$ that $\nabla_{\bar{x}_r} \bar{y}_r = \nabla_{\bar{x}_r} \cdot \bar{y}_r$. Then for $r = i$,

$$\nabla_{\bar{x}_i} \bar{y}_i = \frac{1}{i} [\nabla_{\bar{x}_{i-1}} \nabla_{\underset{\sim}{x}} \underset{\sim}{y} \wedge \bar{y}_{i-1}] \qquad \text{lemma 2.21(ii)}$$

identity 0.37 $= \frac{1}{i} [\nabla_{\bar{x}_{i-1}} \nabla_{\underset{\sim}{x}} \cdot (\underset{\sim}{y} \wedge \bar{y}_{i-1})]$

identity 0.38 $= \frac{1}{i} [\nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x}) \nabla_{\bar{x}_{i-1}} \bar{y}_{i-1} - \nabla_{\bar{x}_{i-1}} \dot{y}(\underset{\sim}{x}) \wedge (\nabla^{\dagger}_{\underset{\sim}{x}} \cdot \bar{y}_{i-1})]$

identity 0.40 $= \frac{1}{i} [\alpha_1 - \nabla_{\bar{x}_{i-2}} \nabla_{\underset{\sim}{x}} y^2_{\dagger}(\underset{\sim}{x}) \wedge \bar{y}_{i-2}]$

cor. A.7(ii) and theorem A.6 $= \frac{1}{i} \{\alpha_1 - \nabla_{\bar{x}_{i-2}} \nabla_{\underset{\sim}{x}} \cdot [y^2_{\dagger}(\underset{\sim}{x}) \wedge \bar{y}_{i-2}]\}$

theorem A.6 $\vdots$
cor. A.7(ii)

$$= \frac{1}{i} [\alpha_s \pm \nabla_{\underset{\sim}{x}} y^i_{\dagger} \underset{\sim}{x}]$$

cor. A.7(ii) and theorem A.6 $= \frac{1}{i} [\alpha_s \pm \nabla_{\underset{\sim}{x}} \cdot y^i_{\dagger} \underset{\sim}{x}]$

XXXX

A proof similar to that of the last theorem is given in Appendix A (theorem A.17).

7. The Identity Mapping

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$, and let $y: \mathcal{X}_m \to \mathcal{X}_m$ be the identity mapping $y(x) \equiv x$.

Theorem 7.1 For $A_r \in \mathcal{D}_x$, (i) $A_r \cdot \nabla_x x = r A_r$

(ii) $A_r \wedge \nabla_x x = (m-r) A_r$ (iii) $\nabla_x x \cdot A_r = r A_r$

(iv) $\nabla_x x \wedge A_r = (m-r) A_r$.

Proof: (i) The proof is by induction on r . The case $r = i$ follows immediately from definition 2.7 with $F(x) \equiv x$. Now assume for $r = i$, that $A_i \cdot \nabla_x x = i A_i$, and for $r = i + 1$ write $A_{i+1} = a \wedge A_i$. Then:

$$A_{i+1} \cdot \nabla_x x = (a \wedge A_i) \cdot \nabla_x x$$

identity 0.38 $= a \wedge (A_i \cdot \nabla_x) x + (-1)^i A_i \, a \cdot \nabla_x x$

identity 0.37 $= a \, A_i \cdot \nabla_x x - a \cdot (A_i \cdot \nabla_x) x + (-1)^i A_i \, a \cdot \nabla_x x$

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = A_r \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - A_r \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}$

using (i) $= A_r i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - rA_r$

property 2.11 $= A_r i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - rA_r$

using (i) $= A_r i_{\underset{\sim}{x}}^{-1} m \, i_{\underset{\sim}{x}} - rA_r$.

$= (m-r) A_r$.

(iii) and (iv) follow from (i) and (ii), using theorem 6.5, if it can be shown that $\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x} = 0$. This is shown below.

$\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x} = [i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} \underset{\sim}{x}]_2$

property 2.11 $= [i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}]_2$

using (i) $= [\, m \,]_2 = 0$.

XXXX

<u>Corollary 7.2</u> $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} = m$, or equivalently $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{x} = m$ and

<u>Lemma 7.3</u> (i) For the mapping $y(x) \equiv \underset{\sim}{x}$ and $A_r \in \mathcal{D}_{\underset{\sim}{x}}$,

$y_\dagger A_r = A_r = y^\dagger A_r$.

(ii) More generally for $A_r \in \mathcal{D}$, $y_\dagger A_r = A_{r_\parallel} = y^\dagger A_r$

where $A_{r_\parallel} \in \mathcal{D}_{\underset{\sim}{x}}$ is the tangential part of A_r to the surface

$\mathcal{X}_m$ in the decomposition $A_r = A_{r_\parallel} + A_{r_\perp}$.

<u>Proof</u> (i) The proof is by induction on r . For $r = 1$, the lemma follows from theorem 7.1(i). Now suppose for $r = i$, $y_\dagger A_i = A_i$, and for $r = i + 1$ write $A_{i+1} = \underset{\sim}{a} \wedge A_i$. Then:

$$y_\dagger A_{i+1} = y_\dagger \underset{\sim}{a} \wedge A_i$$

theorem 3.3(i) $\qquad = y_\dagger \underset{\sim}{a} \wedge y_\dagger A_i$

$$= \underset{\sim}{a} \wedge A_i = A_{i+1} .$$

(ii) The proof of (ii) follows from the decomposition $A_r = A_{r_\parallel} + A_{r_\perp}$ and part (i). I.e.:

$$y_\dagger A_r = y_\dagger (A_{r_\parallel} + A_{r_\perp})$$

$$y_{\dagger} A_{r_{\perp}} = A_{r_{\perp}} \cdot \nabla_{\bar{x}_r} \bar{x}_r \ .$$

property 2.11 $\qquad = 0$.

Thus $\quad y_{\dagger} A_r = A_{r_{\parallel}}$ for any $A_r \in \mathfrak{D}$.

XXXX

Lemma 7.3(i) is used in the proof of the next theorem. Part (ii) of this lemma is later used in section 10.

<u>Theorem 7.4</u> Let $A_r \in \mathfrak{D}^r_{\underset{\sim}{x}}$. Then

$$\text{(i)} \quad A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = \left\{ \begin{matrix} \binom{m-r}{i-r} A_r & \text{for} \quad r \le i \\ \binom{r}{i} A_r & \text{for} \quad r \ge i \end{matrix} \right\} = \nabla_{\bar{x}_i} \bar{x}_i \cdot A_r \ .$$

$$\text{(ii)} \quad A_r \wedge \nabla_{\bar{x}_i} \bar{x}_i = \left\{ \begin{matrix} \binom{m-r}{i} A_r & \text{for} \quad r + i \le m \\ 0 & \text{for} \quad r + i > m \end{matrix} \right\} = \nabla_{\bar{x}_i} \bar{x}_i \wedge A_r \ .$$

<u>Proof</u> (ii) is proved first since it is used in the proof of (i).

(ii) The proof is by induction on i . For $i = 1$, the

$$A_r \wedge \nabla_{\bar{x}_{s+1}} \bar{x}_{s+1} = \frac{1}{s+1} \{[A_r \wedge \nabla_{\bar{x}_s}] \wedge \nabla_{\underset{\sim}{x}_1} \underset{\sim}{x}_1\} \bar{x}_s$$

theorem 7.1(ii) $= \frac{m-(r+s)}{s+1} A_r \wedge \nabla_{\bar{x}_s} \bar{x}_s$

induction hypothesis $= \frac{m-(r+s)}{s+1} \binom{m-r}{s} A_r$

$= \binom{m-r}{s+1} A_r$.

The second equality of theorem 7.4(ii) follows from theorem 6.8(ii).

(i) For $r \leq i$,

$A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = \nabla_{\bar{x}_{i-r}} \bar{x}_{i-r} \wedge y_\dagger A_r$ theorem 3.4(i)

lemma 7.3(i) $= \nabla_{\bar{x}_{i-r}} \bar{x}_{i-r} \wedge A_r$

theorem 7.4(ii) $= \binom{m-r}{i-r} A_r$.

For $r \geq i$,

$$A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = i_{\underset{\sim}{x}}^{-1}(i_{\underset{\sim}{x}} A_r) \wedge \nabla_{\bar{x}_i}$$

Corollary 7.5 $\quad \nabla_{\bar{x}_r} \bar{x}_r = \binom{m}{r}$.

Proof $\quad \nabla_{\bar{x}_r} \bar{x}_r = i_{\underset{\sim}{X}}^{-1} \, i_{\underset{\sim}{X}} \cdot \nabla_{\bar{x}_r} \bar{x}_r$ property 2.11

theorem 7.4(i) $\quad = i_{\underset{\sim}{X}}^{-1} \binom{m}{r} i_{\underset{\sim}{X}}$

$$= \binom{m}{r} .$$

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PART II

MULTIVECTOR FIELDS ON SURFACES

Whereas Part I of this paper only studies the mappings $y_\dagger$ and $y^\dagger$ between the tangent algebras $\mathcal{D}_{\underset{\sim}{x}}$ and $\mathcal{D}_{y(\underset{\sim}{x})}$ for a fixed point $\underset{\sim}{x} \in \mathcal{X}_m$, Part II studies their "field" properties by considering them as mappings of tangent multivector fields on $\mathcal{X}_m$ and $\mathcal{Y}_k$.

8. <u>The Differential and Adjoint Mappings of Multivector Fields</u>

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the m-surface $\mathcal{X}_m$ into the k-surface $\mathcal{Y}_k$.

The adjoint mapping $y^\dagger$, defined and studied in Part I, can be extended pointwise to a mapping of multivector fields on $\mathcal{Y}_k$ into tangent multivector fields on $\mathcal{X}_m$. This is done in the definition below.

<u>Definition 8.1</u> $y^\dagger : \{G(\underset{\sim}{y})\} \to \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ is given by

$F(\underset{\sim}{x}) \equiv y^\dagger G\,[y(\underset{\sim}{x})]$, for each $\underset{\sim}{x} \in \mathcal{X}_m$ and $G(\underset{\sim}{y}) \in \{G(\underset{\sim}{y})\}_{\underset{\sim}{y}}$.

mapping of multivector fields, all properties proved in Part I for $y^{\dagger}$ remain valid.

If the mapping $y\colon \mathcal{X}_m \to \mathcal{Y}_m$ is invertible (one-to-one and non-singular), then the differential mapping $y_{\dagger}$ can also be extended pointwise to a mapping of multivector fields on $\mathcal{X}_m$ into tangent multivector fields on $\mathcal{Y}_m$. This is done in the following definition.

Let $x\colon \mathcal{Y}_m \to \mathcal{X}_m$ denote the inverse mapping of $y(\underset{\sim}{x})$,

i.e.: $\underset{\sim}{x} = x(\underset{\sim}{y})$ iff $\underset{\sim}{y} = y(\underset{\sim}{x})$.

Definition B.2 Let $y(\underset{\sim}{x})$ and $x(\underset{\sim}{y})$ be given as above. Then $y_{\dagger}\colon \{F(\underset{\sim}{x})\} \to \{G(\underset{\sim}{y})\}_{\underset{\sim}{y}}$ is given by $G(\underset{\sim}{y}) \equiv y_{\dagger}F[x(\underset{\sim}{y})]$, for each $F \in \{F(\underset{\sim}{x})\}$, and $\underset{\sim}{y} \in \mathcal{Y}_m$.

The field $G(\underset{\sim}{y})$, where $G(\underset{\sim}{y}) \equiv y_{\dagger}F[x(\underset{\sim}{y})]$ is said to be the "push forward" of the field $F(\underset{\sim}{x})$.

Since the mapping $y_{\dagger}$ is extended pointwise, all properties of $y_{\dagger}$ proved in Part I remain valid.

9. Mapping the Gradient Operator

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the m-surface $\mathcal{X}_m$ into the k-surface $\mathcal{Y}_k$.

Since by property 2.11 the gradient ∇_y behaves like an ordinary vector of $\mathcal{D}^1_y$, the chain rule for the gradient operator (theorem 2.14) can be written in the following instructive way:

$$(9.1) \qquad \nabla_x = y^\dagger \nabla_y \, .$$

Equation (9.1) shows that the gradient ∇_x on the surface $\mathcal{X}_m$ is the gradient ∇_y on the surface $\mathcal{Y}_k$ "pulled back" to the surface $\mathcal{X}_m$.

The next result is theorem 3.3(ii) applied to the gradient ∇_y. It is valid because ∇_y is a vector operator.

Theorem 9.2 $\quad y^\dagger[\nabla_{y_1} \wedge B(y_1)] = y^\dagger \nabla_{y_1} \wedge y^\dagger B(y_1)$, where $B(y)$ is a multivector field on $\mathcal{Y}_k$.

Note that on the right side of the equality in theorem 9.2 that the gradient ∇_y only differentiates $B(y_1)$ and not $y^\dagger$.

<u>Theorem 9.3</u> $\quad y^{\dagger}[\nabla_{\underset{\sim}{y}} \wedge B(\underset{\sim}{y})] = \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} B[y(\underset{\sim}{x})]$

<u>Proof</u> $\quad y^{\dagger}[\nabla_{\underset{\sim}{y}_{s+1}} \wedge B^{s}(\underset{\sim}{y}_{s+1})] = y^{\dagger}\nabla_{\underset{\sim}{y}_{s+1}} \wedge \; y^{\dagger}B^{s}(\underset{\sim}{y}_{s+1})$ theorem 9.2

equation (9.1) $\qquad = \nabla_{\underset{\sim}{x}_{s+1}} \wedge \; \nabla_{\bar{x}_s} \; \bar{y}_s \cdot B^{s}[y(\underset{\sim}{x}_{s+1})]$

property 2.13 $\qquad = \nabla_{\underset{\sim}{x}} \wedge \nabla_{\bar{x}_s} \; \bar{y}_s \cdot B^{s}[y(\underset{\sim}{x})]$

$$= \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} \; B^{s}$$

XXXX

The corresponding statement of theorem 9.3 for dots is false, i.e.: $y^{\dagger}\nabla_{\underset{\sim}{y}} \cdot B(\underset{\sim}{y}) \neq \nabla_{\underset{\sim}{x}} \cdot y^{\dagger}B(\underset{\sim}{y})$. However, in section 13 of this paper it is shown that under certain conditions

$y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot A(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger} \; A(\underset{\sim}{x})$, where $A(\underset{\sim}{x})$ is a multivector field on $\mathcal{X}_m$.

Theorem 9.3 is used in Appendix D to show that the d - operator on differential forms commutes with the pull back mapping of forms.

<u>Theorem 9.4</u> $(y_{\dagger}A_r)\cdot\nabla_{\underset{\sim}{y}_r} B^s(\underset{\sim}{y}_r) = y_{\dagger}(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x}_r)]$,

where $A_r \in \mathfrak{D}$, and $B^s(\underset{\sim}{y}) \in \{F(\underset{\sim}{y})\}$.

Note that $\nabla_{\underset{\sim}{x}_r}$ does not differentiate $y_{\dagger}$ in theorem 9.4 above. The following theorem allows $\nabla_{\underset{\sim}{x}}$ to differentiate $y_{\dagger}$. Its proof depends upon property 2.13.

<u>Theorem 9.5</u> $(y_{\dagger}A_r)\cdot\nabla_{\underset{\sim}{y}} B^s(\underset{\sim}{y}) = y_{\dagger}(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x})]$.

<u>Proof</u> $(y_{\dagger}A_r)\cdot\nabla_{\underset{\sim}{y}} B^s(\underset{\sim}{y}) = y_{\dagger}(A_r\cdot\nabla_{\underset{\sim}{x}_r}) B^s[y(\underset{\sim}{x}_r)]$ lemma 9.4

$= (A_r\cdot\nabla_{\underset{\sim}{x}_r})\cdot\nabla_{\bar{x}_{r-1}} \bar{y}_{r-1} B^s[y(\underset{\sim}{x}_r)]$

property 2.13 $= A_r\cdot(\nabla_{\underset{\sim}{x}}\wedge\nabla_{\bar{x}_{r-1}})\bar{y}_{r-1} B^s[y(\underset{\sim}{x})]$

identity 0.42 $= (A_r\cdot\nabla_{\underset{\sim}{x}})\cdot\nabla_{\bar{x}_{r-1}} \bar{y}_{r-1} B^s[y(\underset{\sim}{x})]$

$= y_{\dagger}(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x})]$

XXXX

Theorem 9.5 is a generalization of the chain rule for the gradient operator.

(9.6) $\quad y_{\dagger}\, i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{y}} \nabla_{\underset{\sim}{y}}$,

where $i_{\underset{\sim}{x}}$ is a pseudoscalar field on $\mathcal{X}_m$, and $i_{\underset{\sim}{y}} \equiv y_{\dagger}\, i_{\underset{\sim}{x}}$ is the corresponding pseudoscalar field on $\mathcal{Y}_m$.

Equation 9.6 shows that the operator $i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}}$ on the surface $\mathcal{X}_m$ is "pushed forward" by $y_{\dagger}$ into the operator $i_{\underset{\sim}{y}} \nabla_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_m$.

Equation 9.6 can also be immediately derived from theorem 9.5 by letting $A_i = i_{\underset{\sim}{x}}$ in that theorem.

10. Lie Brackets

Fundamental to the study of multivector fields on surfaces is the Lie bracket, or bracket operation. In section 10a the definition of the Lie bracket operation of tangent multivector fields is given and its basic properties are studied. Most importantly, it is shown that the Lie bracket of tangent multivector fields is a tangent multivector field, and that the divergence of a tangent multivector field is a tangent multivector field. In section 10b it is shown that the Lie bracket of tangent multivector fields is preserved under the differential mapping.

a) Definition and Basic Properties

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

Definition 10.1 $[A_r,B_s] \equiv (A_r \cdot \nabla_{\underset{\sim}{x}}) \wedge B_s(\underset{\sim}{x}) - A_r(\underset{\sim}{x}) \wedge (\nabla_{\underset{\sim}{x}}^{\dagger} \cdot B_s)$,

where $A_r(\underset{\sim}{x}), B_s(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and $\nabla_{\underset{\sim}{x}}^{\dagger}$ is understood to differentiate only to the left.

The following theorem gives fundamental properties of the bracket operation. Let $\underset{\sim}{a}(\underset{\sim}{x})$, $A_r(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x})$, $B_s(\underset{\sim}{x})$, $C_t(\underset{\sim}{x})$, $\varepsilon \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Theorem 10.2</u> (i) $[A_r+B_s, C_t] = [A_r,C_t] + [B_s,C_t]$, and $[A_r, B_s+C_t] = [A_r,B_s] + [A_r,C_t]$.

(ii) $[A_r\wedge\underset{\sim}{a}, B_s] = A_r\wedge[\underset{\sim}{a},B_s] + (-1)^r\underset{\sim}{a}\wedge[A_r,B_s]$, and $[A_r, \underset{\sim}{b}\wedge B_s] = [A_r,\underset{\sim}{b}]\wedge B_s + (-1)^s [A_r,B_s]\wedge\underset{\sim}{b}$.

(iii) $[A_r,B_s] = -[B_s^\dagger, A_r^\dagger]^\dagger$.

<u>Proof</u> (i) The proof is trivial and is omitted.

(ii) $[A_r\wedge\underset{\sim}{a}, B_s] = [(A_r\wedge\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{x}}]\wedge B_s(x) - [A_r(\underset{\sim}{x})\wedge\underset{\sim}{a}(\underset{\sim}{x})]\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)$

summary 0.38 $= A_r\wedge[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}} B_s(\underset{\sim}{x})] + (-1)^r\underset{\sim}{a}\wedge(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge B_s(\underset{\sim}{x})$

$- A_r\wedge[\underset{\sim}{a}(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)] - (-1)^r\underset{\sim}{a}\wedge[A_r(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)]$

$= A_r\wedge[\underset{\sim}{a},B_s] + (-1)^r\underset{\sim}{a}\wedge[A_r,B_s]$.

The other part of (ii) is proved in a similar way.

$$= \{B_s^\dagger(\underset{\sim}{x})\Lambda(\nabla_{\underset{\sim}{x}}^\dagger\cdot A_r^\dagger) - (B_s^\dagger\cdot\nabla_{\underset{\sim}{x}})\Lambda A_r(\underset{\sim}{x})\}^\dagger$$

$$= -[B_s^\dagger, A_r^\dagger]^\dagger .$$

Similarly, the other part of (iii) is proved.

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<u>Corollary 10.3</u> $[A_r, B_s] = -(-1)^{(r-1)(s-1)}[B_s, A_r]$

<u>Proof</u> The proof follows immediately by substituting

$$A_r^\dagger = (-1)^{\frac{r(r-1)}{2}} A_r ,$$

$$B_s^\dagger = (-1)^{\frac{s(s-1)}{2}} B_s , \text{ and}$$

$$[B_s, A_r] = (-1)^{\frac{(r+s-1)(r+s-1)}{2}} [B_s^\dagger, A_r^\dagger] ,$$

into the right side of theorem 10.3(iii).

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A special case of corollary 10.3 is:

<u>Corollary 10.4</u> $[\underset{\sim}{a}, B_s] = -[B_s, \underset{\sim}{a}]$.

The bracket operation is also defined for directional derivatives. The definition is now given.

The theorem below gives the important relationship between the bracket operation of vector fields, and the bracket operation of directional derivatives. It is further discussed in Appendix E in connection with the "curvature" of a surface.

Theorem 10.6 For any $a(x)$, $b(x) \in \mathcal{D}^1_x$,

(i) $(a \wedge b)\cdot(\nabla_x \wedge \nabla_x) \equiv [a,b]\cdot\nabla_x - [a\cdot\nabla_x, b\cdot\dot{\nabla}_x]$

(ii) $[a,b]\cdot\nabla_x - [a\cdot\nabla_x, b\cdot\nabla_x] \equiv 0$.

Proof (i) The proof of (i) is direct.

$$(a \wedge b)\cdot(\nabla_x \wedge \nabla_x) = a\cdot[b\cdot\nabla_x \; \nabla_x] - b\cdot[a\cdot\nabla_x \; \nabla_x]$$

$$= b\cdot\nabla_x a\cdot\nabla_x - (b\cdot\nabla_x \; a)\cdot\nabla_x - a\cdot\nabla_x b\cdot\nabla_x + (a\cdot\nabla_x \; b)\cdot\nabla_x$$

$$= [a,b]\cdot\nabla_x - [a\cdot\nabla_x, b\cdot\nabla_x] \; .$$

(ii) The proof of (ii) follows from (i) since by property 2.13.

$$(a \wedge b)\cdot(\nabla_x \wedge \nabla_x) \equiv 0 \; .$$

XXXX

Parts (i) and (ii) of theorem 10.6 are kept separate because

Lemma 10.7 $[\underset{\sim}{a},\underset{\sim}{b}] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ if $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Proof The lemma is readily proved by operating on $\underset{\sim}{x}$ by theorem 10.6(ii) and noting that

$$[\underset{\sim}{a},\underset{\sim}{b}]\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x} = [\underset{\sim}{a},\underset{\sim}{b}]_{\parallel}$$

by lemma 7.3(ii), and

$$[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}},\ \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}]\underset{\sim}{x} = [\underset{\sim}{a},\underset{\sim}{b}]$$

by using definition 10.5 and lemma 7.3(i).

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Lemma 10.8 If $\underset{\sim}{a}$, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, then $[\underset{\sim}{a},B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Proof The proof is by induction on s . For $s = 1$, lemma 10.8 reduces to lemma 10.7. Now suppose for $s = i$ that $[\underset{\sim}{a},B_i] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ and for $s = i + 1$, write $B_{i+1} = \underset{\sim}{b} \wedge B_i$, where $\underset{\sim}{b} \in \{F(\underset{\sim}{x})\}^1_{\underset{\sim}{x}}$. Then

$$[\underset{\sim}{a},\ B_{i+1}] = [\underset{\sim}{a},\ \underset{\sim}{b}\wedge B_i]$$

theorem 10.2(ii) $$= [\underset{\sim}{a},\underset{\sim}{b}]\wedge B_i + (-1)^i[\underset{\sim}{a},B_i]\wedge\underset{\sim}{b} .$$

<u>Theorem 10.9</u> If $A_r, B_s \in \{F(\mathbf{x})\}_{\mathbf{X}}$, then $[A_r, B_s] \in \{F(\mathbf{x})\}_{\mathbf{X}}$.

<u>Proof</u> The proof is by induction on r, for a fixed s. For $r = 1$, $[\mathbf{a}, B_s] \in \{F(\mathbf{x})\}_{\mathbf{X}}$ by the previous lemma. Now suppose for $r = i$, $[A_i, B_s] \in \{F(\mathbf{x})\}_{\mathbf{X}}$, and for $r = i + 1$ write $A_{i+1} = A_i \wedge \mathbf{a}$, where $\mathbf{a} \in \{F(\mathbf{x})\}_{\mathbf{X}}$. Then

$$[A_{i+1}, B_s] = [A_i \wedge \mathbf{a}, B_s]$$

theorem 10.2(ii)
$$= A_i \wedge [\mathbf{a}, B_s] + (-1)^i \mathbf{a} \wedge [A_i, B_s] \; .$$

Since both terms after the last equality are in $\{F(\mathbf{x})\}_{\mathbf{X}}$, $[A_{i+1}, B_s] \in \{F(\mathbf{x})\}_{\mathbf{X}}$, and the proof is complete.

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An alternative proof of lemma 10.8 and theorem 10.9 can be obtained by using lemma 10.7 and the following decomposition theorem.

<u>Theorem 10.10</u> (i) $[\mathbf{a}, B_s] = \sum_{i=1}^{s} \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_{i-1} \wedge$

Proof Since (i) is a special case of (ii), only (ii) is proved. The proof is direct.

$$[A_r,B_s] = (A_r\cdot\nabla_{\underset{\sim}{x}})\wedge\underset{\sim}{b}_1 \wedge\ldots\wedge \underset{\sim}{b}_s - A_r\wedge[\nabla^{\dagger}_{\underset{\sim}{x}}\cdot(\underset{\sim}{b}_1 \wedge\ldots\wedge \underset{\sim}{b}_s)]$$

identity 0.40 $$= \sum_{i=1}^{s} (-1)^{i+1}[(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge\underset{\sim}{b}_i(\underset{\sim}{x})]\wedge\underset{\sim}{b}_1 \wedge\ldots\wedge \check{\underset{\sim}{b}}_i \wedge\ldots\wedge \underset{\sim}{b}_s$$

$$-\sum_{i=1}^{s} (-1)^{i+1}[A_r(\underset{\sim}{x})\ \nabla^{\dagger}_{\underset{\sim}{x}}\cdot\underset{\sim}{b}_i]\wedge\underset{\sim}{b}_1 \wedge\ldots\wedge\check{\underset{\sim}{b}}_i \wedge\ldots\wedge \underset{\sim}{b}_s$$

$$= \sum_{i=1}^{s} (-1)^{i+1}[A_r,\underset{\sim}{b}_i]\wedge[\underset{\sim}{b}_1 \wedge\ldots\wedge \check{\underset{\sim}{b}}_i \wedge\ldots\wedge \underset{\sim}{b}_s] .$$

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Theorem 10.10 is used in Appendix D on differential forms.

An important consequence of theorem 10.9 is that the divergence of a tangent multivector field on $\mathcal{X}_m$ is itself a tangent multivector field on $\mathcal{X}_m$. This is proved in the next theorem using the following lemma.

Lemma 10.11 (i) $\nabla_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\wedge A_i) = [\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x})]A_i - \underset{\sim}{a}\wedge[\nabla_{\underset{\sim}{x}}\cdot A_i(\underset{\sim}{x})] +$ $[\underset{\sim}{a},A_i]$, for $\underset{\sim}{a},A_i \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Proof Since (i) is a special case of (ii), only (ii) is proved.

$$\text{(ii)}\quad \nabla_x\cdot(A_r\wedge B_s) = [\nabla_x\cdot A_r(x)]\wedge B_s(x) + (-1)^r A_r(x)\wedge[\dot\nabla_x\cdot B_s(x)]$$

identity 0.38

$$= (\nabla_x\cdot A_r)\wedge B_s + (-1)^{r+1}(A_r\cdot\nabla_x)\wedge B_s + (-1)^r A_r \wedge (\nabla_x^{\dagger}\cdot B_s) + (-1)^r A_r\wedge(\dot\nabla_x\cdot B_s)$$

$$= (\nabla_x\cdot A_r)\wedge B_s + (-1)^r A_r\wedge(\nabla_x\cdot B_s) + (-1)^{r+1}[A_r, B_s]$$

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In the proof above $\dot\nabla_x$ means that the gradient operator differentiates both ways, and $\nabla_x^{\dagger}$ means that it differentiates only to the left.

Theorem 10.12 If $A_r \in \{F(x)\}_x$, then $\nabla_x\cdot A_r(x) \in \{F(x)\}_x$.

Proof The proof is by induction on r. For $r = 1$ the theorem is true, since $\nabla_x\cdot a$ is a scaler. Suppose now for $r = i$ that $\nabla_x\cdot A_i \in \{F(x)\}_x$, and for A_{i+1} write $A_{i+1} = a \wedge A_i$. Then

Since all terms of the last sum are in $\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, it follows that $\nabla_{\underset{\sim}{x}}\cdot A_{i+1} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

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b) The Lie Bracket Under the Differential Mapping

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$. Only tangent multivector fields on $\mathcal{X}_m$ and $\mathcal{Y}_m$ are considered here.

The following lemmas are needed to prove that the Lie bracket is preserved under the differential mapping.

Lemma 10.13 $[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}] = [(y_{\dagger}\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{y}}, (y_{\dagger}\underset{\sim}{b})\cdot\nabla_{\underset{\sim}{y}}]$

Proof $[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}] = [\underset{\sim}{a}\cdot y^{\dagger}\nabla_{\underset{\sim}{y}}, \underset{\sim}{b}\cdot y^{\dagger}\nabla_{\underset{\sim}{y}}]$ using eqn. (9.1)

theorem 9.4 $= [(y_{\dagger}\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{y}}, (y_{\dagger}\underset{\sim}{b})\cdot\nabla_{\underset{\sim}{y}}]$

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Lemma 10.14 $y_{\dagger}[\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}} = [y_{\dagger}\underset{\sim}{a}, y_{\dagger}\underset{\sim}{b}]_{\underset{\sim}{y}}$, for $\underset{\sim}{a}$, $\underset{\sim}{b} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Proof $y_{\dagger}[\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}} = [\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

Lemma 10.15 $\quad y_\dagger[\underset{\sim}{a}, B_s]_{\underset{\sim}{x}} = [y_\dagger \underset{\sim}{a}, y_\dagger B_s]_{\underset{\sim}{y}}$ for $\underset{\sim}{a}$, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$

Proof The proof is by induction on s. For $s = 1$, lemma 10.15 is lemma 10.14. Now suppose for $s = i$ that $y_\dagger[\underset{\sim}{a}, B_i]_{\underset{\sim}{x}} = [y_\dagger \underset{\sim}{a}, y_\dagger B_i]_{\underset{\sim}{y}}$ and for $s = i + 1$, write $B_{i+1} = \underset{\sim}{b} \wedge B_i$. Then

$$y_\dagger[\underset{\sim}{a}, B_{i+1}]_{\underset{\sim}{x}} = y_\dagger[\underset{\sim}{a}, \underset{\sim}{b}\wedge B_i]_{\underset{\sim}{x}}$$

theorem 10.9(ii) $\quad = y_\dagger\{[\underset{\sim}{a}, \underset{\sim}{b}]\wedge B_i + (-1)^i [\underset{\sim}{a}, B_i]\wedge \underset{\sim}{b}\}$

theorem 3.3(i) and lemma 10.14 $\quad = [y_\dagger \underset{\sim}{a}, y_\dagger \underset{\sim}{b}]\wedge y_\dagger B_i + (-1)^i [y_\dagger \underset{\sim}{a}, y_\dagger B_i]\wedge y_\dagger \underset{\sim}{b}$

theorem 10.9(ii) $\quad = [y_\dagger \underset{\sim}{a}, y_\dagger \underset{\sim}{b}\wedge y_\dagger B_i]_{\underset{\sim}{y}}$

theorem 3.3(i) $\quad = [y_\dagger \underset{\sim}{a}, y_\dagger B_{i+1}]_{\underset{\sim}{y}}$

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It now can be shown that the Lie bracket is preserved under the differential mapping.

Theorem 10.19 $\quad y_\dagger[A_r, B_s]_{\underset{\sim}{x}} = [y_\dagger A_r, y_\dagger B_s]_{\underset{\sim}{y}}$, for A_r, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

11. Frames

a) Definitions and Basic Properties

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

Definition 11.1 A set $\{\underset{\sim}{e}_i(\underset{\sim}{x}) \mid i = i, \ldots, m\}$ of m-linearly independent tangent vector fields on $\mathcal{X}_m$ is called a frame on $\mathcal{X}_m$.

Once a frame $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ is chosen on $\mathcal{X}_m$, it is convenient to construct a reciprocal frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ on $\mathcal{X}_m$. It is defined below.

Definition 11.2 The reciprocal frame to $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ is the unique frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ on $\mathcal{X}_m$ satisfying the relations:

$$\underset{\sim}{e}_i(\underset{\sim}{x}) \cdot \underset{\sim}{e}^j(\underset{\sim}{x}) = \delta_i^j \quad \text{for all} \quad i,j \leq m \, .$$

A particularly important frame on a surface is a "coordinate"

$\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ with the property that for each $\underset{\sim}{e}^i(\underset{\sim}{x}) \in \{\underset{\sim}{e}^i(\underset{\sim}{x})\}$, there is a scalar field $\psi^i(\underset{\sim}{x})$ such that $\underset{\sim}{e}^i(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \psi^i(\underset{\sim}{x})$.

For a discussion of the above definitions, and a construction of the reciprocal frames, see [1], p. 83].

<u>Theorem 11.4</u> If $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ is a frame and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ its reciprocal frame, then $[\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{e}_j(\underset{\sim}{x})] \cdot \underset{\sim}{e}^t = - \underset{\sim}{e}_j \cdot [\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{e}^t(\underset{\sim}{x})]$.

<u>Proof</u> The proof follows immediately from differentiating the relationship $\underset{\sim}{e}_j(\underset{\sim}{x}) \cdot \underset{\sim}{e}^t(\underset{\sim}{x}) = \delta_j^t$ by $\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}$.

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<u>Theorem 11.5</u> If $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ is a coordinate frame and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ its reciprocal frame, then: (i) $\nabla_{\underset{\sim}{x}} \wedge e^t(\underset{\sim}{x}) = 0$

(ii) $[\underset{\sim}{e}_i, \underset{\sim}{e}_j] = 0$

(iii) $[\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}, \underset{\sim}{e}_j \cdot \nabla_{\underset{\sim}{x}}] = 0$.

<u>Proof</u> (i) Since $\underset{\sim}{e}^t(\underset{\sim}{x})$ is a coordinate vector, there exists a scalar field $\psi^t(x)$ such that $\nabla_x \psi^t(x) = e^t(x)$. Thus

(ii) By (i), $\nabla_{\underline{x}} \wedge e^t(x) = 0$. This implies:

$$0 = (e_i \wedge e_j)\cdot(\nabla_{\underline{x}} \wedge e^t)$$

$$= e_i\cdot(e_j\cdot\nabla_{\underline{x}}\, e^t) - e_j\cdot(e_i\cdot\nabla_{\underline{x}}\, e^t)$$

theorem 11.5

$$= -(e_j\cdot\nabla_{\underline{x}}\, e_i)\cdot e^t + (e_i\cdot\nabla_{\underline{x}}\, e_j)\cdot e^t$$

$$= [e_i, e_j]\cdot e^t \quad .$$

Since this is true for each $e^t(x) \in \{e^i(x)\}$, and $[e_i, e_j] \in \{F(x)\}_{\underline{x}}$ by lemma 10.7 , $[e_i, e_j] = 0$ for each $i, j \leq m$.

(iii) By theorem 10.6, $[e_i\cdot\nabla_{\underline{x}}, e_j\cdot\nabla_{\underline{x}}] = [e_i, e_j]\cdot\nabla_{\underline{x}}$. But by (ii) , $[e_i, e_j] = 0$. Hence $[e_i\cdot\nabla_{\underline{x}}, e_j\cdot\nabla_{\underline{x}}] = 0$

b) Representation of the Gradient Operator

The gradient operator is now expressed in terms of a frame $\{e^i(x)\}$ and its reciprocal frame $\{e_i(x)\}$. Let $i_{\underline{x}} = e_1(x) \wedge\ldots\wedge e_m(x)$, and $i_{\underline{x}}^{-1} = e^m(x) \wedge\ldots\wedge e^1(x)$. By using the identity 0.41, it is easily seen that $i_{\underline{x}}\cdot i_{\underline{x}}^{-1} = 1$.

$$\nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{x}}^{-1}\, i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}}$$

$$= \underset{\sim}{e}^{m} \wedge \ldots \wedge \underset{\sim}{e}^{1} (\underset{\sim}{e}_1 \wedge \ldots \wedge \underset{\sim}{e}_m) \cdot \nabla_{\underset{\sim}{x}}$$

identity 0.40 $$= \sum_{i=1}^{m} (-1)^{i} (\underset{\sim}{e}^{1} \wedge \ldots \wedge \underset{\sim}{e}^{m}) \cdot (\underset{\sim}{e}_m \wedge \ldots \wedge \check{\underset{\sim}{e}}_i \wedge \ldots \wedge \underset{\sim}{e}_1) \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}$$

identity 0.42 $$= \sum_{i=1}^{m} \underset{\sim}{e}^{i}\, \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \; .$$

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c) Frames Under Mappings

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping (non-singular and one-to-one) between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$.

Let $\{\underset{\sim}{e}^{i}(\underset{\sim}{x})\}$ and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$, and $\{\underset{\sim}{f}^{i}(\underset{\sim}{y})\}$ and $\{\underset{\sim}{f}_i(\underset{\sim}{y})\}$, be frames and their reciprocals on $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

<u>Theorem 11.7</u> (i) $\{\underset{\sim}{f}_i(\underset{\sim}{x}) = y_{\dagger}\underset{\sim}{e}_i(\underset{\sim}{y})\}$ iff $\{\underset{\sim}{e}^{i}(\underset{\sim}{x}) = y^{\dagger}\underset{\sim}{f}^{i}(\underset{\sim}{y})\}$

(ii) $\{\underset{\sim}{e}^{i}(\underset{\sim}{x}) \equiv y^{\dagger}\underset{\sim}{f}^{i}(\underset{\sim}{y})\}$ is a coordinate frame on $\mathcal{X}_m$ iff $\{f^{i}(y)\}$ is a coordinate frame on $\mathcal{Y}_m$.

Proof. (i) Suppose for every i, $f_i = y_{\dagger} e_i$. Then

$$= \underset{\sim}{f}^j \cdot y_{\dagger}\, \underset{\sim}{e}_i$$

cor. 3.6 $$= (y^{\dagger}\, \underset{\sim}{f}^j) \cdot \underset{\sim}{e}_i \quad .$$

This implies that $\underset{\sim}{e}^j = y^{\dagger}\, \underset{\sim}{f}^j$, since reciprocal frames are unique. By reversing the above steps the second half of (i) is proved.

(ii) Suppose that $\{\underset{\sim}{f}^i(\underset{\sim}{y})\}$ is a coordinate frame on $\mathcal{Y}_m$. Then for each i there is a $\psi^i(\underset{\sim}{y})$ such that $\underset{\sim}{f}^i(\underset{\sim}{y}) = \nabla_{\underset{\sim}{y}}\, \psi(\underset{\sim}{y})$.

But then,

$$\underset{\sim}{e}^i \equiv y^{\dagger}\, \underset{\sim}{f}^i$$

$$= y^{\dagger}\, \nabla_{\underset{\sim}{y}}\, \psi^i(\underset{\sim}{y})$$

eqn. (9.1) $$= \nabla_{\underset{\sim}{x}}\, \psi^i[y(\underset{\sim}{x})] \; .$$

Thus $\{\underset{\sim}{e}^i \equiv y^{\dagger}\underset{\sim}{f}^i\}$ is a coordinate frame on $\mathcal{X}_m$. The above steps can be reversed to show the second half of (ii).

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Theorem 11.7 shows that frames and their reciprocals "map" in different directions, and further that coordinate frames when pulled back remain coordinate frames. The first part of the theorem is analogous to theorem 5.1 of section 5.

12. The Divergence of a Field

This section relates the divergences of tangent vector fields on different surfaces. In addition it gives the necessary and sufficient condition for the differential mapping to commute with the divergence operation. Finally properties particular to a coordinate frame are studied.

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between $\mathcal{X}_m$ and $\mathcal{Y}_m$. Let $i_{\underset{\sim}{x}} = i(\underset{\sim}{x})$ be a pseudoscalar field on $\mathcal{X}_m$, and $i_{\underset{\sim}{y}} \equiv y_{\dagger} i_{\underset{\sim}{x}}$ the corresponding pseudoscalar field on $\mathcal{Y}_m$.

Definition 12.1 $g_{\underset{\sim}{x}} \equiv |i_{\underset{\sim}{x}}|^2$, or $\sqrt{g_{\underset{\sim}{x}}} \equiv |i_{\underset{\sim}{x}}|$.

The $\sqrt{g_{\underset{\sim}{x}}}$ is called the "density" or volume element of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, with respect to the pseudoscalar field $i_{\underset{\sim}{x}}$.

Related versions of the following lemma are needed in

<u>Lemma 12.2</u> (i) $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} = 2\, i^{\dagger}_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}})$

(ii) $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} = 2\,(i^{\dagger}_{\underset{\sim}{x}}\,\underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}})$.

<u>Proof</u> The proofs of (i) and (ii) are found in the string of equalities below.

$$\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} = \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, |i_{\underset{\sim}{x}}|^2$$

$$\equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i^{\dagger}_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}}$$

$$= 2\, i^{\dagger}_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}})$$

identity 0.43 $$= 2\,[(i^{\dagger}_{\underset{\sim}{x}}\underset{\sim}{a})\wedge\nabla_{\underset{\sim}{x}}]\cdot i_{\underset{\sim}{x}}$$

identity 0.42 $$= 2\,(i^{\dagger}_{\underset{\sim}{x}}\underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}}) .$$

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Let $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and define $\underset{\sim}{b}(\underset{\sim}{y}) \equiv y_{\dagger}\,\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{y})\}_{\underset{\sim}{y}}$.

The following theorem gives the divergence of $\underset{\sim}{a}(\underset{\sim}{x})$ in terms of the divergence of $\underset{\sim}{b}(\underset{\sim}{y})$.

<u>Theorem 12.3</u> $\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}}\cdot\underset{\sim}{b}(\underset{\sim}{y}) + \frac{1}{2g_{\underset{\sim}{x}}}\,\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} - \frac{1}{2g_{\underset{\sim}{y}}}\,\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\, g_{\underset{\sim}{y}}$.

<u>Proof</u> $\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}}\cdot(i_{\underset{\sim}{x}}\, i^{-1}_{\underset{\sim}{x}}\,\underset{\sim}{a})$

But by a version of lemma 12.2(ii),

$$(i_{\underset{\sim}{x}}^{-1}\ \underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}}) = \frac{1}{2g_{\underset{\sim}{x}}}\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\ g_{\underset{\sim}{x}}\ .$$

On the other hand,

$$(i_{\underset{\sim}{x}}\ \nabla_{\underset{\sim}{x}})\cdot(\underset{\sim}{a}\ i_{\underset{\sim}{x}}^{-1}) = i_{\underset{\sim}{x}}\cdot[\nabla_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot y^{\dagger}\ i_{\underset{\sim}{y}}^{-1})]$$

theorem 3.5(ii), eqn. (9.1) $\quad = i_{\underset{\sim}{x}}\cdot\{y^{\dagger}\nabla_{\underset{\sim}{y}}\wedge y^{\dagger}[(y_{\dagger}\underset{\sim}{a})\cdot i_{\underset{\sim}{y}}^{-1}]\}$

theorem 9.2, cor. 3.6 $\quad = i_{\underset{\sim}{y}}\cdot[\nabla_{\underset{\sim}{y}}\wedge(\underset{\sim}{b}\cdot i_{\underset{\sim}{y}}^{-1})]$

identity 0.43 $\quad = \nabla_{\underset{\sim}{y}}\cdot b(\underset{\sim}{y}) + i_{\underset{\sim}{y}}\cdot(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\ i_{\underset{\sim}{y}}^{-1})$

version of lemma 12.2(i) $\quad = \nabla_{\underset{\sim}{y}}\cdot\underset{\sim}{b}(\underset{\sim}{y}) - \frac{1}{2g_{\underset{\sim}{y}}}\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\ g_{\underset{\sim}{y}}\ .$

Taking the sum of the last two expressions completes the proof of the theorem.

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By an easy computation, using the chain rule and the identity

$$|J_{\bar{\underset{\sim}{y}}_m}(\underset{\sim}{x})| \equiv |i_{\underset{\sim}{x}}^{-1}|\ |i_{\underset{\sim}{y}}| = \frac{\sqrt{g_{\underset{\sim}{y}}}}{\sqrt{g_{\underset{\sim}{x}}}}\quad ,$$

theorem 12.3 is equivalent to:

<u>Theorem 12.4</u> $\quad \nabla_{\underset{\sim}{x}}\cdot a(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}}\cdot b(\underset{\sim}{y}) - \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\ |J_{\bar{\underset{\sim}{y}}_m}(\underset{\sim}{x})|$, where

A trivial but important consequence of theorem 12.4 is:

<u>Corollary 12.5</u> $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$ for each $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ iff $|J_{\bar{y}_m}(\underset{\sim}{x})|$ is constant.

Note that $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$ in corollary 12.5 can be written as $(y^{\dagger}\, \nabla_{\underset{\sim}{y}}) \cdot \underset{\sim}{a} = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$. This is a statement of corollary 3.6 for the gradient operator. It is now shown that under the same conditions as in corollary 12.5, theorem 3.5(i) can be applied to the gradient operator to get:

<u>Theorem 12.6</u> $y_{\dagger}[\nabla_{\underset{\sim}{x}} \cdot A_r] = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\, A_r$ for each $r \geq 1$ and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^r_{\underset{\sim}{x}}$, iff $|J_{\bar{y}_m}|$ is constant.

<u>Proof</u> Because of corollary 12.5 it is sufficient to show that $y_{\dagger}\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}(\underset{\sim}{x})$ for each $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ implies that $y_{\dagger}\nabla_{\underset{\sim}{x}} \cdot A_r = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}A_r$ for each $r \geq 1$, and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^r_{\underset{\sim}{x}}$.

The proof of this is by induction on $r \geq m$. For $r = 1$

$$\nabla_{\underset{\sim}{y}} \cdot y_{\dagger} A_{i+1} = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}(\underset{\sim}{a} \wedge A_i)$$

theorem 3.3(i) $= \nabla_{\underset{\sim}{y}} \cdot (y_{\dagger}\underset{\sim}{a} \wedge y_{\dagger}A_i)$

lemma 10.11 $= [\nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}] y_{\dagger}A_i - (y_{\dagger}\underset{\sim}{a}) \wedge (\nabla_{\underset{\sim}{y}} \cdot y_{\dagger}A_i) + [y_{\dagger}\underset{\sim}{a}, y_{\dagger}A_i]_{\underset{\sim}{y}}$

lemma 10.15 $= (\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}) y_{\dagger}A_i - y_{\dagger}\underset{\sim}{a} \wedge y_{\dagger}\nabla_{\underset{\sim}{x}} \cdot A_i + y_{\dagger}[\underset{\sim}{a}, A_i]_{\underset{\sim}{x}}$

theorem 3.3(i) $= y_{\dagger}\{(\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a})A_i - \underset{\sim}{a} \wedge (\nabla_{\underset{\sim}{x}} \cdot A_i) + [\underset{\sim}{a}, A_i]\}$

lemma 10.11 $= y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot (\underset{\sim}{a} \wedge A_i) = y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot A_{i+1}$.

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Note that any linear mapping $y(\underset{\sim}{x})$ satisfies the conditions of the last theorem.

Now let $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ be a coordinate frame on $\mathcal{X}_m$ and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ its reciprocal frame. For the remainder of this section let $i(\underset{\sim}{x}) = \underset{\sim}{e}_1(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{e}_m(\underset{\sim}{x})$ be the pseudoscalar field under consideration.

The pseudoscalar field $i^{-1}(\underset{\sim}{x}) = \underset{\sim}{e}^m(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{e}^1(\underset{\sim}{x})$ is called the coordinate pseudoscalar field on $\mathcal{X}_m$ with respect to the coordinate frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$.

The following theorem shows that <u>any</u> pseudoscalar field $h(\underset{\sim}{x})$ on $\mathcal{X}_m$ is curl free.

$\nabla_x \psi(x) \in \mathcal{D}_x$ when $\psi(x)$ is a scalar field.

Since $h(x)$ and i_x^{-1} are both pseudoscalar fields on $\mathcal{X}_m$, $h(x) = \psi(x)\, i_x^{-1}$ for some scalar valued $\psi(x)$. But then

$$\nabla_x \wedge h(x) = \nabla_x \wedge \psi(x) i_x^{-1}(x)$$

$$= [\nabla_x \psi(x)] \wedge i_x^{-1} + \psi(x)\, \dot{\nabla}_x \wedge i_x^{-1}$$

theorem 11.5(i) $\qquad = 0$.

Hence $\nabla_x h(x) = \nabla_x \cdot h(x) + \nabla_x \wedge h(x) = \nabla_x \cdot h(x)$.

XXXX

The theorem given below is well known. For a more usual formulation and proof see [14, p. 130].

<u>Theorem 12.8</u> $\quad \nabla_x \cdot e_i(x) = \dfrac{1}{2g_x}\, e_i \cdot \nabla_x g_x = e_i \cdot \nabla_x \ln \sqrt{g_x}$.

<u>Proof</u> $\quad e_i \cdot \nabla_x g = 2g\,(e_i \nabla_x i_x) \cdot i_x^{-1}$ version of lemma 12.2(i)

$$= 2g\,[e_i \cdot \nabla_x e_1 \wedge \ldots \wedge e_m] \cdot [e^m \wedge \ldots \wedge e^1]$$

theorem 11.5(ii) $= 2g \sum_{j=1}^{m} e^j \cdot (e_j \cdot \nabla_x e_i)$

theorem 11.6 $= 2g \nabla_x \cdot e_i(x)$.

Thus $e_i \cdot \nabla_x g = 2g_x \nabla_x \cdot e_i(x)$. The remainder of the proof is trivial.

XXXX

<u>Theorem 12.9</u> The following statements are equivalent:

(i) g_x is constant.

(ii) $\nabla_x \cdot e_i(x) = 0$ for each $i \leq m$.

(iii) $\nabla_x \cdot i_x \equiv 0$.

<u>Proof</u> That (i) $\leftrightarrows$ (ii) is a direct consequence of theorem 12.8.

That (i) $\leftrightarrows$ (iii) is a direct consequence of a version of lemma 12.2(ii).

XXXX

13. The Shape Operator

The results of the preceding sections require only that the surfaces under consideration be sufficiently smooth. The shapes of the surfaces have not been a point of interest. In this final section, a "shape operator" is defined, which provides a measure of the shape of a surface.

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$, and let $p_{\underset{\sim}{x}} \equiv p(\underset{\sim}{x})$ be a unit pseudoscalar field on $\mathcal{X}_m$.

<u>Definition 13.1</u> Call $S(\underset{\sim}{a}) \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, p(\underset{\sim}{x})$ for each $\underset{\sim}{a} \in \mathcal{D}^1_x$, the shape operator of the surface $\mathcal{X}_m$ with respect to the pseudoscalar field $p_{\underset{\sim}{x}}$.

(Note that a unit pseudoscalar field on a surface is unique up to an orientation.)

A few basic properties of the shape operator are given in the following theorem.

Let $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Theorem 13.2

(i) $S(\mathbf{a}) \wedge \mathbf{b} = -\, p_{\mathbf{x}} \wedge [\mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b}]$.

(ii) $S(\mathbf{a}) \wedge \mathbf{b} = S(\mathbf{b}) \wedge \mathbf{a}$.

(iii) $S(\mathbf{a})\cdot p_{\mathbf{x}} = 0$.

(iv) $\nabla_{\bar{\mathbf{x}}_2}\, p(\mathbf{x}_1)\cdot p(\mathbf{x}_2) = 0$.

Proof (i) Since $\mathbf{b} \in \{F(\mathbf{x})\}_{\mathbf{x}}$, $p_{\mathbf{x}} \wedge b = 0$. Thus,

$$
\begin{aligned}
0 &= \mathbf{a}\cdot\nabla_{\mathbf{x}}\, p_{\mathbf{x}}\wedge\mathbf{b} \\
&= (\mathbf{a}\cdot\nabla_{\mathbf{x}}\, p_{\mathbf{x}})\wedge\mathbf{b} + p_{\mathbf{x}}\wedge(\mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b}) \\
&= S(\mathbf{a})\wedge\mathbf{b} + p_{\mathbf{x}}\wedge(\mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b})
\end{aligned}
$$

Hence $S(\mathbf{a}) \wedge \mathbf{b} = -\, p_{\mathbf{x}} \wedge [\mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b}]$.

(ii)
$$
\begin{aligned}
S(\mathbf{a}) \wedge \mathbf{b} - S(\mathbf{b}) \wedge \mathbf{a} &= -\, p_{\mathbf{x}}\wedge(\mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b}) + p_{\mathbf{x}}\wedge(\mathbf{b}\cdot\nabla_{\mathbf{x}}\, \mathbf{a}) \\
&= -\, p_{\mathbf{x}}\wedge(\mathbf{a}\cdot\nabla_{\mathbf{x}}\mathbf{b} - \mathbf{b}\cdot\nabla_{\mathbf{x}}\mathbf{a}) \\
&= -\, p_{\mathbf{x}}\wedge[\mathbf{a},\mathbf{b}] = 0 ,
\end{aligned}
$$

since by lemma 10.7, $[a,b] \in \mathcal{D}^{1}_{\mathbf{x}}$. This is sufficient to prove (ii)

(iv) The proof of (iv) is similar to the proof of lemma 2.21(i), and is omitted.

XXXX

The shape operator $S(\underset{\sim}{a})$ is a generalization of the Weingarten mapping to surfaces which are not necessarily hypersurfaces. See [12, p. 21, 77]. The Weingarten mapping is further discussed in Appendix E.

APPENDICES

Appendix A. Linear Mappings

In this appendix basic ideas of linear algebra are reformulated in terms of the geometric language developed in this paper. The use of geometric algebra in the study of linear mappings makes the introduction of matrix algebra largely unnecessary.

Let $y\colon \mathcal{E}_n \to \mathcal{E}_n$ be a linear mapping from $\mathcal{E}_n$ into $\mathcal{E}_n$. Since the mapping $y(\underset{\sim}{x})$ is from $\mathcal{E}_n$ into $\mathcal{E}_n$, $\mathcal{D}_{\underset{\sim}{x}} = \mathcal{D} = \mathcal{D}_{y(\underset{\sim}{x})}$ for all $\underset{\sim}{x} \in \mathcal{E}_n$. Also $\mathcal{E}_n = \mathcal{D}^1$, i.e., vectors which are names for points in $\mathcal{E}_n$ are identified with tangent vectors of $\mathcal{D}^1$.

In this appendix the mapping $y(\underset{\sim}{x})$ is always taken to be linear.

a) Basic Definitions and Properties

Definition A.1 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be linear, provided for all scalars α, β, and points $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$,

$$y(\alpha\underset{\sim}{x}_1+\beta\underset{\sim}{x}_2) = \alpha y(\underset{\sim}{x}_1) + \beta y(\underset{\sim}{x}_2).$$

Definition A.3 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be skew-symmetric if for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$, $y(\underset{\sim}{x}_1)\cdot\underset{\sim}{x}_2 = -\underset{\sim}{x}_1\cdot y(\underset{\sim}{x}_2)$.

The following theorem shows that a linear mapping is equivalent to its differential mapping at each point $\underset{\sim}{x} \in \mathcal{E}_n$.

Theorem A.4 If $y_{\dagger}$ is the differential mapping of $y(\underset{\sim}{x})$ at any point $\underset{\sim}{x} \in \mathcal{E}_n$, then $y_{\dagger}(\underset{\sim}{a}) = y(\underset{\sim}{a})$ for all $\underset{\sim}{a} \in \mathcal{D}^1_{\underset{\sim}{x}} = \mathcal{E}_n$

Proof $\quad y_{\dagger}\underset{\sim}{a} = \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

def. 2.7 $\quad = |\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \dfrac{y(\underset{\sim}{x}+\Delta\underset{\sim}{x}) - y(\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$

$$= |\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \frac{y(\Delta\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$$

$$= y\left(|\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \frac{\Delta\underset{\sim}{x}}{|\Delta\underset{\sim}{x}|} \right)$$

def. 2.7 $\quad = y\,[|\underset{\sim}{a}|\,\hat{\underset{\sim}{a}}] = y(\underset{\sim}{a})$

XXXX

Lemma A.5

Proof (i) $\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x}\cdot y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}_1}\, \underset{\sim}{x}_1\cdot y(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1}\, y(\underset{\sim}{x}_1)\cdot \underset{\sim}{x}$

$$= y_{\dagger}\underset{\sim}{x} + y^{\dagger}\underset{\sim}{x}$$

(ii) $\underset{\sim}{x}\cdot[\nabla_{\underset{\sim}{x}_1}\wedge y(\underset{\sim}{x}_1)] = \underset{\sim}{x}\cdot\nabla_{\underset{\sim}{x}_1} y(\underset{\sim}{x}_1) - \nabla_{\underset{\sim}{x}_1} y(\underset{\sim}{x}_1)\cdot\underset{\sim}{x}$ identity 0.39

$$= y_{\dagger}\underset{\sim}{x} - y^{\dagger}\underset{\sim}{x}$$

XXXX

Theorem A.6 The following statements are equivalent:

(i) $y(\underset{\sim}{x})$ is symmetric

(ii) $y_{\dagger} \equiv y^{\dagger}$

(iii) $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

Proof It is shown that (i) $\rightleftarrows$ (ii) $\rightleftarrows$ (iii) .

(i) $\rightleftarrows$ (ii). If $y(\underset{\sim}{x})$ is symmetric, then for all $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2 \in \mathcal{E}_n$,

$$y(\underset{\sim}{x}_1)\cdot\underset{\sim}{x}_2 = \underset{\sim}{x}_1\cdot y(\underset{\sim}{x}_2)$$

theorem A.4 $\qquad = \underset{\sim}{x}_1\cdot y_{\dagger}\,\underset{\sim}{x}_2$

cor. 3.6 $\qquad = (y^{\dagger}\underset{\sim}{x}_1)\cdot\underset{\sim}{x}_2$

(ii) $\rightleftarrows$ (iii) follows trivially from lemma A.5(ii).

XXXX

Corollary A.7 (i) If $y(\underset{\sim}{x})$ and $w(\underset{\sim}{x})$ are symmetric linear mappings, and $y \circ w \equiv w \circ y$, then $y \circ w$ is symmetric.

(ii) If $y(\underset{\sim}{x})$ is symmetric, then $y^i(\underset{\sim}{x}) \equiv y \overset{1}{\circ} \ldots \overset{i-1}{\circ} y(\underset{\sim}{x})$ is symmetric.

Proof (i) Because of theorem A.6(ii) it is sufficient to show that $(y \circ w)_\dagger = (y \circ w)^\dagger$.

$$(y \circ w)_\dagger = y_\dagger \circ w_\dagger \qquad \text{theorem 4.2(i)}$$

theorem A.4 $\qquad = w_\dagger \circ y_\dagger$

theorem A.6(ii) $\qquad = w^\dagger \circ y^\dagger$

theorem 4.2(ii) $\qquad = (y \circ w)^\dagger$

(ii) is a trivial consequence of (i).

XXXX

The analogous theorem to A.6 for skew-symmetric mappings is:

Theorem A.8 The following statements are equivalent:

The next theorem decomposes $y(\underset{\sim}{x})$ into the sum of a symmetric mapping and a skew-symmetric mapping. Its proof is an easy consequence of the preceding theorems, and is omitted.

Theorem A.9 $y(\underset{\sim}{x}) = \frac{1}{2} \nabla_{\underset{\sim}{x}} \underset{\sim}{x} \cdot y(\underset{\sim}{x}) + \frac{1}{2} \underset{\sim}{x} \cdot [\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]$, where the first term on the right is symmetric and the second term skew-symmetric.

Definition A.10 Call $\nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x})$ the trace of $y(\underset{\sim}{x})$.

This is equivalent to the definition of trace in matrix theory.

Theorem A.11 If $y(\underset{\sim}{x})$ is skew-symmetric, then $\nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x}) = 0$.

Proof By theorem A.9, $y(\underset{\sim}{x}) = \frac{1}{2} \underset{\sim}{x} \cdot [\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]$. Thus,

$$\nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x}) = \frac{1}{2} \nabla_{\underset{\sim}{x}} \cdot \{\underset{\sim}{x} \cdot [\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]\}$$

identity 0.42 $= \frac{1}{2} [\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x}] \cdot [\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x})]$

cor. 7.2 $= 0$.

XXXX

<u>Theorem A.13</u> The following statements are equivalent:

(i) $y(\underset{\sim}{x})$ is orthogonal.

(ii) $y^\dagger(y_\dagger \underset{\sim}{x}) = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in \mathcal{E}_n^1$.

(iii) $(y_\dagger \underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

<u>Proof</u> It will be shown that (i) → (ii) → (iii) → (i) .

(i) → (ii) If $y(\underset{\sim}{x}_1)\cdot y(\underset{\sim}{x}_2) = \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2$ for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$, then by using theorem A.4 and corollary 3.6, $\underset{\sim}{x}_1\cdot y^\dagger y_\dagger \underset{\sim}{x}_2 = \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2$ for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$. This implies $y^\dagger(y_\dagger \underset{\sim}{x}_2) = \underset{\sim}{x}_2$ for all $\underset{\sim}{x}_2 \in \mathcal{E}_n$.

(ii) → (iii) If $y^\dagger y_\dagger \underset{\sim}{x} = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in \mathcal{E}_n$, then

$$\underset{\sim}{x}^2 = \underset{\sim}{x}\cdot y^\dagger y_\dagger \underset{\sim}{x}$$

cor. 3.6 $$= y_\dagger \underset{\sim}{x}\cdot y_\dagger \underset{\sim}{x} \ .$$

Hence, $(y_\dagger \underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

(iii) → (i) If $(y_\dagger \underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$, then

$$y_\dagger \underset{\sim}{x}_1\cdot y_\dagger \underset{\sim}{x}_2 = \frac{1}{2}\{[y_\dagger(\underset{\sim}{x}_1+\underset{\sim}{x}_2)]^2 - (y_\dagger \underset{\sim}{x}_1)^2 - (y_\dagger \underset{\sim}{x}_2)^2\}$$

$$= \frac{1}{2}\{(\underset{\sim}{x}_1+\underset{\sim}{x}_2)^2 - \underset{\sim}{x}_1^2 - \underset{\sim}{x}_2^2\}$$

b) The Characteristic Polynomial

<u>Definition A.14</u> For the linear mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$, let $c(\underset{\sim}{x}) = y(\underset{\sim}{x}) - \lambda\underset{\sim}{x}$ for each $\underset{\sim}{x} \in \mathcal{E}_n$, and where λ is a scalar. Then the characteristic polynomial of $\underset{\sim}{y} = y(\underset{\sim}{x})$ in the variable λ is $\Psi(\lambda) \equiv J_{\bar{c}_n}$, and its characteristic equation in the variable λ is $\Psi(\lambda) = 0$.

<u>Theorem A.15</u> $\Psi(\lambda) = \sum_{i=0}^{n} (-1)^i \, [J_{\bar{y}_{n-i}}]_0 \, \lambda^i$.

<u>Proof</u> In theorem 4.3(ii) let $g(\underset{\sim}{x}) = y(\underset{\sim}{x})$, and $h(\underset{\sim}{x}) = -\lambda\underset{\sim}{x}$. Then $\Psi(\lambda) \equiv J_{\bar{c}_n}$

theorem 4.3(ii) $= \sum_{i=0}^{n} \nabla_{\bar{x}_i} \Lambda \nabla_{\bar{x}_{n-i}} \, \bar{y}_{n-i} \Lambda \overline{(-\lambda x)}_i$

$= \sum_{i=0}^{n} (-1)^i \lambda^i \nabla_{\bar{x}_{n-i}} \Lambda \nabla_{\bar{x}_i} \, \bar{x}_i \Lambda \bar{y}_{n-i}$

identity 0.42 $= \sum_{i=0}^{n} (-1)^i \lambda^i \, \nabla_{\bar{x}_{n-i}} \cdot [\nabla_{\bar{x}_i} \bar{x}_i \Lambda \bar{y}_{n-i}]$

Theorem A.15 identifies the scalar parts of the characteristic multivectors of a mapping as being the coefficients or scalar invariants of its characteristic equation. By theorem 6.9, these are the only parts when $y(\underset{\sim}{x})$ is symmetric.

Theorem A.16 A linear mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ satisfies its characteristic equation.

Proof It must be shown that $\nabla_{\bar{x}_n} \cdot \bar{y}_n \underset{\sim}{x} - \nabla_{\bar{x}_{n-1}} \cdot \bar{y}_{n-1} y(\underset{\sim}{x}) + \ldots + (-1)^n y^n(\underset{\sim}{x}) \equiv 0$ for all $\underset{\sim}{x} \in \mathcal{E}_n$. The first term on the left will be decomposed into the negative of the remaining terms.

$$\nabla_{\bar{x}_n} \cdot \bar{y}_n \underset{\sim}{x} = \underset{\sim}{x}\, \nabla_{\bar{x}_n} \cdot \bar{y}_n$$

$$= \underset{\sim}{x} \cdot \nabla_{\bar{x}_n} \bar{y}_n$$

theorem 3.4(i) $$= \nabla_{\bar{x}_{n-1}} \ddot{y}_{n-1} \wedge y_{+}\underset{\sim}{x}$$

theorem A.4 $$= \nabla_{\bar{x}_{n-1}} \cdot [\bar{y}_{n-1} \wedge y(\underset{\sim}{x})]$$

[11, p.13, 3.12] $$= \nabla_{\bar{x}_{n-1}} \cdot \bar{y}_{n-1}\, y(\underset{\sim}{x}) - \frac{1}{(n-1)!} \nabla_{\bar{x}_{n-1}} \cdot (y_2 \wedge \ldots \wedge y_{n-1} \wedge y)\, y_1$$

The last steps follow by expanding $[\nabla_{\bar{x}_{n-1}} \cdot (\underset{\sim}{y}_2 \wedge \ldots \wedge \underset{\sim}{y}_{n-1} \wedge \underset{\sim}{y})]\underset{\sim}{y}_1$, and making repeated use of the fact that $y^i(\underset{\sim}{x}) \cdot \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = y^{i+1}(\underset{\sim}{x})$.

XXXX

A known result in matrix theory is that the scalar invariants of the characteristic polynomial of a matrix can be expressed in terms of traces of powers of the matrix. The final theorem of this section shows the equivalent by a recursive decomposition of $[J_{\bar{y}_r}]_0$.

<u>Theorem A.17</u> For the mapping $y(\underset{\sim}{x})$, $[J_{\bar{y}_1}]_0 \equiv \nabla_{\underset{\sim}{x}} \cdot y$, and $[J_{\bar{y}_r}]_0 = \frac{1}{r} \{\nabla_{\underset{\sim}{x}} \cdot y [J_{\bar{y}_{r-1}}]_0 - \nabla_{\underset{\sim}{x}} \cdot y^2 [J_{\bar{y}_{r-2}}]_0 + \ldots + (-1)^{r+1} \nabla_{\underset{\sim}{x}} \cdot y^r\}$, for $r \leq n$.

<u>Proof</u> $[J_{\bar{y}_r}]_0 = \nabla_{\bar{x}_r} \cdot \bar{y}_r$

$$= \frac{1}{r} (\nabla_{\bar{x}_{r-1}} \wedge \nabla_{\underset{\sim}{x}}) \cdot (y \wedge \bar{y}_{r-1})$$

identity 0.38
$$= \frac{1}{r} \{\nabla_{\underset{\sim}{x}} \cdot y \; \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - [(\nabla_{\bar{x}_{r-1}} \cdot y) \wedge \nabla_{\underset{\sim}{x}}^{\dagger}] \cdot \bar{y}_{r-1}\}$$

identity 0.40
$$= \frac{1}{r} \{\nabla_{\underset{\sim}{x}} \cdot y \; \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - (\nabla_{\bar{x}_{r-2}} \wedge \nabla_{\underset{\sim}{x}}) \cdot (y^2 \wedge \bar{y}_{r-2})\}$$

$\vdots$

$$= \frac{1}{r} \{\nabla_{\underset{\sim}{x}} \cdot y \; \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - \nabla_{\underset{\sim}{x}} \cdot y^2 \; \nabla_{\bar{x}_{r-2}} \cdot \bar{y}_{r-2} \ldots$$

The proof is now complete.

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The proof of this theorem is almost identical to that of theorem 6.7.

c) Invariant Linear Subspaces

Definition A.18 A point set $\mathcal{I}_r \subset \mathcal{E}_n$ is called an r-plane of $\mathcal{E}_n$ through the origin if there is a simple r-vector $I_r \in \mathcal{D}^r$ such that,

$$\mathcal{I}_r = \mathcal{I}_r(I_r) \equiv \{\underset{\sim}{x} \in \mathcal{E}_n \mid \underset{\sim}{x} \wedge I_r = 0\} \quad .$$

The simple r-vector I_r is said to define the r-plane $\mathcal{I}(I_r)$.

As a "flat" r-surface in $\mathcal{E}_n$, as defined in section 2, $\mathcal{I}_r$ has a geometric algebra $\mathcal{D}_{\mathcal{I}_r}$ of 2^r-dimensions. The r-vector I_r is a pseudoscalar of $\mathcal{D}_{\mathcal{I}_r}$, and as such is unique up to a scalar multiple. The r-plane $\mathcal{I}_r$ can also be regarded as a linear subspace of $\mathcal{E}_n$.

addition $\lambda \neq 0$, then λ is called a proper r-value and I_r is called a proper invariant r-vector.

Invariant 1-vectors and 1-values are also called eigenvectors and eigenvalues.

The following theorem gives the relationships between invariant r-vectors, and invariant linear subspaces of a linear mapping $y(\underset{\sim}{x})$.

Theorem A.20

(i) If $\mathcal{J}_r$ is an invariant linear subspace, and $\mathcal{J}_r = \mathcal{J}(I_r)$, then I_r is an invariant r-vector.

(ii) If I_r is a proper invariant r-vector, then $\mathcal{J}_r \equiv \mathcal{J}_r(I_r)$ is an invariant subspace, and the mapping $y(\underset{\sim}{x})$ when restricted to $\mathcal{J}_r$ is non-singular.

Proof (i) Since I_r is a pseudoscalar element of $\mathcal{D}_{\mathcal{J}_r}$, there are vectors $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r \in \mathcal{J}_r$ such that theorem A.4 $\quad = y(\underset{\sim}{x}_1) \wedge \ldots \wedge y(\underset{\sim}{x}_r)$

$$= \lambda I_r ,$$

for some scalar λ, since $y(\underset{\sim}{x}_i) \in \mathcal{J}_r$ for each i, and pseudoscalar elements are unique up to a scalar multiple.

(ii) Let $\underset{\sim}{x} \in \mathcal{J}_r$. It must be shown that $y(\underset{\sim}{x}) \in \mathcal{J}_r$, or equivalently that $y(\underset{\sim}{x}) \wedge I_r = 0$. Since $\underset{\sim}{x} \in \mathcal{J}_r$, $\underset{\sim}{x} \wedge I_r = 0$, and thus

$$0 = y_{\dagger} (\underset{\sim}{x} \wedge I_r)$$

theorem 3.3(i) $\quad = y_{\dagger}\underset{\sim}{x} \wedge y_{\dagger} I_r$

theorem A.4 $\quad = y(\underset{\sim}{x}) \wedge y_{\dagger} I_r$.

But since $y_{\dagger} I_r = \lambda I_r$ where $\lambda \neq 0$, it follows that $y(\underset{\sim}{x}) \wedge I_r = 0$.

Finally the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to $\mathcal{J}_r$ is non-singular because,

$$J_{\bar{y}_{\mathcal{J}_r}} \equiv I_r^{-1} \, I_r \cdot \nabla_{\bar{x}_r} \, \bar{y}_r$$

$$= I_r^{-1} \, y_{\dagger} \, I_r$$

The final theorem of this appendix factors the characteristic polynomial of $\underset{\sim}{y} = y(\underset{\sim}{x})$ into the product of characteristic polynomials of $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to invariant linear subspaces.

Let I be a pseudoscalar element of $\mathcal{G}$, the geometric algebra of $\mathcal{E}_n$, and suppose $I = I_{r_1} \wedge \ldots \wedge I_{r_k}$, where I_{r_i} are invariant r_i-vectors of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. Let $\Psi(\lambda)$ be the characteristic polynomial of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$, and let $\Psi_{r_i}(\lambda)$ be the characteristic polynomials of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to the invariant subspaces $\mathcal{L}(I_{r_i})$.

<u>Theorem A.21</u> $\Psi(\lambda) = \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda)$

<u>Proof</u> Let $c(\underset{\sim}{x}) = y(\underset{\sim}{x}) - \lambda\underset{\sim}{x}$. Then by definition,

$$\Psi(\lambda) = \nabla_{\bar{x}_n} \bar{c}_n$$

$$= I^{-1}\, I \cdot \nabla_{\bar{x}_n} \bar{c}_n$$

$$= I^{-1}(I_{r_1} \wedge \ldots \wedge I_{r_k}) \cdot \nabla_{\bar{x}_n} \bar{c}_n$$

$$= I^{-1}[(I_{r_1} I_{r_1}^{-1} I_{r_1} \cdot \nabla_{\bar{x}_{r_1}} \bar{c}_{r_1}) \wedge$$

$$\ldots \wedge (I_{r_k} I_{r_k}^{-1} I_{r_k} \cdot \nabla_{\bar{x}_{r_k}} \bar{c}_{r_k})]$$

$$= I^{-1}[I_{r_1} \Psi_{r_1}(\lambda) \wedge \ldots \wedge I_{r_k} \Psi_{r_k}(\lambda)]$$

$$= I^{-1} I \, \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda)$$

$$= \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda) .$$

XXXX

Appendix B. Jacobians and Transformations of Integrals

The purpose of this appendix is to show how the methods of this paper can be used to derive formulas from advanced calculus relating to Jacobians and transformations of integrals. In part (a) the relation of the characteristic multivector $\mathcal{J}_{\bar{y}_m}$ to the Jacobian is discussed. In part (b) differential statements proved in Part II are rewritten as transformation formulas for integrals.

a) The Jacobian of a Mapping

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be a mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ in $\mathcal{E}_n$. The following is a more general definition of the Jacobian than is given in advanced calculus books.

Definition B.1 Call $J_{\bar{y}_m}(\underset{\sim}{x})$ the Jacobian of the mapping $y(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$.

It will be shown below that this definition is equivalent to the usual definition of the Jacobian when $m = n$, i.e., when

$\mathcal{E}_n$. In terms of this frame the mapping $y(\underset{\sim}{x})$ can be written as

$$y(\underset{\sim}{x}) = \sum_{i=1}^{n} y_i(\underset{\sim}{x})\, \underset{\sim}{e}_i \text{ , where } \underset{\sim}{x} = \sum_{i=1}^{n} x_i\, \underset{\sim}{e}_i \text{ .}$$

The following is the usual definition of the Jacobian in terms of the partial derivatives of its components $y_i(\underset{\sim}{x})$. See for example [3, p.139].

Definition B.2 The Jacobian of the mapping $y(\underset{\sim}{x})$ is:

$$J_y(\underset{\sim}{x}) \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1}, & \ldots, & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1}, & \ldots, & \frac{\partial y_n}{\partial x_n} \end{vmatrix} .$$

Theorem B.3 When $\mathcal{X}_m$ and $\mathcal{Y}_m$ are n-surfaces in $\mathcal{E}_n$, $J_y(\underset{\sim}{x}) \equiv J_{\bar{y}_n}(\underset{\sim}{x})$.

Proof The proof follows directly from the definition of

$J_{\bar{y}}(\underset{\sim}{x})$:

theorem 3.3(i) $= [e_n \wedge \ldots \wedge e_1]\cdot[(e_1\cdot\nabla_x y) \wedge \ldots \wedge (e_n\cdot\nabla_x y)]$

identity 0.41
$$= \begin{vmatrix} e_1\cdot(e_1\cdot\nabla_x y) \ , & \ldots & , \ e_n\cdot(e_1\cdot\nabla_x y) \\ \vdots & & \vdots \\ e_1\cdot(e_n\cdot\nabla_x y) \ , & \ldots & , \ e_n\cdot(e_n\cdot\nabla_x y) \end{vmatrix}$$

$$= J_y(x) \ ,$$

since $e_j\cdot[e_i\cdot\nabla_x \, y(x)] = e_j\cdot[\sum_k \frac{\partial y_k}{\partial x_i} e_k] = \sum_k \frac{\partial y_k}{\partial x_i} \delta_{jk} = \frac{\partial y_j}{\partial x_i}$

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The key to the interpretation of $J_{\bar{y}_m}(x)$ is the following identity: (for $m \le n$)

$$\text{(B.4)} \quad J_{\bar{y}_m}(x) = i_x^{-1} \, i_x\cdot\nabla_{\bar{x}_m} \bar{y}_m = i_x^{-1} \, i_y \ ,$$

where i_x is a directed volume element of the surface $\mathcal{X}_m$ at the point x, and $i_y \equiv y_+ i_x$ is the corresponding directed volume element of the surface $\mathcal{Y}_m$ at the point $y = y(x)$.

In words (B.4) says that the Jacobian of the mapping $y(x)$ is the ratio of corresponding directed volume elements on the surfaces.

Finally note that

Considering the surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ to be embedded in $\mathcal{E}_n$ allows not only the comparison in magnitudes (B.5), but a comparison in directions as well (B.4).

b) Integral Transformations

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ in $\mathcal{E}_n$, and let $F(\underset{\sim}{y})$ be a multivector field on $\mathcal{Y}_m$. (Note that it is not required that $F(\underset{\sim}{y})$ be a <u>tangent</u> multivector field on $\mathcal{Y}_m$.)

Property 2.12 is a differential statement of the chain rule. It can also be represented in the following integral form: Let $\mathcal{C}_{\underset{\sim}{x}}$ be any (smooth) curve in $\mathcal{X}_m$, and $\mathcal{C}_{\underset{\sim}{y}}$ the (smooth) curve in $\mathcal{Y}_m$ which is the image of $\mathcal{C}_{\underset{\sim}{x}}$ under the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$.

$$(B.6) \quad \int_{\mathcal{C}_{\underset{\sim}{x}}} d\underset{\sim}{x} \cdot \nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = \int_{\mathcal{C}_{\underset{\sim}{y}}} d\underset{\sim}{y} \cdot \nabla_{\underset{\sim}{y}} F(\underset{\sim}{y}) \quad ,$$

where $d\underset{\sim}{y} = d\underset{\sim}{x} \cdot \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})$ is the differential vector of arc on the curve $\mathcal{C}_{\underset{\sim}{y}}$ corresponding to $d\underset{\sim}{x}$, the differential vector of arc on the curve $\mathcal{C}_{\underset{\sim}{x}}$. (As a reference, see [5, p.367].)

Now let $\mathcal{S}^r$ be an r-subsurface of $\mathcal{X}_m$ and $\mathcal{S}^r$ the

<u>Theorem B.7</u> $$\int_{\mathcal{A}_{\underset{\sim}{y}}^r} dY_r\, F(\underset{\sim}{y}) = \int_{\mathcal{A}_{\underset{\sim}{x}}^r} dX_r \cdot \nabla_{\bar{X}_r}\, \bar{y}_r\, F[y(\underset{\sim}{x})],$$

where $dY_r \equiv y_{\dagger} dX_r = dX_r \cdot \nabla_{\bar{X}_r}\, \bar{y}_r$ is the differential r-vector of directed area on the surface $\mathcal{A}_{\underset{\sim}{y}}^r$ corresponding to dX_r, the differential r-vector of directed area on the surface $\mathcal{A}_{\underset{\sim}{x}}^r$.

<u>Corollary B.8</u> $$\int_{\mathcal{A}_{\underset{\sim}{y}}^r} |dY_r|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}_{\underset{\sim}{x}}^r} |dX_r \cdot \nabla_{\bar{X}_r}\, \bar{y}_r|\, F[y(\underset{\sim}{x})]$$

<u>Corollary B.9</u> $$\int_{\mathcal{A}_{\underset{\sim}{y}}^m} |dY_m|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}_{\underset{\sim}{x}}^m} |dX_m|\, |J_{\bar{y}_m}(\underset{\sim}{x})|\, F[y(\underset{\sim}{x})],$$

where $\mathcal{A}_{\underset{\sim}{x}}^m$ and $\mathcal{A}_{\underset{\sim}{y}}^m$ are m-surfaces in $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

Corollary B.9 is a statement of the change of variables formula for integrals found in advanced calculus books. See for example [3, p.273].

<u>Theorem B.10</u> $$\int dY_r \cdot \nabla_y\, F(\underset{\sim}{y}) = \int dX_r \cdot \nabla_{\bar{x}}\, \bar{y}_{r-1}\, F[y(\underset{\sim}{x}_r)].$$

<u>Corollary B.11</u> $$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} dY_m \nabla_{\underset{\sim}{y}} F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} dX_m \nabla_{\bar{x}_m} \bar{y}_{m-1} F[y(\underset{\sim}{x}_m)] ,$$

where $\mathcal{A}^m_{\underset{\sim}{x}}$ and $\mathcal{A}^m_{\underset{\sim}{y}}$ are m-surfaces in $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

Corollary B.11 is the integral statement of equation (9.6), the "dual" chain rule for the gradient operator.

Appendix C. Examples of Mappings

This appendix provides explicit calculations for two kinds of mappings. In part (a), mappings are studied which are of the kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$, where $\psi(\underset{\sim}{x})$ is a scalar valued function. In part (b), mappings are studied which are of the kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\underset{\sim}{p}$, where $\psi(\underset{\sim}{x})$ is a scalar valued function, and $\underset{\sim}{p}$ is a constant vector.

a) Mappings of the Kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$.

Let $y\colon \mathcal{X}_m \to \mathcal{Y}_m$ be given by $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$, where $\psi = \psi(\underset{\sim}{x})$ is a scalar valued function.

<u>Theorem C.1</u> For the mapping above, and tangent multivectors $A_r \in \mathcal{D}^r_{\underset{\sim}{x}}$, and $B^r \in \mathcal{D}^r_{\underset{\sim}{y}}$,

(i) $y_{\dagger}A_r = \psi^{r-1}\,[\psi A_r + (A_r \cdot \nabla_{\underset{\sim}{x}}\psi)\wedge \underset{\sim}{x}]$

(ii) $y^{\dagger}B^r = \psi^{r-1}\,[\psi B^r + (\nabla_{\underset{\sim}{x}}\psi)\wedge(\underset{\sim}{x}\cdot B^r)]$

(iii) $J_{\bar{y}_m} = \psi^{m-1}\,[\psi + (\nabla_{\underset{\sim}{x}}\psi)\cdot \underset{\sim}{x}]$

<u>Proof</u> (i) $y_\dagger A_r \equiv A_r\cdot\nabla_{\bar{x}_r}\,\bar{y}_r$

$$= A_r\cdot\nabla_{\bar{x}_r}\frac{1}{r!}\;\psi_1\underset{\sim}{x}_1\wedge\ldots\wedge\psi_r\underset{\sim}{x}_r$$

$$= A_r\cdot[(\nabla_{\underset{\sim}{x}_r}\psi_r + \psi\nabla_{\underset{\sim}{x}_r})\wedge\ldots\wedge(\nabla_{\underset{\sim}{x}_1}\psi + \psi\,\nabla_{\underset{\sim}{x}_1})]\bar{x}_r$$

$$= \psi^r A_r\cdot\nabla_{\bar{x}_r}\bar{x}_r + \psi^{r-1}A_r\cdot[(\nabla_{\underset{\sim}{x}}\psi)\wedge\nabla_{\bar{x}_{r-1}}]\bar{x}_{r-1}\wedge\underset{\sim}{x}$$

theorem 7.4(i) $$= \psi^r A_r + \psi^{r-1}[A_r\cdot(\nabla_{\underset{\sim}{x}}\psi)]\cdot\nabla_{\bar{x}_{r-1}}\bar{x}_{r-1}\wedge\underset{\sim}{x}$$

theorem 7.4(i) $$= \psi^{r-1}[\psi A_r + (A_r\cdot\nabla_{\underset{\sim}{x}}\psi)\wedge\underset{\sim}{x}].$$

(ii) is proved in a similar way to (i).

(iii) is proved by using (i) in the identity

$$J_{\bar{y}_m} \equiv i_{\underset{\sim}{x}}^{-1}y_\dagger i_{\underset{\sim}{x}}\;.$$

(iv) is proved by using (i) in (9.6), and the fact that

$$J_{\bar{y}_m}^{-1} = i_{\underset{\sim}{y}}^{-1}\;i_{\underset{\sim}{x}}\;,$$

which follows from (B.4).

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An example of this kind of mapping is the following: Let $y\colon \mathcal{E}_3 - \{0\} \to \mathcal{E}_3$ be given by $y(\underset{\sim}{x}) = \frac{1}{\underset{\sim}{x}^2}\underset{\sim}{x}$. (The mapping $y(\underset{\sim}{x})$ is an inversion of $\mathcal{E}_3$ through the 2-sphere of radius one cen-

Corollary C.2 For the mapping $y(\underset{\sim}{x})$ given above,

(i) $y_{\dagger}A_r = \left(\frac{1}{\underset{\sim}{x}^2}\right)^3 [A_r - \frac{2}{\underset{\sim}{x}^2}(A_r \cdot \underset{\sim}{x})\Lambda \underset{\sim}{x}] = y^{\dagger}A_r$.

(ii) $J_{\bar{y}_3} = -\left(\frac{1}{\underset{\sim}{x}^2}\right)^3$

(iii) $\nabla_{\underset{\sim}{y}} = \underset{\sim}{x}^2\nabla_{\underset{\sim}{x}} - 2\underset{\sim}{x}\,\underset{\sim}{x}\cdot\nabla_{\underset{\sim}{x}} = -\underset{\sim}{x}\,\nabla_{\underset{\sim}{x}}\overset{\frown}{\underset{\sim}{x}}$, where $\overset{\frown}{\underset{\sim}{x}}$ indicates that the gradient operator is not to differentiate the $\underset{\sim}{x}$.

Proof The proof is a straight forward calculation using theorem C.1. It is helpful to note that since $\nabla_{\underset{\sim}{x}} \Lambda\, y(\underset{\sim}{x}) = 0$, $y_{\dagger}A_r = y^{\dagger}A_r$ by theorem 6.2.

XXXX

b) Mappings of the Kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\,\underset{\sim}{p}$.

Let $y\colon \mathcal{X}_m \to \mathcal{Y}_m$ be given by $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\,\underset{\sim}{p}$, where $\psi(x)$ is a scalar valued function, and $\underset{\sim}{p}$ is a constant vector in $\mathcal{D}$.

(i) $y_{\dagger}A_r = A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi)\Lambda \underset{\sim}{p}$

(ii) $y^{\dagger}B^r = B^r + (\nabla_{\underset{\sim}{x}}\psi)\Lambda(\underset{\sim}{p}\cdot B^r)$

(iii) $J_{\bar{y}_m} = 1 - \underset{\sim}{p}_{\perp} \nabla_{\underset{\sim}{x}}\psi + \underset{\sim}{p}_{\parallel} \cdot \nabla_{\underset{\sim}{x}}\psi$, where

$p_{\parallel} \in \mathcal{D}^1_{\underset{\sim}{x}}$, is the tangential component of $\underset{\sim}{p}$ to the surface $\mathcal{X}_m$,

$\underset{\sim}{p}_{\perp} = \underset{\sim}{p} - \underset{\sim}{p}_{\parallel}$ is the normal component of $\underset{\sim}{p}$ to the surface $\mathcal{X}_m$.

(iv) $\nabla_{\underset{\sim}{y}} = J_{\bar{y}_m}^{-1} \{\nabla_{\underset{\sim}{x}} - \underset{\sim}{p}_{\perp}(\nabla_{\underset{\sim}{x}}\psi)\Lambda\nabla_{\underset{\sim}{x}} + \underset{\sim}{p}_{\parallel}\cdot[(\nabla_{\underset{\sim}{x}}\psi)\Lambda\nabla_{\underset{\sim}{x}}]\}$,

where $\underset{\sim}{p}_{\parallel}$ and $\underset{\sim}{p}_{\perp}$ are given as in (iii).

<u>Proof</u> (i) $y_{\dagger}A_r \equiv A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r$

$= A_r \cdot \nabla_{\bar{x}_r} \frac{1}{r!} (\underset{\sim}{x}_1 + \psi_1 \underset{\sim}{p}) \Lambda \ldots \Lambda (\underset{\sim}{x}_r + \psi_r \underset{\sim}{p})$

$= A_r \cdot \nabla_{\bar{x}_r} \bar{x}_r + A_r \cdot [(\nabla_{\underset{\sim}{x}}\psi)\Lambda\nabla_{\bar{x}_{r-1}}] \bar{x}_{r-1}\Lambda\underset{\sim}{p}$

theorem 7.4(i) $= A_r + [A_r \cdot (\nabla_{\underset{\sim}{x}}\psi)] \cdot \nabla_{\bar{x}_{r-1}} \bar{x}_{r-1}\Lambda\underset{\sim}{p}$

theorem 7.4(i) $= A_r + [A_r \cdot (\nabla_{\underset{\sim}{x}}\psi)]\Lambda\underset{\sim}{p}$

(ii) The proof of (ii) is similar to (i) .

(iii) The proof of (iii) follows by using (i) in the

(iv) The proof of (iv) follows by using (i) in (9.6), the fact that $J_{\bar{y}_m}^{-1} = i_{y}^{-1} i_{x}$, and an algebraic simplification.

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An example of this kind of mapping is the following: Let $y: \mathcal{X}_2 \subset \mathcal{E}_3 \to \mathcal{Y}_2 \subset \mathcal{E}_3$ be given by $y(x) = x + \sqrt{1 + x^2}\, p$, where:

(i) $\mathcal{X}_2$ is a unit disc centered at the origin,

(ii) p is a normal unit vector to the disc $\mathcal{X}_2$,

(iii) $\mathcal{Y}_2$ is the hemisphere having $\mathcal{X}_2$ as its base.

<u>Corollary C.4</u> For the mapping given above, and tangent multivectors $A_r \in \mathcal{D}_{x}^{r}$, and $B^r \in \mathcal{D}_{y}^{r}$,

(i) $y_{\dagger} A_r = A_r - \frac{1}{\sqrt{1-x^2}} (A_r \cdot x)\, p$

(ii) $y^{\dagger} B_r = B^r - \frac{1}{\sqrt{1-x^2}}\, x \wedge (p \cdot B^r)$

(iii) $J_{\bar{y}_2} = 1 + \frac{1}{\sqrt{1-x^2}}\, p\, x$

<u>Proof</u> The proof is a straight forward calculation using theorem C.3 . Note that since $\underset{\sim}{p} = \underset{\sim}{p}_{\parallel} + \underset{\sim}{p}_{\perp}$ is perpendicular to $\mathcal{X}_2$, $\underset{\sim}{p}_{\parallel} = 0$.

XXXX

As a final example of the kind of mapping in theorem C.3, let $y: \mathcal{X}_2 \subset \mathcal{E}_3 \to \mathcal{Y}_2 \subset \mathcal{E}_3$ be given by $y(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{x}^2 \underset{\sim}{p}$,

where:

(i) $\mathcal{X}_2$ is a plane through the origin,

(ii) $\underset{\sim}{p}$ is a normal unit vector to the plane $\mathcal{X}_2$,

(iii) $\mathcal{Y}_2$ is a paraboloid having $\mathcal{X}_2$ as a tangent plane at the origin.

<u>Corollary C.5</u> For the mapping given above, and tangent multivectors $A_r \in \mathcal{D}^r_{\underset{\sim}{x}}$ and $B^r \in \mathcal{D}^r_{\underset{\sim}{y}}$,

(i) $y_{+}A_r = A_r + 2\, A_r \cdot \underset{\sim}{x}\, \underset{\sim}{p}$

(ii) $y^{\dagger} B^r = B^r + 2\, \underset{\sim}{x} \wedge (\underset{\sim}{p} \cdot B^r)$

(iii) $J_{\bar{x}} = 1 + 2xp$

<u>Proof</u> The proof is a straight forward calculation using theorem C.3. Again note that since $\underset{\sim}{p} = \underset{\sim}{p}_{\parallel} + \underset{\sim}{p}_{\perp}$ is normal to $\mathcal{X}_2$, $\underset{\sim}{p}_{\parallel} = 0$.

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Appendix D. Differential Forms

In this appendix the exact relationship between differential forms and geometric algebra is revealed. The algebraic and differential operators which are piecewise introduced on differential forms to enrich their algebraic character, are all simply and directly expressed in terms of geometric algebra together with its one vector differential operator.

A table of the relationships between the two algebraic systems is given in the summary of this paper.

a) Definitions and Basic Properties

The following formal definition of an r-form is used. It is equivalent to that given in [12, p.50] or [4, p.62] .

Definition D.1 A differential r-form on a surface $\mathcal{X}_m$ is a function $f^r(\underset{\sim}{x})$ which assigns to each point $\underset{\sim}{x} \in \mathcal{X}_m$ the real valued function $f^r_{\underset{\sim}{x}} = f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots ; \underset{\sim}{w}_r)$ of the r vector variables $w_1, \ldots, w_r \in \mathcal{M}_m$, with the following properties:

i.e.: $f^r_{\underset{\sim}{x}}$ is antisymmetric over any interchange of its vector variables.

(ii) $f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ is linear in each of its vector variables.

Since $f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ is a function of the r vector variables $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r$, it can be differentiated by $\nabla_{\bar{w}_r}$, the gradient operator with respect to the r-vector variable $\bar{w}_r$ of the tangent m-plane to the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$. Note that $\nabla_{\bar{x}_r} \neq \nabla_{\bar{w}_r}$ unless the surface $\mathcal{X}_m$ is flat at the point $\underset{\sim}{x}$.

Differentiating $f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ by $\nabla_{\bar{w}_r}$ is the key idea to the following theorem which gives the one-to-one correspondence that exists between r-forms on $\mathcal{X}_m$, and tangent r-vector fields on $\mathcal{X}_m$.

<u>Theorem D.2</u> (i) To each r-form $f^r(\underset{\sim}{x})$, there is an

(ii) Conversely, if an r-vector field $F^r(\underset{\sim}{x})$ is given, then $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv F^r(\underset{\sim}{x}) \cdot V^{\dagger}_r$ is a differential r-form.

Proof (i) The proof is a direct verification. Let $F^r(\underset{\sim}{x})$, and $V_r \in \mathcal{D}^r_{\underset{\sim}{x}}$ be given as in the theorem. Then:

$$F^r(\underset{\sim}{x}) \cdot V^{\dagger}_r = V^{\dagger}_r \cdot F^r(\underset{\sim}{x})$$

$$= \frac{1}{r!} (\underset{\sim}{v}_r \wedge \ldots \wedge \underset{\sim}{v}_1) \cdot (\nabla_{\underset{\sim}{w}_1} \wedge \ldots \wedge \nabla_{\underset{\sim}{w}_r})\, f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$$

identity 0.41
def. D.1(i)

$$= \frac{1}{r!} [r!\, \underset{\sim}{v}_1 \cdot \nabla_{\underset{\sim}{w}_1} \cdots \underset{\sim}{v}_r \cdot \nabla_{\underset{\sim}{w}_r} f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)]$$

theorem A.4

$$= f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) .$$

(ii) It is easy to check that $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv F^r(\underset{\sim}{x}) \cdot V^{\dagger}_r$ is a differential r-form.

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The following are helpful definitions for giving a geometric interpretation to an r-form.

Definition D.3 An r-form $f^r(\underset{\sim}{x})$ is said to be simple

<u>Definition D.4</u> If A_r and B_r are simple r-vectors, then $\cos\theta \equiv \hat{A}_r \cdot \hat{B}_r^\dagger = \dfrac{A_r \cdot B_r^\dagger}{|A_r|\ |B_r|}$ defines the angle θ between them. (See [18, p.56].)

Theorem D.2 along with these definitions make the geometric interpretation of a simple r-form evident: A simple r-form $f^r_{\mathbf{x}}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is a scalar measure of the relative directions of the simple r-vector $F^r = F^r(\mathbf{x})$ and the r-vector variable $V_r = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_r$. In particular, when $V_r = F^r$, $f^r_{\mathbf{x}}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = |F^r|^2$.

The Grassmann, or exterior product $f^r_{\mathbf{x}} \wedge g^s_{\mathbf{x}}$ of forms $f^r_{\mathbf{x}}$ and $g^s_{\mathbf{x}}$ is now defined in the conventional way. (See for example [12, p.51] or [1, p.55].)

<u>Definition D.5</u> $f^r_{\mathbf{x}} \wedge g^s_{\mathbf{x}}(\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_{r+s})$

$$= \left(\begin{matrix} r+s \\ \ \end{matrix}\right) \sum (-1)^{\pi} f^r_{\mathbf{x}} \otimes g^s_{\mathbf{x}} (\mathbf{v}_1, \ldots, \mathbf{v}$$

set $\{1, 2, \ldots, r+s\}$.

The theorem below gives the simple relationship between the exterior product of forms, and the outer product of multivector fields.

Theorem D.6 If $f^r_{\underset{\sim}{X}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r_{\underset{\sim}{X}} \cdot V^\dagger_r$, and $g^s_{\underset{\sim}{X}}(\underset{\sim}{v}_{r+1}, \ldots, \underset{\sim}{v}_{r+s}) = G^s_{\underset{\sim}{X}} \cdot W^\dagger_s$, where $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_s$, and $W_s = \underset{\sim}{v}_{r+1} \wedge \ldots \wedge \underset{\sim}{v}_{r+s}$, then $f^r_{\underset{\sim}{X}} \wedge g^s_{\underset{\sim}{X}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r+s}) = (F^r_{\underset{\sim}{X}} \wedge G^s_{\underset{\sim}{X}}) \cdot (V_r \wedge W_s)^\dagger$.

Proof The proof is an algebraic identity and is omitted.

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Using theorem D.6, the properties of the exterior product of forms follow easily from the properties of the outer product of multivectors in geometric algebra. Some of these properties are now given.

Theorem D.7 (i) The exterior product of forms is bilinear.

Proof Let F^r_x , G^s_x , and H^t_x be the multivectors corresponding to the differential forms f^r_x , g^s_x , and h^t_x respectively, at the point $x \varepsilon \mathcal{X}_m$. The proof of the theorem follows from the following algebraic properties of geometric algebra, and theorem D.6.

(i) The $\wedge$-product of multivectors is bilinear.

(ii) $F^r_x \wedge G^s_x = (-1)^{rs} G^s_x \wedge F^r_x$. (identity D.45)

(iii) $F^r_x \wedge (G^s_x \wedge H^t_x) = (F^r_x \wedge G^s_x) \wedge H^t_x$.

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b) The Exterior Derivative

The exterior derivative d-operator of forms is often defined in the following way. See for example [12, p.89] or [4, p.65].

Definition D.8 Let f^r_x be an r-form. Then

$$df^r_x (v_1, \ldots , v_{r+1}) \equiv \sum_{i=1}^{r+1} (-1)^{i+1} v_i \cdot \nabla_x f^r_x (v_1, \ldots , \check{v}_i, \ldots , v_{r+1})$$

$\check{v}_j$, means the j^{th} vector v_j is omitted.

The theorem below shows that if $F^r(x)$ is the r-vector field corresponding to f^r_x , then $\nabla_x \wedge F^r(x)$ is the (r+1)-vector field corresponding to df^r_x .

Theorem D.9 If $f^r_x(v_1, \ldots, v_r) = F^r(x)\cdot V^\dagger_r$, where $V_r = v_1 \wedge \ldots \wedge v_r$, then $df^r_x(v_1, \ldots, v_{r+1}) = [\nabla_x \wedge F^r(x)]\cdot V_{r+1}$, where $V_{r+1} = v_1 \wedge \ldots \wedge v_{r+1}$.

Proof The proof is a direct verification.

$$[\nabla_x \wedge F^r(x)]\cdot V^\dagger_{r+1} = [v_{r+1}\wedge\ldots\wedge v_1]\cdot[\nabla_x \wedge F^r(x)]$$

identity 0.42 $$= \{[v_{r+1}\wedge\ldots\wedge v_1]\cdot\nabla_x\}\cdot F^r(x)$$

identity 0.40 $$= \left\{\sum_{i=1}^{r+1} (-1)^{i+1}[v_{r+1}\wedge\ldots\wedge\check{v}_i\wedge\ldots\wedge v_1]v_i\cdot\nabla_x\right\}\cdot F^r(x$$

$$= \sum_{i=1}^{r+1} (-1)^{i+1} v_i\cdot\nabla_x[v_{r+1}(x)\wedge\ldots\wedge\check{v}_i(x)\wedge\ldots\wedge$$

$$v_1(x)]\cdot F^r(x) + \left[\left[\sum_{i=1} (-1)^i\{v_i\cdot\nabla_x[v_{r+1}(x)\wedge\ldots\wedge\check{v}_i\wedge\ldots\wedge v_1]\}\right]\right]\cdot F^r(x).$$

But,

$$= \sum_{i=1}^{r+1} \sum_{\substack{j=1 \\ j \neq i}}^{r+1} (-1)^{i+j} [\underset{\sim}{v}_{r+1} \wedge \cdots \wedge \check{\underset{\sim}{v}}_j \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1] \wedge [\underset{\sim}{v}_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{v}_j]$$

$$= \sum_{i<j} (-1)^{i+j} [\underset{\sim}{v}_{r+1} \wedge \cdots \wedge \check{\underset{\sim}{v}}_j \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1] \wedge [\underset{\sim}{v}_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{v}_j] + \sum_{i<j}$$

$$(-1)^{i+j} [\underset{\sim}{v}_{r+1} \wedge \cdots \wedge \check{\underset{\sim}{v}}_j \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1] \wedge [\underset{\sim}{v}_j \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{v}_i]$$

$$= \sum_{i<j} (-1)^{i+j} [\underset{\sim}{v}_{r+1} \wedge \cdots \wedge \check{\underset{\sim}{v}}_j \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1] \wedge [\underset{\sim}{v}_i, \underset{\sim}{v}_j] .$$

Thus,

$$[\nabla_x \wedge F^r(x)] \cdot V_{r+1}^{\dagger} = \sum_{i=1}^{r+1} (-1)^{i+1} \underset{\sim}{v}_i \cdot \nabla_{\underset{\sim}{x}} [\underset{\sim}{v}_{r+1}(\underset{\sim}{x}) \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1(\underset{\sim}{x})] \cdot F^r(\underset{\sim}{x}) +$$

$$\sum_{i<j} (-1)^{i+j} \{[\underset{\sim}{v}_{r+1} \wedge \cdots \wedge \check{\underset{\sim}{v}}_j \wedge \cdots \wedge \check{\underset{\sim}{v}}_i \wedge \cdots \wedge \underset{\sim}{v}_1] \wedge [\underset{\sim}{v}_i, \underset{\sim}{v}_j]\} \cdot F^r(\underset{\sim}{x})$$

$$\equiv df^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r+1}) .$$

XXXX

The following properties of the exterior derivative of forms follow easily by using the previous theorem, and corresponding properties of $\nabla_{\underset{\sim}{x}}$.

Theorem D.10. If f^r and g^s are two forms on $\mathcal{X}$, then

(iii) $d(df^r_{\underset{\sim}{x}}) = 0$.

Proof Let $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x}) \cdot V^\dagger_r$, and $g^s_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_s) = G^s(\underset{\sim}{x}) \cdot W^\dagger_s$, where $F^r(\underset{\sim}{x})$, and $G^s(\underset{\sim}{x})$ are the corresponding multivector fields for $f^r_{\underset{\sim}{x}}$ and $g^s_{\underset{\sim}{x}}$ given by theorem D.2(ii). The proof of the theorem follows from the properties of the gradient operator listed below:

(i) $\nabla_{\underset{\sim}{x}} \wedge [F^r(\underset{\sim}{x}) + G^r(\underset{\sim}{x})] = \nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}} \wedge G^r(\underset{\sim}{x})$.

(ii) $\nabla_{\underset{\sim}{x}} \wedge [F^r(\underset{\sim}{x}) \wedge G^s(\underset{\sim}{x})] = [\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] \wedge G^s(\underset{\sim}{x}) + (-1)^r F^r(\underset{\sim}{x}) \wedge [\nabla_{\underset{\sim}{x}} \wedge G^s(\underset{\sim}{x})]$.

(iii) $\nabla_{\underset{\sim}{x}} \wedge [\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] = 0$. (property 2.13)

XXXX

c) The Contraction Operator

The contraction operator $C_{\underset{\sim}{v}}$, for $\underset{\sim}{v} \in \mathcal{D}^1_{\underset{\sim}{x}}$, is a mapping of r-forms into (r-1)-forms. It is defined below. (See [12, p.91] or [4, p.69] for an equivalent definition.)

Let $f^r_{\underset{\sim}{x}}$ be an r-form, and $F^r_{\underset{\sim}{x}}$ be the corresponding r-vector field given by theorem D.2.

<u>Theorem D.12</u> $\quad C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}) = [\underset{\sim}{v} \cdot F^r_{\underset{\sim}{x}}] \cdot V^{\dagger}_{r-1}$,

where $V_{r-1} = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_{r-1}$.

<u>Proof</u> $\quad [\underset{\sim}{v} \cdot F^r_{\underset{\sim}{x}}] \cdot V^{\dagger}_{r-1} = F^r_{\underset{\sim}{x}} \cdot (V^{\dagger}_{r-1} \wedge \underset{\sim}{v})$ identity 0.42

$$\equiv f^r_{\underset{\sim}{x}} (\underset{\sim}{v}, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \ .$$

XXXX

The following theorem gives the basic properties of the contraction operator $C_{\underset{\sim}{v}}$. The proof, which is omitted, follows easily by using theorems D.2 and D.12, and algebraic properties of geometric algebra.

Let $f^r_{\underset{\sim}{x}}$ and $g^s_{\underset{\sim}{x}}$ be forms on $\mathcal{X}_m$. Then:

<u>Theorem D.13</u>

(i) $(C_{\underset{\sim}{v}})^2 f^r_{\underset{\sim}{x}} = 0$

(ii) $C_{\underset{\sim}{v}}[f^r_{\underset{\sim}{x}} + g^s_{\underset{\sim}{x}}] = C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} + C_{\underset{\sim}{v}} g^s_{\underset{\sim}{x}}$

d) The Covariant Derivative

The covariant derivative operator $D_{\underset{\sim}{v}}$ for $\underset{\sim}{v} \in \mathcal{D}^1_{\underset{\sim}{x}}$, is a mapping of r-forms into r-forms. The definition for it given below is equivalent to that found in [12, p.94].

Let $f^r_{\underset{\sim}{x}}$ be an r-form on the surface $\mathcal{X}_m$, and let $\underset{\sim}{v} \in \mathcal{D}_{\underset{\sim}{x}}$. Then:

<u>Definition D.14</u> $D_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, \underset{\sim}{v}_r(\underset{\sim}{x})) - \sum_{i=1}^{r} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{i-1}, \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\underset{\sim}{v}_i(\underset{\sim}{x}), \underset{\sim}{v}_{i+1}, \ldots, \underset{\sim}{v}_r)$.

Let $f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r_{\underset{\sim}{x}}\cdot V^\dagger_r$, where $F^r_{\underset{\sim}{x}}$ is given by theorem D.2(ii), and $V_r = V_r(\underset{\sim}{x}) = \underset{\sim}{v}_1(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{v}_r(\underset{\sim}{x})$. The next theorem relates the covariant derivative of a form to the directional derivative of its corresponding multivector field. Its proof is an easy consequence of the identity

$$\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} V_r(\underset{\sim}{x}) = \sum_{i=1}^{r} \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\underset{\sim}{v}_i(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{v}_r .$$

Proof

$$[\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x})]\cdot V_r^{\dagger} = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}[F^r(\underset{\sim}{x})\cdot V_r^{\dagger}(\underset{\sim}{x})] - F^r\cdot[\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, V_r^{\dagger}(\underset{\sim}{x})]$$

$$\equiv D_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}} \quad ,$$

using the identity given above, and definition D.14.

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e) The Lie Derivative

The Lie derivative $L_{\underset{\sim}{v}}$ is a mapping of r-forms into r-forms. Its definition is given below. See [12, p.93] or [1, p.64].

Definition D.16 $L_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, \underset{\sim}{v}_r(\underset{\sim}{x})) - \sum_{i=1}^{r} f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}, [\underset{\sim}{v},\underset{\sim}{v}_i], \underset{\sim}{v}_{i+1}, \ldots, \underset{\sim}{v}_r)$.

Theorem D.17 (i) $L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}[F^r(\underset{\sim}{x})\cdot V_r^{\dagger}(\underset{\sim}{x})] - F^r(\underset{\sim}{x})\cdot[\underset{\sim}{v},V_r]^{\dagger}$, where $f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x})\cdot V_r^{\dagger}$, and $[\underset{\sim}{v},V_r]$ is the Lie bracket on multivector fields defined in section 10.

(ii) $L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r_{\underset{\sim}{x}} + \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r_{\underset{\sim}{x}}]\}\cdot V_r^{\dagger}$.

$$L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x})\cdot V^\dagger_r(\underset{\sim}{x}) - \sum_{i=1}^{r} F^r(\underset{\sim}{x})\cdot[\underset{\sim}{v}_1 \wedge$$

$$\cdots\wedge\underset{\sim}{v}_{i-1}\wedge[\underset{\sim}{v},\underset{\sim}{v}_i]\wedge\underset{\sim}{v}_{i+1}\wedge\cdots\wedge\underset{\sim}{v}_r]^\dagger$$

theorem 10.10(i) $$= \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x})\cdot V^\dagger_r(\underset{\sim}{x}) - F^r(\underset{\sim}{x})\cdot[\underset{\sim}{v},V_r]^\dagger .$$

(ii) follows from (i) by the short computation given below.

$$L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x})\cdot V^\dagger_r(\underset{\sim}{x}) - F^r\cdot[\underset{\sim}{v},V_r]^\dagger \qquad \text{using (i)}$$

$$= \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x})\cdot V^\dagger_r(\underset{\sim}{x}) - F^r\cdot[\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} V^\dagger_r(\underset{\sim}{x})] +$$

$$\{\nabla_{\underset{\sim}{x}_1}\wedge[\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r]\}\cdot V^\dagger_r$$

$$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1}\wedge[\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r]\}\cdot V_r^{\dagger} .$$

XXXX

Well-known properties of the Lie derivative are given in the next theorem and are proved using theorem D.17. Let $f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x})\cdot V^\dagger_r$, and $g^s_{\underset{\sim}{x}} = G^s(\underset{\sim}{x})\cdot W^\dagger_s$, then :

<u>Theorem D.18</u> (i) $L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = C_{\underset{\sim}{v}}\, df^r_{\underset{\sim}{x}} + d\, C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}$.

[illegible]

identity, [illegible] $= [illegible]$

$$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} f^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1} \Lambda[\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r]\}\cdot V_r^{\dagger}$$

theorem D.17(ii) $\equiv L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r)$.

The proofs of (ii) and (iii) follow from (i) by using the properties proved for $C_{\underset{\sim}{v}}$ and d in theorems D.10 and D.12.

XXXX

f) The Pull Back of Forms

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$. The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ induces a linear mapping y^* called the "pull back," of r-forms $g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_k$ into r-forms $y^* g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{X}_m$. The following definition of y^* is equivalent to that given in [12, p.53].

Definition D.19 [illegible]

Proof

(i) $C_{\underset{\sim}{v}}\, df^r_{\underset{\sim}{x}} + d\, C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}$

$$= \{\underset{\sim}{v}\cdot[\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] + \nabla_{\underset{\sim}{x}} \wedge [\underset{\sim}{v}(\underset{\sim}{x})\cdot F^r(\underset{\sim}{x})]\}\cdot V^{\dagger}_r$$

identity 0.38 $$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x}) - \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}\cdot F^r(\underset{\sim}{x}_1)] + \nabla_{\underset{\sim}{x}} \wedge [\underset{\sim}{v}(\underset{\sim}{x})\cdot F^r(\underset{\sim}{x})]\}\cdot V^{\dagger}_r$$

$$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r]\}\cdot V^{\dagger}_r$$

theorem D.17(ii) $$\equiv L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) .$$

The proofs of (ii) and (iii) follow from (i) by using the properties proved for $C_{\underset{\sim}{v}}$ and d in theorems D.10 and D.12.

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f) The Pull Back of Forms

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Definition D.19

Let the r-form $g^r_{\underset{\sim}{y}}$ be given by $g^r_{\underset{\sim}{y}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r) =$ $G^r(\underset{\sim}{y}) \cdot W^{\dagger}_r$, where $W_r = \underset{\sim}{w}_1 \wedge \ldots \wedge \underset{\sim}{w}_r$, and $G^r(\underset{\sim}{y})$ is the r-vector field for $g^r_{\underset{\sim}{y}}$ given by theorem D.2.

Theorem D.20 $y^* g^r_{\underset{\sim}{y}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = [y^{\dagger} G^r(\underset{\sim}{y})] \cdot V^{\dagger}_r$, where $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_r \in \mathcal{D}^r_{\underset{\sim}{x}}$.

Proof $y^* g^r_{\underset{\sim}{y}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = G^r_{\underset{\sim}{y}} \cdot [y_{\dagger}\underset{\sim}{v}_1 \wedge \ldots \wedge y_{\dagger}\underset{\sim}{v}_r]^{\dagger}$

theorem 3.3(i) $= G^r_{\underset{\sim}{y}} \cdot [y_{\dagger}(\underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_r)]^{\dagger}$

$= G^r_{\underset{\sim}{y}} \cdot y_{\dagger} V^{\dagger}_r$

cor. 3.6 $= (y^{\dagger} G^r_{\underset{\sim}{y}}) \cdot V^{\dagger}_r$

XXXX

The following properties of y^* now follow easily from the properties of $y^{\dagger}$, and the preceding theorems of this appendix.

Theorem D.21 Let $f^r_{\underset{\sim}{y}}$ and $g^s_{\underset{\sim}{y}}$ be forms on $\mathcal{V}_k$. Then,

(i) $y^*(f^r_{\underset{\sim}{y}} + g^r_{\underset{\sim}{y}}) = y^* f^r_{\underset{\sim}{y}} + y^* g^r_{\underset{\sim}{y}}$, for $r = s$.

Appendix E. The Intrinsic Gradient and Curvature

Throughout this paper the tangential gradient has been used. In this appendix another gradient called the intrinsic gradient is introduced. The relationships between the gradient ∇ on $\mathcal{E}_n$, the tangential gradient $\nabla_{\underset{\sim}{x}}$ on $\mathcal{X}_m$, and the intrinsic gradient $\not\nabla_{\underset{\sim}{x}}$ on $\mathcal{X}_m$ are studied, and a new formulation of the Gauss curvature equation is given.

a) The Gradients ∇, $\nabla_{\underset{\sim}{x}}$ and $\not\nabla_{\underset{\sim}{x}}$

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

The gradient ∇_x on the surface $\mathcal{X}_m$ is related to the gradient ∇ on $\mathcal{E}_n$ by the following equation:

(E.1) $$\nabla = \nabla_{\parallel} + \nabla_{\perp}\ , \text{ where } \nabla_{\underset{\sim}{x}} \equiv \nabla_{\parallel} \quad .$$

Equation (E.1) shows that if the gradient ∇ of $\mathcal{E}_n$ is decomposed into a tangential component $\nabla_{\parallel}$ and a normal component $\nabla_{\perp}$ to the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, then $\nabla_{\underset{\sim}{x}}$ is the tangential component.

The identification of ∇_x as being the tangential component surface $\mathcal{X}_m$ at the point $\mathbf{x}$.) However, it differs from ∇ in one crucial respect, and that is it doesn't preserve tangent multivector fields on $\mathcal{X}_m$. I.e., if $F(\mathbf{x})$ is a tangent multivector field on $\mathcal{E}_n$, then $\nabla F(\mathbf{x})$ will be also, but if $F(\mathbf{x})$ is a tangent multivector field on $\mathcal{X}_m$, $\nabla_{\mathbf{x}} F(\mathbf{x})$ will in general have both tangential and normal components to the surface $\mathcal{X}_m$.

The following equation decomposes $\nabla_{\mathbf{x}} F(\mathbf{x})$ into tangential and normal components, and at the same time identifies the intrinsic gradient applied to $F(\mathbf{x})$.

$$\text{(E.2)} \qquad \nabla_{\mathbf{x}} F(\mathbf{x}) = [\nabla_{\mathbf{x}} F(\mathbf{x})]_{\parallel} + [\nabla_{\mathbf{x}} F(\mathbf{x})]_{\perp} ,$$

where $\not\nabla_{\mathbf{x}} F(\mathbf{x}) \doteq [\nabla_{\mathbf{x}} F(\mathbf{x})]$.

In words, (E.2) says that if $\nabla_{\mathbf{x}} F(\mathbf{x})$ is decomposed into tangential and normal components to the surfaces $\mathcal{X}_m$ at the point $\mathbf{x}$, then the intrinsic gradient of $F(\mathbf{x})$ is defined to be the tangential part.

Thus where $\nabla_{\mathbf{x}} F(\mathbf{x})$ suffers the "defect" of not preserving tangent fields on $\mathcal{X}_m$, $\not\nabla_{\mathbf{x}}$ removes this defect by "throwing away" the normal part to the surface.

A more formal definition of $\not\nabla_{\mathbf{x}}$ in terms of $\nabla_{\mathbf{x}}$ is now given.

Important special cases of this definition are:

(E.3a) $\quad \not\nabla_{\underset{\sim}{x}} \cdot F(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}_1} \cdot [F(\underset{\sim}{x}_1) \cdot p_{\underset{\sim}{x}}\ p_{\underset{\sim}{x}}^{\dagger}]$

(E.3b) $\quad \not\nabla_{\underset{\sim}{x}} \wedge F(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}_1} \wedge [F(\underset{\sim}{x}_1) \cdot p_{\underset{\sim}{x}}\ p_{\underset{\sim}{x}}^{\dagger}]$

(E.3c) $\quad A_r \cdot \not\nabla_{\underset{\sim}{x}}\ F(\underset{\sim}{x}) = A_r \cdot \nabla_{\underset{\sim}{x}_1} [F(\underset{\sim}{x}_1) \cdot p_{\underset{\sim}{x}}] p_{\underset{\sim}{x}}^{\dagger}$

(E.3d) $\quad A_r \wedge \not\nabla_{\underset{\sim}{x}}\ F(\underset{\sim}{x}) = A_r \wedge \nabla_{\underset{\sim}{x}_1} [F(\underset{\sim}{x}_1) \cdot p_{\underset{\sim}{x}}] p_{\underset{\sim}{x}}^{\dagger}$;

The theorem below relates properties of the intrinsic gradient $\not\nabla_{\underset{\sim}{x}}$ to properties of the tangential gradient $\nabla_{\underset{\sim}{x}}$.

Let $[A_r/B_s]$ denote the Lie bracket operation defined in section 10, but with respect to the intrinsic gradient.

<u>Theorem E.4</u> (i) $\quad \not\nabla_{\underset{\sim}{x}} \cdot F(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \cdot F(\underset{\sim}{x})$

(ii) $\quad [A_r/B_s] = [A_r, B_s]$, for A_r, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$

(iii) $\quad \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}\ \underset{\sim}{b}(\underset{\sim}{x}) - \underset{\sim}{a} \cdot \not\nabla_{\underset{\sim}{x}}\ \underset{\sim}{b}(\underset{\sim}{x}) = -[\underset{\sim}{b} \wedge S(\underset{\sim}{a})] p_{\underset{\sim}{x}}^{\dagger}$,

for $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and where $S(\underset{\sim}{a})$ is the shape operator defined in section 13.

<u>Proof</u> (i) The proof of (i) follows immediately from

$$\nabla_x \cdot (A_r \wedge B_s) = (\nabla_x \cdot A_r) \wedge B_s + (-1)^r A_r \wedge (\nabla_x \cdot B_s) + (-1)^{r+1} [A_r, B_s] \, .$$

The same decomposition applied to $\not\nabla_x$ gives

$$\not\nabla_x \cdot (A_r \wedge B_s) = (\not\nabla_x \cdot A_r) \wedge B_s + (-1)^r A_r \wedge (\not\nabla_x \cdot B_s) + (-1)^{r+1} [A_r / B_s] \, .$$

But by (i), $\not\nabla_x \cdot (A_r \wedge B_s) = \nabla_x \cdot (A_r \wedge B_s)$, $\not\nabla_x \cdot A_r = \nabla_x \cdot A_r$, and $\not\nabla_x \cdot B_s = \nabla_x \cdot B_s$. Hence it follows that $[A_r / B_s] = [A_r, B_s]$.

(iii) $\quad a \cdot \nabla_x \, b(x) - a \cdot \not\nabla_x \, b(x)$

$$= a \cdot \nabla_{x_1} \, b(x_1) \cdot p_x \, p_x^\dagger + a \cdot \nabla_{x_1} \, b(x_1) \wedge p_x \, p_x^\dagger - a \cdot \not\nabla_x \, b(x)$$

def. E.3c $\quad = a \cdot \nabla_{x_1} \, b(x_1) \wedge p_x \, p_x^\dagger$

theorem 13.2(i) $\quad = -[b \wedge S(a)] p_x^\dagger$

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Part (iii) of the last theorem shows that the difference between the tangential and intrinsic directional derivatives of a vector field is completely determined by the shape of the surface. (See [12, p.75] for a similar result.)

and definition E.3.

<u>Theorem E.5</u> $\not\nabla_{\underline{x}} \wedge \not\nabla_{\underline{x}} F(\underline{x}) = \nabla_{\bar{\underline{x}}_2} [F(\underline{x})\cdot p(\underline{x}_1)]\cdot p^{\dagger}(\underline{x}_2)$,

where $F(\underline{x}) \in \{F(\underline{x})\}_{\underline{x}}$, and $p(\underline{x})$ is a unit pseudoscalar field.

<u>Proof</u> $\not\nabla_{\underline{x}} \wedge \not\nabla_{\underline{x}} F(\underline{x}) = \not\nabla_{\underline{x}} \wedge \not\nabla_{\underline{x}} [F(\underline{x})\cdot p(\underline{x})]\cdot p^{\dagger}(\underline{x})$

def. E.3 $= \not\nabla_{\underline{x}} \wedge \nabla_{\underline{x}_1} [F(\underline{x}_1)\cdot p(\underline{x})]\cdot \dot{p}^{\dagger}(\underline{x})$

def. E.3 $= \nabla_{\underline{x}_2} \wedge \nabla_{\underline{x}_1} [F(\underline{x}_1)\cdot p(\underline{x})]_2\cdot p^{\dagger}(\underline{x})$

property 2.13 $= \nabla_{\bar{\underline{x}}_2} [F(\underline{x}_1)\cdot p(\underline{x}_2)]\cdot p^{\dagger}(\underline{x})$

$= \nabla_{\bar{\underline{x}}_2} [F(\underline{x}_1)\cdot p(\underline{x}_2)]\cdot p^{\dagger}(\underline{x}_1)$

$\quad - \nabla_{\bar{\underline{x}}_2} [F(\underline{x})\cdot p(\underline{x}_2)]\cdot p^{\dagger}(\underline{x}_1)$

$= \nabla_{\bar{\underline{x}}_2} F(\underline{x}_1) [p(\underline{x}_2)\cdot p^{\dagger}(\underline{x}_1)]$

$\quad - \nabla_{\bar{\underline{x}}_2} [F(\underline{x})\cdot p(\underline{x}_2)]\cdot p^{\dagger}(\underline{x}_1)$

theorem 13.2(iii),(iv) $= \nabla_{\bar{\underline{x}}_2} [F(\underline{x})\cdot p(\underline{x}_1)]\cdot p^{\dagger}(\underline{x}_2)$.

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Theorem E.5 shows that $\not\nabla_{\underline{x}} \wedge \not\nabla_{\underline{x}} F(\underline{x})$ is completely determined by the shape of the surface, and is independent of the field

Corollary E.6 $\partial_x \wedge \partial_x \, v(x) = \frac{1}{2} \nabla_{\bar{x}_2} \, v \cdot [p(x_1) \, p^{\dagger}(x_2)]_{bi}$,

where $v(x) \in \{F(x)\}^1_x$, and "bi" stands for bivector part.

Proof The proof is an algebraic simplification of theorem E.5 with $F(x) = v(x)$.

XXXX

The following definition for curvature is analogous to that given in [12, p.59] .

Definition E.7 Call $R(a,b) \equiv (b \wedge a) \cdot (\partial_x \wedge \partial_x)$ the curvature operator of the vectors a , $b \in \mathcal{D}^1_x$.

Applying theorem 10.6(i) to the intrinsic gradient ∂_x gives the following identity for $R(a,b)$, when $a(x), b(x) \in \{F(x)\}_x$

(E.8) $R(a,b) \equiv [a \cdot \partial_x , b \cdot \partial_x] - [a,b] \cdot \partial_x$.

Applying $R(a,b)$ to a vector field $v(x)$ and using corollary E.6, gives a form of what is known as the Gauss curvature equation for a surface in $\mathcal{E}_n$. (See [12, p.76].)

Proof The proof is direct using corollary E.6.

$$R(a,b)\, v = (b\wedge a)\cdot(\dot{\nabla}_x\wedge\dot{\nabla}_x)\, v(x)$$

cor. E.6
$$= \frac{1}{2}\,(b\wedge a)\cdot\nabla_{\bar{x}_2}\, v\cdot[p(x_1)\, p^{\dagger}(x_2)]_2$$

$$= \frac{1}{2}\, v\cdot[S(b)\, S^{\dagger}(a) - S(a)\, S^{\dagger}(b)]_2$$

$$= [S(a)\, S^{\dagger}(b)]_2\cdot v \quad .$$

XXXX

Finally theorem E.9 will be applied to a hypersurface $\mathcal{X}_{n-1}$ of $\mathcal{E}_n$ to show more clearly the relationship of this theorem to more usual formulations. Let I be a unit pseudoscalar element of $\mathcal{E}_n$, then $n(x) \equiv p(x)I$ is an orthonormal vector field to $\mathcal{X}_{n-1}$.

The following definition and theorem are given in [12, p.77].

Definition E.10 Call $L(a) \equiv a\cdot\nabla_x\, n = S(a)\, I$ the Weingarten mapping for $a \in \mathcal{D}^1_x$.

Theorem E.11 For the hypersurface $\mathcal{X}_{n-1}$,

$$R(a,b)\, v = v\cdot L(b)\, L(a) - v\cdot L(a)\, L(b) \quad .$$

$$= [L(\underset{\sim}{a}) \wedge L(\underset{\sim}{b})] \cdot \underset{\sim}{v}$$

identity 0.39 $$= \underset{\sim}{v} \cdot L(\underset{\sim}{b})\, L(\underset{\sim}{a}) - \underset{\sim}{v} \cdot L(\underset{\sim}{a})\, L(\underset{\sim}{b}) \, .$$

XXXX

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BIBLIOGRAPHY

1. Abraham, Ralph, Foundations of Mechanics, W.A. Benjamin, Inc., New York, 1967.

2. Apostol, Tom M., Calculus, Vol. II, Blaisdell Publishing Co., New York, 1962.

3. ________, Mathematical Analysis, Addison Wesley Publishing Co., Massachusetts, 1957.

4. Bishop, Richard L., and Crittenden, Richard J., Geometry of Manifolds, Academic Press, New York, 1964.

5. Buck, R. Creighton, Advanced Calculus, 2nd Edition, McGraw-Hill Book Company, New York, 1965.

6. Gantmacher, F. R., Matrix Theory, Vol. I, Chelsea Publishing Company, New York, 1960.

7. Greub, W.H., Linear Algebra, 3rd Edition, Springer-Verlag New York Inc., New York, 1967.

8. ________, Multilinear Algebra, Springer-Verlag New York Inc., New York, 1967.

9. Hestenes, David, "Multivector Calculus," Journal of Mathematical Analysis and Applications, Vol. 24, No. 2, November 1968, Academic Press, New York.

10. ________, "Multivector Functions," Journal of Mathematical

11. ________, Space-Time Algebra, Documents on Modern Physics, Gordon and Breach Science Publishers, Inc., New York, 1966.

12. Hicks, Noel J., Notes on Differential Geometry, D. Van Nostrand Company, Inc., Princeton, 1965.

13. Hu, S. T., Differentiable Manifolds, Holt, Rinehart and Winston, Inc., New York, 1969.

14. Kreyszig, Erwin, Introduction to Differential Geometry and Riemannian Geometry, University of Toronto Press, Toronto, 1968.

15. Lehti, Raimo, "On the Introduction of Dyadics into Elementary Vector Algebra," Commentationes Physico-Mathematicae, XXVI, 2, Helsinki-Helsingfors, 1962.

16. O'Neill, Barrett, Elementary Differential Geometry, Academic Press, New York, 1966.

17. Segercrantz, Jerry, "Note on the Proof of the Cayley-Hamilton Theorem," Commentationes Physico-Mathematicae, Vol 34, No. 3, Helsinki-Helsingfors, 1968.

18. Whitney, Hassler, Geometric Integration Theory, Part I, Princeton University Press, Princeton, New Jersey, 1957.

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