Answers to Exercises 13.1:

1. Consider the image $M = \mathbf{x}(R)$ of the 2-rectangle $R = [-1,1] \times [-1,1]$, where

$$\mathbf{x}(s^1, s^2) = (s^1, s^2, -(s^1)^2 + s^2 + 2),$$

see Figure 11.1.

a) Find $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial s^1}$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial s^2}$

$$\mathbf{x}_1 = (1, 0, -2s^1), \quad \mathbf{x}_2 = (0, 1, 1).$$

b) Find the reciprocal basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ at the point $(s^1, s^2) \in R$.

$$\mathbf{x}^1 = \Big(\frac{2}{4(s^1)^2 + 2}, \frac{2s^1}{4(s^1)^2 + 2}, -\frac{2s^1}{4(s^1)^2 + 2}\Big),$$

$$\mathbf{x}^2 = \Big(\frac{2s^1}{4(s^1)^2 + 2}, \frac{4(s^1)^2 + 1}{4(s^1)^2 + 2}, \frac{4(s^1)^2 + 1}{4(s^1)^2 + 2} - \frac{4(s^1)^2}{4(s^1)^2 + 2}\Big)$$

c) Find the 2-chain $\beta(M)$ which is the boundary to M.

d) Find the tanget vectors to the surface M at the points

$$\mathbf{x} = \mathbf{x}(\frac{1}{2}, 0), \mathbf{x}(1, \frac{1}{2}), \mathbf{x}(\frac{1}{2}, 1), \mathbf{x}(0, \frac{1}{2}).$$

$$\mathbf{x}_1(\frac{1}{2}, 0) = (1, 0, -1), \quad \mathbf{x}_2(\frac{1}{2}, 0) = (0, 1, 1)$$

$$\mathbf{x}_1(1, \frac{1}{2}) = (1, 0, -2), \quad \mathbf{x}_2(1, \frac{1}{2}) = (0, 1, 1)$$

$$\mathbf{x}_1(\frac{1}{2}, 1) = (1, 0, -1), \quad \mathbf{x}_2(\frac{1}{2}, 1) = (0, 1, 1)$$

$$\mathbf{x}_1(0, \frac{1}{2}) = (1, 0, 0), \quad \mathbf{x}_2(0, \frac{1}{2}) = (0, 1, 1)$$

2. Let $R = [-1,1] \times [-1,1]$ be a 2-square in $\mathbb{R}^2$ and let $M = \{\mathbf{x}(R)\} \subset \mathbb{R}^3$ where $\mathbf{x}(x,y) = (x, y, x^2 + y^2)$. See Figure 11.2.

a) Find $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x}$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial y}$

b) Find the reciprocal basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ at the point $\mathbf{x}(x,y) \in M$.

ans: $\mathbf{x}^1 = (\frac{4y^2+1}{4x^2+4y^2+1}, -\frac{4xy}{4x^2+4y^2+1}, \frac{2x}{4x^2+4y^2+1})$,
$\mathbf{x}^2 = (-\frac{4xy}{4x^2+4y^2+1}, \frac{4x^2+1}{4x^2+4y^2+1}, \frac{2y}{4x^2+4y^2+1})$.

c) Find the tangent vectors to the boundary of the surface M at the points $\mathbf{x} = \mathbf{x}(1,0), \mathbf{x}(0,1), \mathbf{x}(-1,0), \mathbf{x}(0,-1)$.

d) Find the 2-chain $\beta(M)$ which is the boundary to M.

3. Let $R = [-1,1] \times [-1,1]$ be a 2-square in $\mathbb{R}^2$ and let $M = \{\mathbf{x}(R)\} \subset \mathbb{R}^3$ where $\mathbf{x}(x,y) = (x, y, x^2 - y^2 + 1)$. See Figure 11.3.

a) Find $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x}$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial y}$
b) Find the reciprocal basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ at the point $\mathbf{x}(x, y) \in M$.
ans: $\mathbf{x}^1 = \left\{ \frac{4y^2+1}{4x^2+4y^2+1}, \frac{4xy}{4x^2+4y^2+1}, \frac{2x}{4x^2+4y^2+1} \right\}$,
$\mathbf{x}^2 = \left\{ \frac{4xy}{4x^2+4y^2+1}, \frac{4x^2+1}{4x^2+4y^2+1}, -\frac{2y}{4x^2+4y^2+1} \right\}$.
c) Find the tangent vector to the boundary of the surface M at the points $\mathbf{x} = \mathbf{x}(1, 0), \mathbf{x}(0, 1), \mathbf{x}(-1, 0), \mathbf{x}(0, -1)$.
d) Find the 2-chain $\beta(M)$ which is the boundary to M.

4. Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \cdot \\ \cdot \\ \mathbf{a}_k \end{pmatrix}$, $\mathbf{B} = (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_k)$, and C be a real $k \times k$ matrix.

Show that $(\mathbf{A} \cdot \mathbf{B})C = \mathbf{A} \cdot (\mathbf{B}C)$, and $C(\mathbf{A} \cdot \mathbf{B}) = (C\mathbf{A}) \cdot \mathbf{B}$, where the product $\mathbf{A} \cdot \mathbf{B}$ is defined as in equation.

One way of proving this result is to check that the (jm) entry is the same for both sides of the respective equation. This can be accomplished by multiply on the left by the row matrix $(\delta_{j1} \quad \delta_{j2} \quad \cdots \quad \delta_{jk})$, and on the right by the column matrix $(\delta_{1m} \quad \delta_{2m} \quad \cdots \quad \delta_{km})^T$ to get $(c_{j1}\mathbf{a}_1 + \cdots + c_{jk}\mathbf{a}_k) \cdot \mathbf{b}_m$ for both the left and right sides of the respective equation.