Answers to Exercises 13.2:

1. Consider the image $M = \mathbf{x}(R)$ of the 2-rectangle $R = [-1, 1] \times [-1, 1]$, where

$$\mathbf{x}(s^1, s^2) = (s^1, s^2, -(s^1)^2 + s^2 + 2).$$

See Figure 13.1.

a) Find the area $|M|$ of the patch M by calculating the integral $\int_M |d\mathbf{x}_{(2)}| = \int_R |\mathbf{x}_{(2)}| ds^{(2)}$, where $ds^{(2)} = ds^1 ds^2$.

$$|M| = \int_{-1}^{1}\int_{-1}^{1} |\mathbf{x}_1 \times \mathbf{x}_2| ds^1 ds^2 = \int_{-1}^{1}\int_{-1}^{1} \sqrt{2 + 4(s^1)^2} ds^1 ds^2$$

$$= 2\sqrt{6} - 2\sinh^{-1}\sqrt{2} = 7.19141$$

b) Find the length $|\partial M|$ of the boundary ∂M of M by evaluating the integral

$$|\partial M| = \int_{\beta(M)} |d\mathbf{x}|.$$

$$|\partial M| = \int_{-1}^{1} |\mathbf{x}_1(s^1, -1)| ds^1 + \int_{-1}^{1} |\mathbf{x}_2(1, s^2)| ds^2$$

$$+ \int_{-1}^{1} |\mathbf{x}_1(s^1, 1)| ds^1 + \int_{-1}^{1} |\mathbf{x}_2(-1, s^2)| ds^2$$

$$= 4\sqrt{2} + 2\sqrt{5} + \sinh^{-1}(2) = 11.5726$$

2. Let $R = [-1, 1] \times [-1, 1]$ be a 2-square in $\mathbb{R}^2$ and let $M = \{f(R)\} \subset \mathbb{R}^3$ where $f(s1, s2) = (s1, s2, s1^2 + s2^2)$. See Figure 13.2.

a) Find the area $|M|$ of the patch M by calculating the integral

$$\int_M |d\mathbf{x}_{(2)}| = \int_R |\mathbf{x}_{(2)}| ds^{(2)} = \int\int_R \sqrt{g} ds^1 ds^2,$$

where $g = -(\mathbf{x}_1 \wedge \mathbf{x}_2)^2 = \det[g]$, and $[g]$ is the Gramian matrix.

$$|M| = \int_{-1}^{1}\int_{-1}^{1} |\mathbf{x}_1 \times \mathbf{x}_2| ds^1 ds^2 = \int_{-1}^{1}\int_{-1}^{1} \sqrt{4(s^1)^2 + 4(s2)^2 + 1}$$

$$= 4 - \frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) + \frac{7\log(5)}{3} = 7.44626$$

b) Find the length $|\partial M|$ of the boundary ∂M of M by evaluating the integral

$$|\partial M| = \int_{\beta(M)} |d\mathbf{x}|.$$

3. Let $R = [-1,1] \times [-1,1]$ be a 2-square in $I\!R^2$ and let $M = \{f(R)\} \subset I\!R^3$ where $f(x,y) = (x, y, x^2 - y^2 + 1)$. See Figure 13.3.

a) Find the area $|M|$ of the patch M by calculating the integral $\int_M |d\mathbf{x}_{(2)}| = \int_R |\mathbf{x}_{(2)}| ds^{(2)}$, where $ds^{(2)} = ds^1 ds^2$. *ans.* $4 - \frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) + \frac{7\log(5)}{3} = 7.44626$.

b) Find the length $|\partial M|$ of the boundary ∂M of M by evaluating the integral

$$|\partial M| = \int_{-1}^{1} |\mathbf{x}_1(s^1,-1)| ds^1 + \int_{-1}^{1} |\mathbf{x}_2(1,s^2)| ds^2$$

$$+ \int_{-1}^{1} |\mathbf{x}_1(s^1,1)| ds^1 + \int_{-1}^{1} |\mathbf{x}_2(-1,s^2)| ds^2$$

$$= 4\sqrt{5} + 2\sinh^{-1}(2) = 11.8315.$$

4. Use the fundamental theorem of calculus to show that

a) $\int_M d\mathbf{x}_{(2)} = \frac{1}{2}\int_{\beta(M)} d\mathbf{x} \wedge \mathbf{x}$,

b) $\int_{\beta(M)} d\mathbf{x} \cdot \mathbf{x} = 0$.

The fundamental theorem of calculus gives $\int_M d\mathbf{x}_{(2)} \partial_{\mathbf{x}} \mathbf{x} = \int_{\beta(M)} d\mathbf{x}\mathbf{x}$. This immediately implies a) and b).