Answers to Exercises 13.3:

1. If the acceleration of a particle at any time $t > 0$ is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 5\cos 2t\mathbf{e}_1 - \sin t\mathbf{e}_2 + 2t\mathbf{e}_3,$$

and the velocity $\mathbf{v}$ and position vector $\mathbf{x}$ are 0 at $t = 0$, find $\mathbf{v}(t)$ and $\mathbf{x}(t)$ at any time $t > 0$.

$$\mathbf{v} = \Big(\frac{5}{2}\sin(2t), \cos(t) - 1, t^2\Big), \quad \mathbf{x} = \Big(\frac{5}{2} - \frac{5\cos^2(t)}{2}, \sin(t) - t, \frac{t^3}{3}\Big).$$

2. Evaluate the integral $\int \mathbf{a} \wedge \frac{d^2\mathbf{a}}{dt^2} dt$.

$$\int \mathbf{a} \wedge \frac{d^2\mathbf{a}}{dt^2} dt = \int \frac{d}{dt}\Big(\mathbf{a} \wedge \frac{d\mathbf{a}}{dt}\Big) dt = \mathbf{a} \wedge \frac{d\mathbf{a}}{dt} + C.$$

3. Let $\mathbf{a} = (3x + 2y)\mathbf{e}_1 + 2yz\mathbf{e}_2 + 5xz^2\mathbf{e}_3$.

a) Calculate $\int_C \mathbf{a} \cdot d\mathbf{x}$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ where C is the curve defined by $x = t$, $y = -t^2 + 2t$, $z = 2t^2 - t$.

$$\int_C \mathbf{a} \cdot d\mathbf{x} = \int_0^1 \mathbf{a} \cdot \frac{d\mathbf{x}}{dt} dt = 1853/420 = 4.4119$$

b) Calculate the integral in a) where C is the straight line joining $(0, 0, 0)$ and $(1, 1, 1,)$.

$$\int_C \mathbf{a} \cdot d\mathbf{x} = \int_0^1 \mathbf{a} \cdot \frac{d\mathbf{x}}{dt} dt = 53/12 = 4.41667$$

4. If $\mathbf{a} = \partial_{\mathbf{x}}\phi(\mathbf{x})$, show that $\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{a} \cdot d\mathbf{x}$ is independent of the curve C from the point $\mathbf{x}_1$ to the point $\mathbf{x}_2$.

Let $\mathbf{x}(t)$ be any curve such that $\mathbf{x}(t_1) = \mathbf{x}_1$ and $\mathbf{x}(t_2) = \mathbf{x}_2$, and let $\phi(\mathbf{x})$ be a scalar valued function. Then by the chain rule, $\frac{d\phi}{dt} = \frac{d\mathbf{x}}{dt} \cdot \partial_{\mathbf{x}}\phi(\mathbf{x})$, so that

$$\phi(\mathbf{x}_2) - \phi(\mathbf{x}_1) = \int_{t_1}^{t_2} \frac{d\phi}{dt} dt = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{a} \cdot d\mathbf{x}.$$

5. Evaluate the integral

$$\int\int_{\mathcal{S}} \mathbf{a} \cdot \mathbf{n} |d\mathbf{x}_{(2)}|,$$

where $\mathbf{a} = 2z\mathbf{e}_1 + \mathbf{e}_2 + 3y\mathbf{e}_3$, $\mathcal{S}$ is that part of the plane $2x + 3y + 6z = 12$ which is bounded by the coordinate planes, and $\mathbf{n}$ is the normal unit vector to this plane.

Clearly $\mathbf{n} = \frac{1}{7}(2, 3, 6)$, and $|d\mathbf{x}_{(2)}| = \frac{1}{\mathbf{e}_3 \cdot \mathbf{n}} dydx = \frac{7}{6} dydx$, so that

$$\int\int_{\mathcal{S}} \mathbf{a} \cdot \mathbf{n} |d\mathbf{x}_{(2)}| = \frac{1}{6}\int_0^6 \int_0^{4-\frac{2x}{3}} (4z + 18y + 3) dydx = 356$$

6. Let $\mathbf{f}(\mathbf{x}) = M(\mathbf{x})\mathbf{e}_1 + N(\mathbf{x})\mathbf{e}_2$, where $M(\mathbf{x}), N(\mathbf{x})$ are C^1 real-valued functions for $\mathbf{x} \in R \subset I\!R^2$, where R is a closed region in $I\!R^2$ bounded by the simple curve C. Using the fundamental theorem of calculus, prove

$$\int_C d\mathbf{x} \cdot \mathbf{f} = \int\int_R d\mathbf{x}_{(2)}(\partial_{\mathbf{x}} \wedge \mathbf{f}) = \int\int \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dxdy$$

which is classically known as **Green's Theorem** in the plane. Note that $d\mathbf{x} = dx\mathbf{e}_1 + dy\mathbf{e}_2$, and $d\mathbf{x}_{(2)} = dxdy\mathbf{e}_{21}$ since we are integrating *counter-clockwise* around the curve C in the xy-plane.

Using the convention given in the definition, the fundamental theorem for the plane is

$$\int_C d\mathbf{x}\mathbf{f} = \int\int_R d\mathbf{x}_{(2)}\partial_{\mathbf{x}}\mathbf{f},$$

where the simple closed curve C is oriented clockwise. Taking the scalar parts of both sides of this equation then gives

$$\int_C \mathbf{f} \cdot d\mathbf{x} = -\int\int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dydx.$$

The minus sign is removed by agreeing to integrate on the left counterclockwise.

7. Let S be an open oriented 2-surface in $I\!R^3$ bounded by a closed simple curve C, and let $\mathbf{f}(\mathbf{x})$ be a C^1 vector-valued function on S. Using the fundamental theorem of calculus, prove

$$\int_C d\mathbf{x} \cdot \mathbf{f} = \int\int_R d\mathbf{x}_{(2)} \cdot (\partial_{\mathbf{x}} \wedge \mathbf{f}) = \int\int_R (\partial_{\mathbf{x}} \times \mathbf{f}) \cdot \hat{\mathbf{n}}|d\mathbf{x}_{(2)}|,$$

which is classically known as **Stokes' Theorem** for a surface $S \subset I\!R^3$. *The fundamental theorem for a 2-surface in $I\!R^3$ has the form*

$$\int_C d\mathbf{x}\mathbf{f} = \int\int_R d\mathbf{x}_{(2)}\partial_{\mathbf{x}}\mathbf{f}.$$

Taking the scalar parts of both sides of this equation gives

$$\int_C d\mathbf{x} \cdot \mathbf{f} = \int\int_R d\mathbf{x}_{(2)} \cdot (\partial_{\mathbf{x}} \wedge \mathbf{f}) = -\int\int_R |d\mathbf{x}_{(2)}|\hat{\mathbf{n}} \cdot (\partial_{\mathbf{x}} \times \mathbf{f})$$

By agreeing to the counterclockwise rule for integrating around the boundary, we can eliminate the minus sign.

8. Let V be a volume bounded by a closed surface S in $I\!R^3$, and $\mathbf{f}(\mathbf{x}) \in C^1$ be a vector-valued function. Using the fundamental theorem of calculus, prove

$$\int\int_S \mathbf{f} \cdot \hat{\mathbf{n}}|d\mathbf{x}_{(2)}| = \int\int\int_V (\partial_{\mathbf{x}} \cdot \mathbf{f})|d\mathbf{x}_{(3)}|$$

which is classically known as **Gauss' Divergence Theorem** in $I\!R^3$. *Choosing $g = 1$, the fundamental theorem for a closed 2-surface in $I\!R^3$ is*

$$\int_M d\mathbf{x}_{(3)}\partial\mathbf{f} = \int_{\beta(M)} d\mathbf{x}_{(2)}\mathbf{f}$$

Multiplying both sides of this equation by $-I = \mathbf{e}_{321}$*, and taking the scalar part of both sides of the resulting equation, gives the desired result that*

$$\int\int\int_V (\partial_{\mathbf{x}} \cdot \mathbf{f})|d\mathbf{x}_{(3)}| = \int\int_S \mathbf{f} \cdot \hat{\mathbf{n}}|d\mathbf{x}_{(2)}|.$$

9. Using the fundamental theorem of calculus, show that the vector derivative $\partial_{\mathbf{x}} f(\mathbf{x})$ of a geometric valued function $f(\mathbf{x}) \in \mathbb{G}_3$ can be defined by

$$\partial_{\mathbf{x}} f(\mathbf{x}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int\int_{\Delta S} |d\mathbf{x}_{(2)}|\hat{\mathbf{n}} f(\mathbf{x}),$$

where the small volume element ΔV around the point $\mathbf{x}$ is bounded by the closed surface ΔS, and $\hat{\mathbf{n}}$ is the outward unit normal vector at each point $\mathbf{x} \in \Delta S$. *Choosing* $g = 1$*, the fundamental theorem for a closed 2-surface in* $\mathbb{R}^3$ *is*

$$\int_M d\mathbf{x}_{(3)} \partial f = \int_{\beta(M)} d\mathbf{x}_{(2)} f$$

Let ΔV *be a small directed element of volume at the point* $\mathbf{x}$*. Then, approximately*

$$\Delta V \partial_{\mathbf{x}} f \doteq \int_M d\mathbf{x}_{(3)} \partial f = \int_{\beta(M)} d\mathbf{x}_{(2)} f,$$

from which the result follows.