Answers to Exercises 15.1:

0.1 Exercises

1. Show that $g_{ij,k} := \mathbf{x}_k \cdot \partial_{\mathbf{x}} g_{ij} = \frac{\partial g_{ij}}{\partial s^k} = \mathbf{x}_{ik} \cdot \mathbf{x}_j + \mathbf{x}_i \cdot \mathbf{x}_{jk} = g_{lj}\Gamma_{ik}^{\ l} + g_{li}\Gamma_{jk}^{\ l}$.

$\mathbf{x}_k \cdot \partial_{\mathbf{x}} g_{ij} = \frac{\partial \mathbf{x}_i \cdot \mathbf{x}_j}{\partial s^k} = \mathbf{x}_{ik} \cdot \mathbf{x}_j + \mathbf{x}_i \cdot \mathbf{x}_{jk} = g_{lj}\Gamma_{ik}^{\ l} + g_{li}\Gamma_{jk}^{\ l}$.

2. Show that $g^{ij}g_{jk} = \delta^i_k$ implies $g^{ij}_{\ ,k} = -g^{il}\Gamma^j_{lk} - g^{lj}\Gamma^i_{lk}$.

$g^{ij}_{\ ,l}g_{jk} + g^{ij}g_{jk,l} = 0$ implies that $g^{ij}_{\ ,l}g_{jk} = -g^{ij}g_{jk,l} = -g^{ij}\mathbf{x}_{lj} \cdot \mathbf{x}_k - \Gamma^i_{kl}$. Multiplying both sides of this last equation by g^{km} (and summing over k) gives

$$g^{im}_{\ ,l} = -g^{ij}\Gamma^m_{jl} - g^{km}\Gamma^i_{kl},$$

which is equivalent to the above expression to be proved.

3. Show that for $\mathbf{a} \in I\!R^n$,

$$P_i(\mathbf{a}) = P_i(\mathbf{a}_\parallel) + P_i(\mathbf{a}_\perp) = \mathbf{a} \cdot \mathbf{L}_{ij}\mathbf{x}^j + \mathbf{a} \cdot \mathbf{x}^j \mathbf{L}_{ij}.$$

$P^2 = P$ implies that $P_iP + PP_i = P_i$. Now write $\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp$. It follows that

$$P_i\mathbf{a} = P_i\mathbf{a}_\parallel + P_i\mathbf{a}_\perp, \quad PP_i\mathbf{a}_\parallel = 0, \quad PP_i\mathbf{a}_\perp = P_i\mathbf{a}_\perp.$$

We now calculate $P_i\mathbf{a} = \mathbf{x}^k_{\ ,i}\mathbf{x}_k \cdot \mathbf{a} + \mathbf{x}^k\mathbf{x}_{ki} \cdot \mathbf{a}$, from which it follows that $P_i\mathbf{a}_\perp = \mathbf{x}^k\mathbf{x}_{ki} \cdot \mathbf{a}_\perp = \mathbf{x}^k\mathbf{L}_{ki} \cdot \mathbf{a}$. Finally,

$$P_i\mathbf{a}_\parallel = \mathbf{x}^k_{\ ,i}\mathbf{x}_k \cdot \mathbf{a}_\parallel + \mathbf{x}^k\Gamma_{ki} \cdot \mathbf{a}_\parallel = (g^{kl}_{\ ,i}\mathbf{x}_l + g^{kl}\mathbf{x}_{li})\mathbf{x}_k \cdot \mathbf{a}_\parallel + \mathbf{x}^k\Gamma_{ki} \cdot \mathbf{a}_\parallel = \mathbf{a} \cdot \mathbf{x}^l\mathbf{L}_{li},$$

from which the result follows.

Let $\mathbf{x} = \mathbf{x}(s^1, s^2)$ be a 2-surface in $I\!R^3$. Then the *chain rule* gives the 1-differential

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial s^1}ds^1 + \frac{\partial \mathbf{x}}{\partial s^2}ds^2 = \mathbf{x}_1 ds^1 + \mathbf{x}_2 ds^2$$

in the direction defined by (ds^1, ds^2).

4. Show that $ds^2 = d\mathbf{x}^2 = g_{11}(ds^1)^2 + 2g_{12}ds^1ds^2 + g_{22}(ds^2)^2$. The expression ds^2 is called the *first fundamental form* of the surface $\mathbf{x}(s^1, s^2)$ in the direction defined by (ds^1, ds^2).

5. Show that $g_{11} > 0, g_{22} > 0$ and $g_{11}g_{22} - g_{12}^2 = (\mathbf{x}_1 \times \mathbf{x}_2)^2 > 0$. The result follows from

$$(\mathbf{x}_1 \times \mathbf{x}_2)^2 = (\mathbf{x}_1 \wedge \mathbf{x}_2) \cdot (\mathbf{x}_2 \wedge \mathbf{x}_1) = \mathbf{x}_1^2\mathbf{x}_2^2 - (\mathbf{x}_1 \cdot \mathbf{x}_2)^2.$$

6. Let $\mathbf{x} = \mathbf{x}(s^1, s^2)$ be a 2-surface in $I\!R^3$. Show that the 2^{nd} *differential*, in the direction (ds^1, ds^2), is given by

$$d^2\mathbf{x} = \mathbf{x}_{11}(ds^1)^2 + 2\mathbf{x}_{12}ds^1ds^2 + \mathbf{x}_{22}(ds^2)^2.$$

7. Let $\mathbf{x} = \mathbf{x}(s^1, s^2)$ be a 2-surface in $I\!R^3$, and $\hat{\mathbf{n}} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}$ be the unit orthonormal vector to $\mathbf{x}(s^1, s^2)$. It follows that $d\mathbf{x} \cdot \hat{\mathbf{n}} = 0$. Using problem 6, show that

$$\hat{\mathbf{n}} \cdot d^2\mathbf{x} = -d\mathbf{x} \cdot d\hat{\mathbf{n}} = L_{11}(ds^1)^2 + 2L_{12}ds^1ds^2 + L_{22}(ds^2)^2.$$

$-d\mathbf{x} \cdot d\hat{\mathbf{n}}$ is called the *second fundamental form* of $\mathbf{x}(s^1, s^2)$. The result follows from the fact that

$$\hat{\mathbf{n}} \cdot d^2\mathbf{x} = d(\hat{\mathbf{n}} \cdot d\mathbf{x}) - d\hat{\mathbf{n}} \cdot d\mathbf{x} = -d\hat{\mathbf{n}} \cdot d\mathbf{x}.$$

8. Show that Taylor's theorem at the point $\mathbf{x}(s_0^1, s_0^2)$ in the direction of (ds^1, ds^2), can be expressed as

$$\mathbf{x}(s_0^1 + ds^1, s_0^2 + ds^2) = \mathbf{x}(s_0^1, s_0^2) + \sum_{i=1}^{k} \frac{1}{k!} d^k\mathbf{x}(ds^1, ds^2) + o[((ds^1)^2 + (ds^2)^2)^{\frac{k}{2}}],$$

where $d^k\mathbf{x}$ is the k^{th} differential of $\mathbf{x}$ at the point $\mathbf{x}_0 = \mathbf{x}(s_0^1, s_0^2)$ in the direction of (ds^1, ds^2), and the "small oh" notation $\mathbf{f}(\mathbf{x}) = o[((ds^1)^2 + (ds^2)^2)^{\frac{k}{2}}]$ means that

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\mathbf{f}(\mathbf{x})}{((ds^1)^2 + (ds^2)^2)^{\frac{k}{2}}} = 0.$$