Answers to Exercises 15.2:

1. Given the surface $\mathbf{x}(s^1, s^2) = (s^1, s^2, (s^1)^2 s^2 + s^1 s^2 - (s^2)^2 + 2)$ of the last example. a) Calculate the tangent vectors $\mathbf{x}_1, \mathbf{x}_2$ and the mixed derivatives $\mathbf{x}_{11}, \mathbf{x}_{12}$ and $\mathbf{x}_2 2$ at the point $\mathbf{x}(\frac{1}{2}, 0)$.

$$\mathbf{x}_1(\frac{1}{2}, 0) = (1, 0, 0),\ \mathbf{x}_2(\frac{1}{2}, 0) = (0, 1, \frac{3}{4}),\ \mathbf{x}_{11}(\frac{1}{2}, 0) = (0, 0, 0) = \mathbf{x}_{22}(\frac{1}{2}, 0),$$

$$\mathbf{x}_{12}(\frac{1}{2}, 0) = (0, 0, 2).$$

b) Calculate the tangent plane at the point $\mathbf{x}(\frac{1}{2}, 0)$.

Since $\mathbf{x}(\frac{1}{2}, 0) = (\frac{1}{2}, 0, 2)$ and $\hat{\mathbf{n}}(\frac{1}{2}, 0) = \frac{1}{5}(0, -3, 4)$,

$$[(x, y, z) - (\frac{1}{2}, 0, 2)] \cdot (0, -3, 4) = 0.$$

c) Calculate the osculating paraboloid at the point $\mathbf{x}(\frac{1}{2}, 0)$.

The osculating paraboloid is best represented by taking the first three terms of the Taylor expansion of $\mathbf{x}(s^1, s^2)$ around the point $\mathbf{x}(\frac{1}{2}, 0)$. We find

$$\mathbf{y} = \mathbf{x}(\frac{1}{2}, 0) + \mathbf{x}_1(\frac{1}{2}, 0)s^1 + \mathbf{x}_2(\frac{1}{2}, 0)s^2 + \frac{1}{2}[\mathbf{x}_{11}(\frac{1}{2}, 0)(s^1)^2 + 2\mathbf{x}_{12}(\frac{1}{2}, 0)s^1 s^2 + \mathbf{x}_{22}(\frac{1}{2}, 0)(s^2)^2]$$

$$= (\frac{1}{2} + s^1, s^2, 2 + \frac{3s^2}{4} - 2s^1 s^2).$$

d) Classify the surface at the point $\mathbf{x}(\frac{1}{2}, 0)$.

Since $L_{11}L_{22} - L_{12} = -L_{12} = -\frac{8}{5} < 0$, the surface $\mathbf{x}(s^1, s^2)$ is hyperbolic at the point $\mathbf{x}(\frac{1}{2}, 0)$.

2. For the 2-rectangle $R = [-1, 1] \times [-1, 1]$, define the surface $\mathbf{x}(R)$ by

$$\mathbf{x}(s^1, s^2) = (s^1, s^2, -(s^1)^2 + s^2 + 2),$$

see Figure 11.1. of the Section 11.

a) Calculate the tangent vectors $\mathbf{x}_1, \mathbf{x}_2$ and the mixed derivatives $\mathbf{x}_{11}, \mathbf{x}_{12}$ and $\mathbf{x}_2 2$ at the point $\mathbf{x}(0, 0)$.
b) Calculate the tangent plane at the point $\mathbf{x}(0, 0)$.
c) Calculate the osculating paraboloid at the point $\mathbf{x}(0, 0)$.
d) Classify the surface at the point $\mathbf{x}(0, 0)$.

3. For the 2-rectangle $R = [-1, 1] \times [-1, 1]$, define the surface $\mathbf{x}(R)$ by

$$\mathbf{x}(x, y) = (x, y, x^2 + y^2),$$

see Figure 11.2. of the Section 11.

a) Calculate the tangent vectors $\mathbf{x}_1, \mathbf{x}_2$ and the mixed derivatives $\mathbf{x}_{11}, \mathbf{x}_{12}$ and $\mathbf{x}_2 2$ at the point $\mathbf{x}(0, 0)$.

b) Calculate the tangent plane at the point $\mathbf{x}(0, 0)$.

c) Calculate the osculating paraboloid at the point $\mathbf{x}(0, 0)$.

d) Classify the surface at the point $\mathbf{x}(0, 0)$.

4. For the 2-rectangle $R = [-1, 1] \times [-1, 1]$, define the surface $\mathbf{x}(R)$ by

$$\mathbf{x}(x, y) = (x, y, x^2 - y^2 + 1),$$

see Figure 11.3. of the Section 11.

a) Calculate the tangent vectors $\mathbf{x}_1, \mathbf{x}_2$ and the mixed derivatives $\mathbf{x}_{11}, \mathbf{x}_{12}$ and $\mathbf{x}_2 2$ at the point $\mathbf{x}(0, 0)$.

b) Calculate the tangent plane at the point $\mathbf{x}(0, 0)$.

c) Calculate the osculating paraboloid at the point $\mathbf{x}(0, 0)$.

d) Classify the surface at the point $\mathbf{x}(0, 0)$.