Partial Answers to Exercises 3.2.1:

1. Let $\mathbf{x}_1 = (2,3)$, $\mathbf{x}_2 = (2,-3)$, and $\mathbf{x}_3 = (4,1)$.
   a) $\mathbf{x}_1 \cdot \mathbf{x}_2 = 2 \cdot 2 + 3 \cdot (-3) = -5$.
   b) $\mathbf{x}_1 \wedge \mathbf{x}_2 = \det \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{e}_{12} = -12\mathbf{e}_{12}$.
   c) The Euler form is $\mathbf{x}_2\mathbf{x}_3 \widetilde{=} \sqrt{13 \cdot 17} e^{-1.22777\mathbf{e}_{12}}$.
   d) $\mathbf{x}_1(\mathbf{x}_2 + \mathbf{x}_3) = \mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{x}_1 \wedge (\mathbf{x}_2 + \mathbf{x}_3) = 6 - 22\mathbf{e}_{12}$.
   e) $\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_3 = \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \wedge \mathbf{x}_2 + \mathbf{x}_1 \cdot \mathbf{x}_3 + \mathbf{x}_1 \wedge \mathbf{x}_3 = 6 - 22\mathbf{e}_{12}$.
   f) Graph $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ in $\mathbb{R}^2$.

2. Let $\mathbf{x} = (x,y) = x\mathbf{e}_1 + y\mathbf{e}_2$.
   a) The magnitude $|\mathbf{x}| = \sqrt{\mathbf{x}^2} = \sqrt{x^2 + y^2}$.
   b) The unit vector $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{(x,y)}{\sqrt{x^2+y^2}}$, and

$$\mathbf{x}^{-1} = \frac{1}{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2} = \frac{\hat{\mathbf{x}}}{|\mathbf{x}|},$$

because

$$\mathbf{x}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|} = \frac{|\mathbf{x}|\hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|} = 1$$

c) The equation of the **unit circle** in $\mathbb{R}^2$ with center at the point $\mathbf{a} = (a_1, a_2)$ is

$$(\mathbf{x} - \mathbf{a})^2 = (x - a_1)^2 + (y - a_2)^2 = 1$$

d) For the vectors $\mathbf{x}_1 = (3,4)$, $\mathbf{x}_2 = (9,12)$, the polar form

$$\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_1 \cdot \mathbf{x}_2 = 75.$$

3. Let $w_1 = 5 + 4\mathbf{e}_1$, $w_2 = 5 - 4\mathbf{e}_2$, and $z_3 = 2 + \mathbf{e}_1\mathbf{e}_2$ be geometric numbers.
   a) Find $w_1w_2 - z_3 = 23 + 20\mathbf{e}_1 - 20\mathbf{e}_2 - 17\mathbf{e}_{12}$.
   b) Show that $w_1(w_2z_3) = (w_1w_2)z_3 = 66 + 60\mathbf{e}_1 - 20\mathbf{e}_2 - 7\mathbf{e}_{12}$ (geometric multiplication is associative.)
   c) Show that $w_1(w_2 + z_3) = w_1w_2 + w_1z_3 = 35 + 28\mathbf{e}_1 - 16\mathbf{e}_2 - 11\mathbf{e}_{12}$ (distributive law).

4. Let $w = x + \mathbf{e}_1 y$ and $\overline{w} = x - \mathbf{e}_1 y$. We define the magnitude $|w| = \sqrt{|w\overline{w}|}$.
   a) Show that $|w| = \sqrt{|x^2 - y^2|}$.
   b) Show that the equation of the **unit hyperbola** in the hyperbolic number plane $\mathbb{H}$, with center at the origin, is $|w|^2 = |w\overline{w}| = 1$ and has 4 branches.

c) **Hyperbolic Euler Formula:** Let $x > |y|$. Show that

$$w = x + \mathbf{e}_1 y = |w|(\frac{x}{|w|} + \mathbf{e}_1 \frac{y}{|w|}) = \rho(\cosh\phi + \mathbf{e}_1 \sinh\phi) = \rho e^{\mathbf{e}_1 \phi}$$

where $\rho = |w|$ is the hyperbolic magnitude of w, and ϕ is the hyperbolic angle that w makes with the x-axis. The (ρ, ϕ) are also called the **hyperbolic polar coordinates** of the point $w = (x, y) = x + \mathbf{e}_1 y$. What happens in the case that $y > |x|$?

d) Let $w_1 = \rho_1 \exp(\mathbf{e}_1\phi_1)$ and $w_2 = \rho_2 \exp(\mathbf{e}_1\phi_2)$. Show that $w_1 w_2 = \rho_1\rho_2 \exp(\mathbf{e}_1(\phi_1 + \phi_2))$. What is the geometric interpretation of this result? Illustrate with a figure.

f) Find the *square roots* of the the geometric numbers $w = 5 + 4\mathbf{e}_1$ and $z = 2 + \mathbf{e}_{12}$. *Hint:* First express the numbers in Euler form.

If $w = \rho e^{\phi u}$, then $\sqrt{w} = \sqrt{\rho} e^{\frac{1}{2}\phi u}$. Another way of extracting square roots is to use the spectral basis $u_+ = \frac{1}{2}(1+u)$ and $u_- = \frac{1}{2}(1-u)$. Then for $w = x + yu = (x+y)u_+ + (x-y)u_-$,

$$\sqrt{w} = \pm\sqrt{x+y}u_+ + \pm\sqrt{x-y}u_-.$$

For $w = 5 + 4\mathbf{e}_1$, we find $w = 9u_+ + u_-$, so that

$$\sqrt{w} = \pm 3u_+ \pm u_-,$$

so that w has 4 distinct square roots.

5. Calculate a) $e^{i\theta}\mathbf{e}_1$ and $e^{i\theta}\mathbf{e}_2$, where $i = \mathbf{e}_1\mathbf{e}_2$, and graph the results on the unit circle in $\mathbb{R}^2$.

b) Show that $e^{i\theta}\mathbf{e}_1 = \mathbf{e}_1 e^{-i\theta}$.

$$e^{i\theta}\mathbf{e}_1 = (\cos\theta + i\sin\theta)\mathbf{e}_1 = \mathbf{e}_1(\cos\theta - i\sin\theta) = \mathbf{e}_1 e^{-i\theta}.$$

c) Show that $(e^{i\theta}\mathbf{e}_1) \wedge (e^{i\theta}\mathbf{e}_2) = \mathbf{e}_1 \wedge \mathbf{e}_2 = i$, and explain the geometric significance of this result.

$$(e^{i\theta}\mathbf{e}_1) \wedge (e^{i\theta}\mathbf{e}_2) = \frac{1}{2}[(e^{i\theta}\mathbf{e}_1)(e^{i\theta}\mathbf{e}_2) - (e^{i\theta}\mathbf{e}_2)(e^{i\theta}\mathbf{e}_1)] = \frac{1}{2}(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_{12}.$$

6. Show that $e^{-i\theta}\mathbf{a}$ rotates the vector $\mathbf{a} = (a_1, a_2)$ counter-clockwise in the (x, y)-plane through an angle of θ. We have $\mathbf{a} = (\, \mathbf{e}_1 \quad \mathbf{e}_2 \,) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, so that

$$e^{-i\theta}\mathbf{a} = e^{-i\theta}(\, \mathbf{e}_1 \quad \mathbf{e}_2 \,)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (\, e^{-i\theta}\mathbf{e}_1 \quad e^{-i\theta}\mathbf{e}_2 \,)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= (\, \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \quad -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \,)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

7. Let the *geometric numbers* $A = 1 + 2\mathbf{e}_1 - \mathbf{e}_2 + 3i$ and $B = -2 - \mathbf{e}_1 + 2\mathbf{e}_2 - i$. Calculate the geometric product AB and write it as the sum of its *scalar*, *vector*, and *bivector* parts. $AB = -3 + 5\mathbf{e}_2 - 4\mathbf{e}_{12}$.