Partial Answers to Exercises 3.3.1:

Given the vectors $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3, \mathbf{b} = -\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3, \mathbf{c} = 3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3 \in \mathbb{R}^3$. The following problems are in the geometric algebra $\mathbb{G}_3$. Let $i = \mathbf{e}_{123}$.

1. Verify the coordinate formulas (3.9) and (3.10) for the inner and outer products in $\mathbb{G}_3$.

$$\mathbf{ab} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 + (a_1b_2 - a_2b_1)\mathbf{e}_{12} + (a_1b_3 - a_3b_1)\mathbf{e}_{13} + (a_2b_3 - a_3b_2)\mathbf{e}_{23},$$

and

$$\mathbf{ba} = (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 - (a_1b_2 - a_2b_1)\mathbf{e}_{12} - (a_1b_3 - a_3b_1)\mathbf{e}_{13} - (a_2b_3 - a_3b_2)\mathbf{e}_{23},$$

from which formulas (3.9) and (3.10) follow.

2. a) From (3.10), $\mathbf{a} \wedge \mathbf{b} = I(\mathbf{a} \times \mathbf{b})$. b) Using part a)

$$(\mathbf{a} \wedge \mathbf{b})^2 = [I(\mathbf{a} \times \mathbf{b})]^2 = -(\mathbf{a} \times \mathbf{b})^2 = -|\mathbf{a} \times \mathbf{b}|^2 = -|\mathbf{a} \wedge \mathbf{b}|^2$$

from which it follows that

$$\frac{1}{\mathbf{a} \wedge \mathbf{b}} = -\frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|^2}.$$

3. Calculate $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ using formula (3.12).

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = -\mathbf{a} \times (\mathbf{b} \times \mathbf{d}).$$

4. Calculate $(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{b} \wedge \mathbf{c}) := <(\mathbf{a} \wedge \mathbf{b})(\mathbf{b} \wedge \mathbf{c})>_0$ by using (3.13) and (3.12).

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{b} \wedge \mathbf{c})) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) = -(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}).$$

5. Calculate $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ using the formula (3.11).

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = I(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})) = I\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

6. Calculate $(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{b}\wedge\mathbf{c})$, where $A\otimes B := \frac{1}{2}(AB-BA)$, using the formula (3.10).

$$(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{b}\wedge\mathbf{c}) = (I\mathbf{a}\times\mathbf{b})\otimes(I\mathbf{b}\times\mathbf{c}) = -(\mathbf{a}\times\mathbf{b})\wedge(\mathbf{b}\times\mathbf{c}) = -I(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})$$

7. Show that $(\mathbf{a}\wedge\mathbf{b})(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c}) + (\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{b}\wedge\mathbf{c})$.

$$(\mathbf{a}\wedge\mathbf{b})(\mathbf{b}\wedge\mathbf{c}) = -(\mathbf{a}\times\mathbf{b})(\mathbf{b}\times\mathbf{c}) = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\times\mathbf{b})\wedge(\mathbf{b}\times\mathbf{c})$$

$$= -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c}) - I[(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})]$$

To complete the problem, use problems 4. and 6. above.

8. Noting that

$$\mathbf{b} = (\mathbf{b}\mathbf{a})\mathbf{a}^{-1} = (\mathbf{b}\cdot\mathbf{a})\mathbf{a}^{-1} + (\mathbf{b}\wedge\mathbf{a})\mathbf{a}^{-1},$$

show that $\mathbf{b}_{\parallel}\mathbf{a} = \mathbf{b}_{\parallel}\cdot\mathbf{a}$ and $\mathbf{b}_{\perp}\mathbf{a} = \mathbf{b}_{\perp}\wedge\mathbf{a}$ where $\mathbf{b}_{\parallel} = (\mathbf{b}\cdot\mathbf{a})\mathbf{a}^{-1}$ and $\mathbf{b}_{\perp} = (\mathbf{b}\wedge\mathbf{a})\mathbf{a}^{-1} = \mathbf{b}-\mathbf{b}_{\parallel}$.

$\mathbf{b}_{\parallel}\mathbf{a} = \mathbf{b}_{\parallel}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a}$ because $\mathbf{b}_{\parallel}\wedge\mathbf{a} = 0$. Similarly,
$\mathbf{b}_{\perp}\mathbf{a} = (\mathbf{b}-\mathbf{b}_{\parallel})\mathbf{a} = (\mathbf{b}\mathbf{a}-\mathbf{b}_{\parallel}\mathbf{a}) = \mathbf{b}\wedge\mathbf{a}$.

9. Find vectors $\mathbf{c}_{\parallel}$ and $\mathbf{c}_{\perp}$ such that $\mathbf{c} = \mathbf{c}_{\parallel}+\mathbf{c}_{\perp}$ and $\mathbf{c}_{\parallel}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b})$ and $\mathbf{c}_{\perp}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})$. *Hint:* Use the fact that

$$\mathbf{c} = [\mathbf{c}(\mathbf{a}\wedge\mathbf{b})](\mathbf{a}\wedge\mathbf{b})^{-1} = [\mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b}) + \mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})](\mathbf{a}\wedge\mathbf{b})^{-1}.$$

For problems 9 - 16, let $\mathbf{a},\mathbf{b},\mathbf{c}$ be *any* vectors in $\mathbb{G}_3$.

10. Using (3.10), show that $\mathbf{a}\times\mathbf{b} = -I(\mathbf{a}\wedge\mathbf{b})$ where $I = \mathbf{e}_{123}$.

Multiplying equation (3.10) on both sides by I gives the result.

11. Show that $\mathbf{a}\cdot[I(\mathbf{b}\wedge\mathbf{c})] = I(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})$ where $I = \mathbf{e}_{123}$.

$$\mathbf{a}\cdot[I(\mathbf{b}\wedge\mathbf{c})] =< \mathbf{a}[I(\mathbf{b}\wedge\mathbf{c})] >_0 =< I\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) >_0 = I(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}).$$

12. Show that $\mathbf{a}\wedge[I(\mathbf{b}\wedge\mathbf{c})] = I[\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})]$ where $I = \mathbf{e}_{123}$.

$$\mathbf{a}\wedge[I(\mathbf{b}\wedge\mathbf{c})] =< I[\mathbf{a}(\mathbf{b}\wedge\mathbf{c})] >_2 = I[\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})].$$

13. Show that $\mathbf{a}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\wedge\mathbf{b})$

$$\mathbf{a}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\wedge\mathbf{b}) + \mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\wedge\mathbf{b}).$$

14. Show that $(\mathbf{a}+\mathbf{b}\wedge\mathbf{c})^2 = \mathbf{a}^2 + 2\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})^2$.

$$(\mathbf{a}+\mathbf{b}\wedge\mathbf{c})^2 = \mathbf{a}^2 + \mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + (\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{b}\wedge\mathbf{c})^2 = \mathbf{a}^2 + 2\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})^2.$$

15. Show that $\mathbf{a}(\mathbf{a}\wedge\mathbf{b}) = -(\mathbf{a}\wedge\mathbf{b})\mathbf{a}$.

$$\mathbf{a}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\wedge\mathbf{b}) = -(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{a} = -(\mathbf{a}\wedge\mathbf{b})\mathbf{a}.$$

16. Show that $(\mathbf{a}+\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{a}^2 + (\mathbf{a}\wedge\mathbf{b})^2$.

$$(\mathbf{a}+\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{a}(\mathbf{a}\wedge\mathbf{b}) + (\mathbf{a}\wedge\mathbf{b})\mathbf{a} + (\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{a}^2 + (\mathbf{a}\wedge\mathbf{b})^2.$$

17. Show that $(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c})^2 = (\mathbf{a}\wedge\mathbf{b})^2 + 2(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c}) + (\mathbf{b}\wedge\mathbf{c})^2$.

Use problem 7 and the fact that $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{b}\wedge\mathbf{c})\cdot(\mathbf{a}\wedge\mathbf{b})$.