MT 318 Differential Geometry **Universidad de Las Americas - Puebla** February 15, 2010

Answers to Exercises 3.5.1:

1. Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{R}^n$. Show that

$$\mathbf{abc} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}.$$

$$\mathbf{a}(\mathbf{bc}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\wedge\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) + \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$$

2. Show that

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\wedge\mathbf{d} - \mathbf{b}\wedge[\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d})].$$

This formula is a direct consequence of the identity (3.21).

3. Show that

$$(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{cd}) = [(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\mathbf{d} + \mathbf{c}[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}],$$

where $A\otimes B = \frac{1}{2}(AB - BA)$ is the anti-symmetric bracket operation.

$$(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{cd}) = [(\mathbf{a}\wedge\mathbf{b})\otimes\mathbf{c}]\mathbf{d} + \mathbf{c}[(\mathbf{a}\wedge\mathbf{b})\otimes\mathbf{d}] = [(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\mathbf{d} + \mathbf{c}[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}]$$

4. Show that

$$(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{c}\wedge\mathbf{d}) = [(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\wedge\mathbf{d} + \mathbf{c}\wedge[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}].$$

Follows from problem 3 by noting that $(\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{cd}) = (\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{c}\wedge\mathbf{d})$ for the left side, and for the right side,

$$[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\mathbf{d} + \mathbf{c}[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}] = [(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}]\wedge\mathbf{d} + \mathbf{c}\wedge[(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{d}]$$

5. Show that

$$(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{b})\otimes(\mathbf{c}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}).$$

Follows from problem 4, by noting that the the symmetric part

$$(\mathbf{a}\wedge\mathbf{b})\circ(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}).$$

6. Prove the identity (3.22) by using (3.21). The result can be proved by induction on $r \geq 1$, and where $s \geq 1$. For $r = 1$ is just the identity (3.21). Now assume true for $r = k > 1$, and write $A_{k+1} = A_k \wedge \mathbf{a}_{k+1}$. Then $\mathbf{a}\cdot(A_{k+1}\wedge B_s) =$

$$\mathbf{a}\cdot[A_k\wedge(\mathbf{a}_{k+1}\wedge B_s)] = (\mathbf{a}\cdot A_k)\wedge(\mathbf{a}_{k+1}\wedge B_s) + (-1)^k A_k\wedge[\mathbf{a}\cdot(\mathbf{a}_{k+1}\wedge B_s)]$$

$$= \cdots = [\mathbf{a}\cdot(A_k\wedge\mathbf{a}_{k+1})]\wedge B_s + (-1)^{k+1}(A_k\wedge\mathbf{a}_{k+1})\wedge(\mathbf{a}\cdot B_s)$$