Answers to Exercises 3.5.1:

Calculate the following vector derivatives in $I\!R^n$.

1. For $k \neq 0$, $\partial_{\mathbf{x}}|\mathbf{x}|^k = k|\mathbf{x}|^{k-2}\mathbf{x}$,

$$\partial_{\mathbf{x}}|\mathbf{x}|^k = k|\mathbf{x}|^{k-1}\partial_{\mathbf{x}}|\mathbf{x}| = k|\mathbf{x}|^{k-1}\hat{\mathbf{x}} = k|\mathbf{x}|^{k-2}\mathbf{x}.$$

2. $\partial_{\mathbf{x}}\frac{\mathbf{x}}{|\mathbf{x}|^k} = \frac{n-k}{|\mathbf{x}|^k}$,

$$\partial_{\mathbf{x}}\frac{\mathbf{x}}{|\mathbf{x}|^k} = \frac{\partial_{\mathbf{x}}\mathbf{x}}{|\mathbf{x}|^k} + (\partial_{\mathbf{x}}|\mathbf{x}|^{-k})\mathbf{x} = \frac{n}{|\mathbf{x}|^k} - k|\mathbf{x}|^{-k-1}\hat{\mathbf{x}}\mathbf{x} = \frac{n-k}{|\mathbf{x}|^k}.$$

3. $\partial_{\mathbf{x}}\log|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|^2} = \mathbf{x}^{-1}$.

$$\partial_{\mathbf{x}}\log|\mathbf{x}| = \frac{1}{|\mathbf{x}|}\partial_{\mathbf{x}}|\mathbf{x}| = \frac{1}{|\mathbf{x}|}\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2} = \mathbf{x}^{-1}.$$

4. $\partial_{\mathbf{x}}\sin\mathbf{x} = (\frac{n-1}{|\mathbf{x}|})\sin|\mathbf{x}| + \cos|\mathbf{x}|$. Using problem 2, with $k=1$, we have that $\partial_{\mathbf{x}}\hat{\mathbf{x}} = \frac{n-1}{|\mathbf{x}|}$, so that

$$\partial_{\mathbf{x}}\sin\mathbf{x} = \partial_{\mathbf{x}}\sin(\hat{\mathbf{x}}|\mathbf{x}|) = \partial_{\mathbf{x}}\hat{\mathbf{x}}\sin|\mathbf{x}|$$

$$= (\frac{n-1}{|\mathbf{x}|})\sin|\mathbf{x}| + (\partial_{\mathbf{x}}\sin|\mathbf{x}|)\hat{\mathbf{x}} = (\frac{n-1}{|\mathbf{x}|})\sin|\mathbf{x}| + \cos|\mathbf{x}|.$$

Note what happens when $n=1$.

5. $\partial_{\mathbf{x}}\exp\mathbf{x} = e^{\mathbf{x}} + (\frac{n-1}{|\mathbf{x}|})\sinh|\mathbf{x}|$. Using problem 2 with $k=1$, we get

$$\partial_{\mathbf{x}}\exp\mathbf{x} = \partial_{\mathbf{x}}(\cosh|\mathbf{x}| + \hat{\mathbf{x}}\sinh|\mathbf{x}|) = \hat{\mathbf{x}}\sinh|\mathbf{x}| + (\frac{n-1}{|\mathbf{x}|})\sinh|\mathbf{x}| + \cosh|\mathbf{x}|$$

$$= e^{\mathbf{x}} + (\frac{n-1}{|\mathbf{x}|})\sinh|\mathbf{x}|.$$

6. $\partial_{\mathbf{x}}(\mathbf{x}\wedge\mathbf{b})^2$. First, note that

$$\mathbf{b}^2\mathbf{x}^2 = \mathbf{x}\mathbf{b}\mathbf{b}\mathbf{x} = (\mathbf{x}\cdot\mathbf{b}+\mathbf{x}\wedge\mathbf{b})(\mathbf{x}\cdot\mathbf{b}-\mathbf{x}\wedge\mathbf{b}) = (\mathbf{x}\cdot\mathbf{b})^2-(\mathbf{x}\wedge\mathbf{b})^2 = (\mathbf{x}\cdot\mathbf{b})^2+|\mathbf{x}\wedge\mathbf{b}|^2$$

Taking the vector derivative of this equation gives

$$\partial_{\mathbf{x}}\mathbf{b}^2\mathbf{x}^2 = 2\mathbf{b}^2\mathbf{x} = 2(\mathbf{x}\cdot\mathbf{b})\mathbf{b} - \partial_{\mathbf{x}}(\mathbf{x}\wedge\mathbf{b})^2 = 2(\mathbf{x}\cdot\mathbf{b})\mathbf{b} + 2|\mathbf{x}\wedge\mathbf{b}|\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|,$$

or

$$\partial_{\mathbf{x}}(\mathbf{x}\wedge\mathbf{b})^2 = 2(\mathbf{x}\cdot\mathbf{b})\mathbf{b} - 2\mathbf{x}\mathbf{b}^2 = -2(\mathbf{x}\wedge\mathbf{b})\mathbf{b}.$$

7. $\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|$. Using the answer to problem 6, and problem 7, we find that

$$\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}| = \frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}.$$

8. $\partial_{\mathbf{x}}\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}$. Using the identity $\partial_{\mathbf{x}}\mathbf{x}\wedge\mathbf{b} = \partial_{\mathbf{x}}(\mathbf{x}\mathbf{b} - \mathbf{x}\cdot\mathbf{b}) = (n-1)\mathbf{b}$, we find that

$$\partial_{\mathbf{x}}\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|} = (\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|^{-1})\mathbf{x}\wedge\mathbf{b} + \frac{\partial_{\mathbf{x}}\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}$$

$$= -|\mathbf{x}\wedge\mathbf{b}|^{-2}\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}(\mathbf{x}\wedge\mathbf{b}) + \frac{(n-1)\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|} = \frac{(n-2)\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}.$$

9. $\partial_{\mathbf{x}}\exp(\mathbf{x}\wedge\mathbf{b}) = \mathbf{b}e^{\mathbf{x}\wedge\mathbf{b}} + \frac{n-2}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}\sin|\mathbf{x}\wedge\mathbf{b}|$. Using the identities from problems 7 and 8, we find

$$\partial_{\mathbf{x}}\exp(\mathbf{x}\wedge\mathbf{b}) = \partial_{\mathbf{x}}[\cosh(|\mathbf{x}\wedge\mathbf{b}|\widehat{\mathbf{x}\wedge\mathbf{b}}) + \sinh(|\mathbf{x}\wedge\mathbf{b}|\widehat{\mathbf{x}\wedge\mathbf{b}})]$$

$$= \partial_{\mathbf{x}}[\cos(|\mathbf{x}\wedge\mathbf{b}|) + \widehat{\mathbf{x}\wedge\mathbf{b}}\sin(|\mathbf{x}\wedge\mathbf{b}|)] = -(\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|)\sin|\mathbf{x}\wedge\mathbf{b}| + (\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|)(\widehat{\mathbf{x}\wedge\mathbf{b}})\cos|\mathbf{x}\wedge\mathbf{b}|$$

$$= -\frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}\sin|\mathbf{x}\wedge\mathbf{b}| + \frac{\mathbf{x}\wedge\mathbf{b}}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}(\widehat{\mathbf{x}\wedge\mathbf{b}})\cos|\mathbf{x}\wedge\mathbf{b}| + \frac{n-2}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}\sin|\mathbf{x}\wedge\mathbf{b}|$$

$$= \mathbf{b}e^{\mathbf{x}\wedge\mathbf{b}} + \frac{n-2}{|\mathbf{x}\wedge\mathbf{b}|}\mathbf{b}\sin|\mathbf{x}\wedge\mathbf{b}|.$$