MT 318 Differential Geometry **Universidad de Las Americas - Puebla** March 1, 2010

**Exam 1 over Chapter 3**

1. (**10 pts.**) a) What is the infinite series expansion for $e^x$?

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

b) What is the infinite series expansion for $\sin x$?

$$e^x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}.$$

2. (**10 pts.**) For the complex number $z = 2\sqrt{3} - 2i = re^{i\theta}$, find $r$ and $\theta$ for the polar form.

$$z = 4e^{-\frac{\pi}{6}i}.$$

3. (**10 pts.**) For the hyperbolic number $w = 5 - 3u = \rho e^{u\phi}$, find $\rho$ and $\phi$ for the hyperbolic polar form.

$$w = 4e^{-\phi u}$$

where $\phi = \tanh^{-1}\frac{3}{5} = 0.63147$

4. (**40 pts.**) Given the vectors $\mathbf{a} = 2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$, calculate the following:

a) $\mathbf{a} \cdot \mathbf{b} = 4$.

b) $\mathbf{a} \wedge \mathbf{b} = -6\mathbf{e}_{23} + 2\mathbf{e}_{13} + 5\mathbf{e}_{12}$.

c) $\sqrt{\mathbf{ab}} = 3\hat{\mathbf{a}}\frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|} = \mathbf{a}(\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}) = \frac{13+5\mathbf{e}_{12}+2\mathbf{e}_{13}-6\mathbf{e}_{23}}{\sqrt{26}}$.

d) Find a rotation $R(\mathbf{x})$ with the property that $R(\mathbf{a}) = \mathbf{b}$.

$$R(\mathbf{x}) = (\mathbf{ba}^{-1})^{\frac{1}{2}}\mathbf{x}(\mathbf{ab}^{-1})^{\frac{1}{2}}$$

e) Find a reflection $L(\mathbf{x})$ with the property that $L(\mathbf{a}) = \mathbf{b}$.

$$L(\mathbf{x}) = -(\mathbf{a} - \mathbf{b})\mathbf{x}(\mathbf{a} - \mathbf{b})^{-1}$$

Given the constant vectors $\mathbf{a}$ and $\mathbf{b}$, calculate the following $\mathbf{a}$-directional derivatives in $I\!R^n$.

5. (**10 pts.**) $\mathbf{a}\cdot\partial_{\mathbf{x}}\ \log|\mathbf{x}| = \frac{1}{|\mathbf{x}|}\mathbf{a}\cdot\hat{\mathbf{x}} = \frac{\mathbf{a}\cdot\mathbf{x}}{\mathbf{x}^2}$.

6. (**10 pts.**) Using that

$$\mathbf{a} = \mathbf{a}\cdot\partial_{\mathbf{x}}\ \mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}|\mathbf{x}|\hat{\mathbf{x}} = (\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} + |\mathbf{x}|\mathbf{a}\cdot\partial_{\mathbf{x}}\hat{\mathbf{x}}$$

it follows that

$$\mathbf{a}\cdot\partial_{\mathbf{x}}\hat{\mathbf{x}} = \frac{\mathbf{a}-\mathbf{a}\cdot\hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|} = \frac{(\mathbf{a}\wedge\mathbf{x})\mathbf{x}}{|\mathbf{x}|^3},$$

so

$$\mathbf{a}\cdot\partial_{\mathbf{x}}\ \sin\mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}\ \hat{\mathbf{x}}\sin|\mathbf{x}| = \frac{(\mathbf{a}\wedge\mathbf{x})\mathbf{x}}{|\mathbf{x}|^3}\sin|\mathbf{x}| + (\mathbf{a}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}\cos|\mathbf{x}|$$

$$= \frac{(\mathbf{a}\wedge\mathbf{x})\sin\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\mathbf{x}\cos\mathbf{x}}{\mathbf{x}^2}.$$

7. (**10 pts.**) $\mathbf{a}\cdot\partial_{\mathbf{x}}\ |\mathbf{x}\wedge\mathbf{b}|$. From

$$2\mathbf{b}^2\mathbf{a}\cdot\mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}\mathbf{x}\mathbf{b}\mathbf{b}\mathbf{x} = \mathbf{a}\cdot\partial_{\mathbf{x}}[(\mathbf{x}\cdot\mathbf{b})^2+|\mathbf{x}\wedge\mathbf{b}|^2] = 2(\mathbf{x}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})+2|\mathbf{x}\wedge\mathbf{b}|\mathbf{a}\cdot\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}|,$$

we find that

$$\mathbf{a}\cdot\partial_{\mathbf{x}}|\mathbf{x}\wedge\mathbf{b}| = \frac{\mathbf{b}^2(\mathbf{a}\cdot\mathbf{x})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{x}\cdot\mathbf{b})}{|\mathbf{x}\wedge\mathbf{b}|} = -\frac{(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{x}\wedge\mathbf{b})}{|\mathbf{x}\wedge\mathbf{b}|}.$$