MT 318 Differential Geometry **Universidad de Las Americas - Puebla** March 25, 2010

Answers for Exam 2 over Chapter 13

1. (**20 pts.**) a) Given the vectors $\mathbf{x}_1 = (1, 2, -2)$ and $\mathbf{x}_2 = (2, 2, 1)$ of a 2-dimensional subspace of $I\!R^3$, find the reciprocal basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ of that subspace.

$$[g] = \begin{pmatrix} 9 & 4 \\ 4 & 9 \end{pmatrix}, \ [g]^{-1} = \frac{1}{65}\begin{pmatrix} 9 & -4 \\ -4 & 9 \end{pmatrix}$$

$$\mathbf{x}^1 = \frac{1}{65}(1, 10, -22), \quad \mathbf{x}^2 = \frac{1}{65}(14, 10, 17).$$

b) $\alpha = 3$ and $\beta = -2$ such that $(-1, 2, -8) = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$.

2. (**20 pts.**) a) State the Fundamental Theorem of Calculus for a k-patch M and its $(k-1)$-boundary $\beta(M)$.

$$\int_M g d\mathbf{x}_{(k)} \partial_{\mathbf{x}} f = \int_{\beta(M)} g d\mathbf{x}_{(k-1)} f.$$

b) Show that for a 3-volume patch M in $I\!R^3$ and its 2-boundary patch $\beta(M)$, that

$$\int_M d\mathbf{x}_{(3)} = \frac{1}{3}\int_{\beta(M)} d\mathbf{x}_{(2)} \wedge \mathbf{x}.$$

Choosing $g = 1$, and $f = \mathbf{x}$ gives the result, since $\partial_{\mathbf{x}}\mathbf{x} = 3$ in $I\!R^3$.

3. (**30 pts.**) Let M be the 3-volume bounded by the plane $3x + 2y + 6z = 6$ and the coordinate planes $x = 0$, $y = 0$, and $z = 0$ in $I\!R^3$.

a) Using 2b) above, prove that

$$\int_M |d\mathbf{x}_{(3)}| = \frac{1}{3}\int_{\beta(M)} \hat{\mathbf{n}} \cdot \mathbf{x} |d\mathbf{x}_{(2)}|,$$

where $\hat{\mathbf{n}}$ is the unit outward normal to each face.

Multiplying both sides of the equation 2b) by $I^{-1} = \mathbf{e}_{321}$ gives

$$\int_M |d\mathbf{x}_{(3)}| = \mathbf{e}_{321}\frac{1}{3}\int_{\beta(M)} d\mathbf{x}_{(2)} \wedge \mathbf{x} = \frac{1}{3}\int_{\beta(M)} \hat{\mathbf{n}} \cdot \mathbf{x} |d\mathbf{x}_{(2)}|,$$

since $\mathbf{e}_{321} d\mathbf{x}_{(2)} = |d\mathbf{x}_{(2)}|\hat{\mathbf{n}}$.

b) Directly verify the above result by carrying out the integration on both sides of the equation in part 3a).

Let $\mathbf{x} = (x, y, z)$ so that $\mathbf{x}_1 = \mathbf{e}_1, \mathbf{x}_2 = \mathbf{e}_2, \mathbf{x}_3 = \mathbf{e}_3$. Then

$$\int_M |d\mathbf{x}_{(3)}| = \int_0^1 \int_0^{3(1-z)} \int_0^{2(1-z-\frac{1}{3}y)} |\mathbf{e}_{123}| dxdydz = \frac{1}{6}|2\mathbf{e}_1 3\mathbf{e}_2\mathbf{e}_3| = 1.$$

We must now integrate over the 4 faces bounding M: i) $y = 0$, ii) $z = 0$, iii) $x = 0$, iv) $3x + 2y + 6z = 6$. The integrals over i), ii) and iii) are 0. We now calculate the integral over iv). For this face, we have

$$\hat{\mathbf{n}} = \frac{1}{7}(3, 2, 6),\ \hat{\mathbf{n}} \cdot \mathbf{x} = \frac{1}{7}(3x + 2y + 6z),\ \mathbf{x}_1 = (1, 0, -1/2),\ \mathbf{x}_2 = (0, 1, -1/3),$$

and

$$|d\mathbf{x}_{(2)}| = |\mathbf{x}_1 \times \mathbf{x}_2| dxdy = \frac{7}{6} dxdy$$

Completing the calculation, using that $z = 1 - \frac{1}{2}x - \frac{1}{3}y$, we get

$$\frac{1}{3}\int_{\beta(M)} \hat{\mathbf{n}} \cdot \mathbf{x} |d\mathbf{x}_{(2)}| = \frac{1}{18}\int_0^3 \int_0^{2-\frac{2}{3}y} (3x + 2y + 6z) dxdy = 1$$

4. (**10 pts.**) Show that $\int_C |d\mathbf{x}| = 2\pi$ where C is the closed curve defined by the equation $x^2 + y^2 = 1$.

$$\int_C |d\mathbf{x}| = \int_0^{2\pi} d\theta = 2\pi.$$

5. (**20 pts.**) From the Fundamental Theorem of Calculus applied to a 2-patch M, we know that

$$\int_M d\mathbf{x}_{(2)} = \frac{1}{2}\int_{\beta(M)} d\mathbf{x} \wedge \mathbf{x}$$

Show that this formula is true for the surface of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane bounded by the circle $x^2 + y^2 = 1$.

On the right side, we get $\frac{1}{2}\int_{\beta(M)} d\mathbf{x} \wedge \mathbf{x} = \pi\mathbf{e}_{12}$, and on the left side we have

$$\mathbf{x} = (r\cos\theta, r\sin\theta, 1 - r^2),\ d\mathbf{x}_{(2)} = \mathbf{x}_r \wedge \mathbf{x}_\theta d\theta dr,$$

where $\mathbf{x}_r = (\cos\theta, \sin\theta, -2r)$, $\mathbf{x}_\theta = (-r\sin\theta, r\cos\theta, 0)$, so that

$$\int_M d\mathbf{x}_{(2)} = I\int_0^1 \int_0^{2\pi} (2r^2\cos\theta, -2r^2\sin\theta, r) d\theta dr = I(0, 0, \pi) = \pi\mathbf{e}_{12}.$$