# Final Exam: Part B ANSWERS (50 points)

1. **15 pts.** Given the vectors $\mathbf{a} = (1, 2, -2), \mathbf{b} = (2, 1, 2), \mathbf{c} = (1, -1, 1)$, calculate

   a) $\mathbf{b} \wedge \mathbf{c} = -3\mathbf{e}_{12} + 3\mathbf{e}_{23}$

   b) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 9\mathbf{e}_{123}$

   c) $\sqrt{\mathbf{ab}} = (3 - \mathbf{e}_{12} + 2\mathbf{e}_{13} + 2\mathbf{e}_{23})/\sqrt{2} = \frac{\mathbf{a}(\mathbf{a}+\mathbf{b})}{|\mathbf{a}+\mathbf{b}|}$

   d) $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = 6\mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3$

   e) $[\mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}]^2 = -54$

2. **10 pts.** For $\mathbf{a}$ and $\mathbf{b}$ given in problem 1, find a rotation $R(\mathbf{x})$ for which $R(\mathbf{a}) = \mathbf{b}$.

$$R(\mathbf{x}) = \frac{1}{9}\sqrt{\mathbf{ba}}\mathbf{x}\sqrt{\mathbf{ab}} = \frac{1}{18}(3 + \mathbf{e}_{12} - 2\mathbf{e}_{13} - 2\mathbf{e}_{23})\mathbf{x}(3 - \mathbf{e}_{12} + 2\mathbf{e}_{13} + 2\mathbf{e}_{23})$$

3. **10 pts.** In $I\!R^n$, calculate

   a) $\partial_{\mathbf{x}} \frac{\mathbf{x}}{|\mathbf{x}|^k} = \frac{n-k}{|\mathbf{x}|^k}$

   b) $\partial_{\mathbf{x}} (\mathbf{x} \wedge \mathbf{a})^2 = 2\mathbf{a}(\mathbf{x} \wedge \mathbf{a})$ using that $(\mathbf{x} \wedge \mathbf{a})^2 = (\mathbf{x} \cdot \mathbf{a})^2 - \mathbf{a}^2\mathbf{x}^2$.

4. **5 pts.** State the Fundamental Theorem of Calculus for a $k$-surface $\mathcal{M}$ and its $(k-1)$-boundary $\beta(\mathcal{M})$ in $I\!R^n$.

$$\int_{\mathcal{M}} g d\mathbf{x}_{(k)} \partial_{\mathbf{x}} f = \int_{\beta(\mathcal{M})} g d\mathbf{x}_{(k-1)} f$$

5. **5 pts.** Give the infinite series expansion for $\sinh x$.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

6. **5 pts.** For vectors $\mathbf{a}, \mathbf{b}$, such that $|\mathbf{a}| > |\mathbf{b}|$, show that

$$\frac{1}{\mathbf{a} - \mathbf{b}} = \frac{1}{\mathbf{a}} + \frac{1}{\mathbf{a}}\mathbf{b}\frac{1}{\mathbf{a}} + \frac{1}{\mathbf{a}}\mathbf{b}\frac{1}{\mathbf{a}}\mathbf{b}\frac{1}{\mathbf{a}} + \cdots$$

$$\frac{1}{\mathbf{a} - \mathbf{b}} = \frac{1}{\mathbf{a}(1 - \mathbf{a}^{-1}\mathbf{b})} = \frac{1}{(1 - \mathbf{a}^{-1}\mathbf{b})}\frac{1}{\mathbf{a}}$$

$$= [1 + \mathbf{a}^{-1}\mathbf{b} + (\mathbf{a}^{-1}\mathbf{b})^2 + (\mathbf{a}^{-1}\mathbf{b})^3 + \cdots]\frac{1}{\mathbf{a}}$$