The Riemann Curvature Vector in $I\!R^3$

The Riemann curvature bivector is specified by

$$R(\mathbf{a}\wedge\mathbf{b}) = \partial_\mathbf{v}\wedge P_\mathbf{a}P_\mathbf{b}(\mathbf{v}).$$

In the case of a 2-surface in $I\!R^3$, since $\mathbf{a}\wedge\mathbf{b} = I\mathbf{a}\times\mathbf{b}$, we write

$$I^{-1}R(I\mathbf{a}\times\mathbf{b}) = I^{-1}\partial_\mathbf{v}\wedge P_\mathbf{a}P_\mathbf{b}(\mathbf{v}) = \partial_\mathbf{v}\times P_\mathbf{a}P_\mathbf{b}(\mathbf{v}).$$

We can now express the quantity

$$R(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) =< R(\mathbf{a}\wedge\mathbf{b})I^{-1}I(\mathbf{c}\times\mathbf{d}) >_0= [\partial_\mathbf{v}\times P_\mathbf{a}P_\mathbf{b}(\mathbf{v})]\cdot(\mathbf{c}\times\mathbf{d}). \quad (1)$$

In terms of the basis $\{\mathbf{x}_1,\mathbf{x}_2\}$ and reciprocal basis $\{\mathbf{x}^1,\mathbf{x}^2\}$, the classical components $R^i{}_{jkl}$ are given by

$$R^i{}_{jkl} = (\mathbf{x}^i\wedge\mathbf{x}_j)\cdot R(\mathbf{x}_k\wedge\mathbf{x}_l) = (\mathbf{x}^i\times\mathbf{x}_j)\cdot[\partial_\mathbf{v}\times P_kP_l(\mathbf{v})].$$

We call

$$\mathbf{R}_{ij} = \partial_\mathbf{v}\times P_iP_j(\mathbf{v}) = \sum_{m=1}^{2}\mathbf{x}^m\times P_iP_j(\mathbf{x}_m) \quad (2)$$

the *Riemann curvature vector* of the 2-dimensional surface $\mathbf{x}(u,v)$ in $I\!R^3$. The Riemann curvature vector is a scalar multiple of the unit normal vector $\mathbf{n}$ to the surface.

The *contraction* $R(\mathbf{b}) = \partial_\mathbf{a}\cdot R(\mathbf{a}\wedge\mathbf{b})$ of the Riemann curvature bivector is called the *Ricci tensor*. We calculate

$$R(\mathbf{b}) = \partial_\mathbf{a}\cdot R(\mathbf{a}\wedge\mathbf{b}) = P_\mathbf{a}P_\mathbf{b}(\partial_\mathbf{a}) - P_\mathbf{b}P_\mathbf{a}(\partial_\mathbf{a})$$

$$= P_{\mathbf{x}_1}P_\mathbf{b}(\mathbf{x}^1) + P_{\mathbf{x}_2}P_\mathbf{b}(\mathbf{x}^2) - P_\mathbf{b}P_{\mathbf{x}_1}(\mathbf{x}^1) - P_\mathbf{b}P_{\mathbf{x}_2}(\mathbf{x}^2). \quad (3)$$

Curvature of the Torus

We now find the various kinds of curvature for a torus. The torus can be parameterized by

$$\mathbf{x}(u,v) = (x,y,z) = (\,(c+a\cos v)\cos u, (c+a\cos v)\sin u, a\sin v\,)\,,$$

where both $u,v\in[0,2\pi]$. Each point $\mathbf{x}(u,v)$ on the torus satisfies the equation $(c-\sqrt{x^2+y^2})^2+z^2 = a^2$. The torus $\mathbf{x}(u,v)$ is pictured in the figure.

Calculating the basis and reciprocal basis of the tangent space, we find

$$\mathbf{x}_1 = (-(c+a\cos(v))\sin(u), \cos(u)(c+a\cos(v)), 0)$$

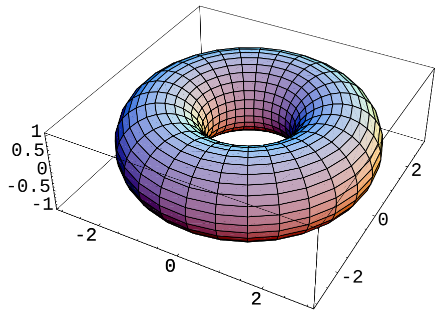

Figure 1: Plot of torus $\mathbf{x}(u,v)$ for $c=2, a=1$.

$$\mathbf{x}_2 = (-a\cos(u)\sin(v), -a\sin(u)\sin(v), a\cos(v))$$
$$\mathbf{x}_{11} = (-\cos(u)(c+a\cos(v)), -(c+a\cos(v))\sin(u), 0)$$
$$\mathbf{x}_{12} = (a\sin(u)\sin(v), -a\cos(u)\sin(v), 0)$$
$$\mathbf{x}_{22} = (-a\cos(u)\cos(v), -a\cos(v)\sin(u), -a\sin(v))$$

The metric $[g]$ is found to be

$$[g] = \begin{pmatrix} (c+a\cos(v))^2 & 0 \\ 0 & a^2 \end{pmatrix}$$

with the inverse

$$[g]^{-1} = \begin{pmatrix} \frac{1}{(c+a\cos(v))^2} & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix}$$

from which we calculate

$$\mathbf{x}^1 = \Big(-\frac{\sin(u)}{c+a\cos(v)}, \frac{\cos(u)}{c+a\cos(v)}, 0 \Big)$$

and

$$\mathbf{x}^2 = \Big(-\frac{\cos(u)\sin(v)}{a}, -\frac{\sin(u)\sin(v)}{a}, \frac{\cos(v)}{a} \Big).$$

The projection operator

$$P_{\mathbf{x}}(\mathbf{a}) = \mathbf{a}\cdot\mathbf{x}^1\mathbf{x}_1 + \mathbf{a}\cdot\mathbf{x}^2\mathbf{x}_2,$$

and the unit normal to the surface is $\mathbf{n} = \frac{\mathbf{x}_1\times\mathbf{x}_2}{|\mathbf{x}_1\times\mathbf{x}_2|}$, or

$$\mathbf{n} = \Big(\cos(u)\cos(v)), \cos(v)\sin(u), \sin(v) \Big)$$

Since $P_{\mathbf{x}}$ projects onto the tangent space, and $\mathbf{n}$ is orthogonal to the tangent space, it follows that

$$P_{\mathbf{x}}(\mathbf{n}) = 0 \Longrightarrow \dot{P}_{\mathbf{x}}(\mathbf{n}) + P_{\mathbf{x}}(\dot{\mathbf{n}}) = 0 \Longrightarrow \dot{P}_{\mathbf{x}}(\mathbf{n}) = -P_{\mathbf{x}}(\dot{\mathbf{n}}),$$

from which follow the special cases

$$\dot{P}_{\mathbf{x}_1}(\mathbf{n}) = -P_{\mathbf{x}}(\underline{\mathbf{n}}(\mathbf{x}_1)) = -\underline{\mathbf{n}}(\mathbf{x}_1) = (-\cos(v)\sin(u), \cos(u)\cos(v), 0)$$

and

$$\dot{P}_{\mathbf{x}_2}(\mathbf{n}) = -\underline{\mathbf{n}}(\mathbf{x}_2) = (-\cos(u)\sin(v), -\sin(u)\sin(v), \cos(v))$$

We also calculate $P_{\mathbf{x}_1}(\mathbf{x}_2) = 0$,

$$\dot{P}_{\mathbf{x}_1}(\mathbf{x}_1) = \big(-\cos(u)\cos^2(v)(c+a\cos(v)), -\cos^2(v)(c+a\cos(v))\sin(u), -\cos(v)(c+a\cos(v))\sin(v)\big)$$

and

$$\dot{P}_{\mathbf{x}_2}(\mathbf{x}_2) = \big(-a\cos(u)\cos(v), -a\cos(v)\sin(u), -a\sin(v)\big).$$

Calculation of curvature

The matrix $[\underline{\mathbf{n}}]$ of the linear mapping $\underline{\mathbf{n}}(\mathbf{a})$ is

$$[\underline{\mathbf{n}}] = \begin{pmatrix} \mathbf{x}^1 \cdot \underline{\mathbf{n}}(\mathbf{x}_1) & \mathbf{x}^1 \cdot \underline{\mathbf{n}}(\mathbf{x}_2) \\ \mathbf{x}^2 \cdot \underline{\mathbf{n}}(\mathbf{x}_1) & \mathbf{x}^2 \cdot \underline{\mathbf{n}}(\mathbf{x}_2) \end{pmatrix} = \begin{pmatrix} -\frac{\cos(v)}{c+a\cos(v)} & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}$$

The *Gausian curvature*

$$K_G = \det[\underline{\mathbf{n}}] = \frac{\cos(v)}{\cos(v)a^2 + ca}.$$

The *Mean curvature*

$$\kappa_m = \frac{1}{2} trace[\underline{\mathbf{n}}] = \frac{-c - 2a\cos(v)}{2a(c + a\cos(v))}.$$

The *principal curvatures* κ_1, κ_2 are the *eigenvalues* of the matrix $[\underline{\mathbf{n}}]$,

$$\{\kappa_1, \kappa_2\} = \left\{-\frac{1}{a}, -\frac{\cos(v)}{c + a\cos(v)}\right\},$$

and the corresponding *eigenvectors* $(0, 1)$ and $(1, 0)$ of $[\underline{\mathbf{n}}]$ are the *principal directions* at the point $\mathbf{x}(u, v)$.

Finally, using (2), we calculate the *Riemann curvature vectors* $\mathbf{R_{11}} = 0 = \mathbf{R_{22}}$, and

$$\mathbf{R_{12}} = \left\{-\cos(u)\cos^2(v), -\cos^2(v)\sin(u), -\cos(v)\sin(v)\right\}.$$

Using (3), we calculate the Ricci tensors

$$R(\mathbf{x}_1) = \left\{\frac{\cos(v)\sin(u)}{a}, -\frac{\cos(u)\cos(v)}{a}, 0\right\}$$

and

$$R(\mathbf{x}_2) = \left\{\frac{\cos(u)\cos(v)\sin(v)}{c + a\cos(v)}, \frac{\cos(v)\sin(u)\sin(v)}{c + a\cos(v)}, -\frac{\cos^2(v)}{c + a\cos(v)}\right\}$$

Note that $R(\mathbf{x}_1)\cdot\mathbf{x}_1 = -\frac{\cos(v)(c+a\cos(v))}{a}$, and $R(\mathbf{x}_2)\cdot\mathbf{x}_2 = -\frac{a\cos(v)}{c+a\cos(v)}$. The Riemannian *scalar curvature* is the constraction $\partial_{\mathbf{a}}\cdot R(\mathbf{a})$ of Ricci curvature. Calculating, we find that

$$\partial_{\mathbf{a}}\cdot R(\mathbf{a}) = \mathbf{x}^1\cdot R(\mathbf{x}_1) + \mathbf{x}^2\cdot R(\mathbf{x}_2) = -\frac{2\cos(v)}{\cos(v)a^2+ca}.$$

The Gauss-Bonnet Theorem

Let $\mathcal{M}$ be a *compact* 2-surface in $I\!R^3$. The quantity

$$\int\int_{\mathcal{M}} K_G|d\mathbf{x}_{(2)}|$$

is called the *total curvature* of $\mathcal{M}$. Let $\mathcal{D}$ be a *rectangular decomposition* of $\mathcal{M}$, and let v, e and f be the number of *vertices, edges*, and *faces* in $\mathcal{D}$. Then the integer $\chi = v - e + f$ is called the *Euler-Poincare characteristic* of $\mathcal{M}$.

Theorem: Gauss-Bonnet Theorem. *If $\mathcal{M}$ is a compact 2-surface $\mathcal{M}$ in $I\!R^3$ with boundary $\beta(\mathcal{M})$ and with the Euler characteristic χ, then*

$$\int\int_{\mathcal{M}} K_G|d\mathbf{x}_{(2)}| + \int_{\beta(\mathcal{M})} k_g|d\mathbf{x}| = 2\pi\chi$$

where K_G is the Gaussian curvature and k_g is the geodesic curvature on the boundary. If the boundary $\beta(\mathcal{M})$ is piecewise smooth, then the integral $\int_{\beta(\mathcal{M})} k_g|d\mathbf{x}|$ is the sum of the integrals along the smooth portions plus the sum of angles turned at the corners of the boundary $\beta(\mathcal{M})$.

For example, the surface of a cube $\mathcal{M}$ has $v = 8, e = 12$ and $f = 6$, so that its Euler-Poincare characteristic is $\chi = v-e+f = 2$. All of the curvature of a cube is concentrated at its vertices, so it is not clear how to integrate over the surface of the cube. However, if the cube is "blown-up" into a sphere, it will evenly spread out its curvature, without changing its Euler-Poincare characteristic. By the Gauss-Bonnet Theorem it follows that for the 2-sphere $\mathcal{S}$,

$$\int\int_{\mathcal{S}} K_G|d\mathbf{x}_{(2)}| = 2\pi\chi = 4\pi.$$