MAPPINGS OF SURFACES IN EUCLIDEAN SPACE

USING GEOMETRIC ALGEBRA

by

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ABSTRACT

A coordinate-free formulation of mappings between surfaces is achieved by utilizing the Geometric Calculus developed by D. Hestenes. Greatly simplifying concepts introduced in this formulation are: (i) differentiation with respect to an r-vector variable; (ii) generalized invariants of a mapping; and (iii) a generalized Lie bracket.

Basic ideas of linear algebra, advanced calculus, differential forms, and differential geometry are then efficiently reformulated in terms of this approach.

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0. Summary

This summary serves several purposes:

(i) It is a listing of the symbols used in this paper, with a brief description of their meanings, and the page numbers on which they first occur.

(ii) It lists some of the basic identities of geometric algebra that will be used repeatedly. (Proofs of most of these identities can be found in [11].)

(iii) It groups properties proved in this paper according to subject area. This serves to bring together related properties that are otherwise apart in the logical exposition of this paper.

An index to the listings by subject headings is found on the next page.

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Symbols Used in This Paper

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.0	XXXX	means "end of proof"	
0.1	$\mathcal{E}_n$	Euclidean n-space	4
0.2	$\mathfrak{G}$	geometric algebra of $\mathcal{E}_n$	4
0.3	$\mathcal{X}_m, \mathcal{Y}_k$	surfaces in $\mathcal{E}_n$	5
0.4	$\mathfrak{G}_{\underset{\sim}{x}}$	geometric algebra of $\mathcal{X}_m$	5
0.5	$\{F(\underset{\sim}{x})\}$	set of multivector fields on $\mathcal{X}_m$	5
0.6	$\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$	set of tangent multivector fields on $\mathcal{X}_m$	5
0.7	$\nabla_{\underset{\sim}{x}}$	tangential gradient operator on $\mathcal{X}_m$	6
0.8	$\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}$	directional derivative on $\mathcal{X}_m$	6
0.9	$y: \mathcal{X}_m \to \mathcal{Y}_k$, $\underset{\sim}{y} = y(\underset{\sim}{x})$	mapping of $\mathcal{X}_m$ into $\mathcal{Y}_k$	6
0.10	$i_{\underset{\sim}{x}} = i(\underset{\sim}{x})$	pseudoscalar field on $\mathcal{X}_m$	6
0.11	$\bar{x}_r$	r-vector variable on $\mathcal{X}_m$	8
0.12	$\nabla_{\bar{x}_r}$	gradient operator w.r.t. $\bar{x}_r$	8

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.13	$J_{\bar{y}_r}$	characteristic multivector of $y(\underset{\sim}{x})$	8
0.14	$y_{\dagger}$	the differential or "push forward" mapping induced by $y(\underset{\sim}{x})$	12
0.15	$y^{\dagger}$	the adjoint or "pull back" mapping induced by $y(\underset{\sim}{x})$	13
0.16	$z \circ y$	composition of mappings	17
0.17	$(\quad)_{\parallel}$	tangential component to surface	29, 89
	$(\quad)_{\perp}$	normal component to surface	
0.18	$\binom{m}{k}$	binomial coefficient	30
0.19	$[\ ,\ ]$	Lie bracket	40
0.20	$\{\underset{\sim}{e}_i\}$	frame on $\mathcal{X}_m$	50
0.21	$\psi^i(\underset{\sim}{x})$	scalar field on $\mathcal{X}_m$	51
0.22	$g_{\underset{\sim}{x}}$	volume element of $\mathcal{X}_m$	55
0.23	$p_{\underset{\sim}{x}} = p(\underset{\sim}{x})$	unit pseudoscalar field on $\mathcal{X}_m$	62
0.24	$S(\underset{\sim}{a})$	shape operator of $\mathcal{X}_m$	62
0.25	$\Psi(\lambda)$	characteristic polynomial	72
0.26	$J_y(\underset{\sim}{x})$	the Jacobian of $y(\underset{\sim}{x})$	80

	SYMBOL	BRIEF DESCRIPTION	PAGE
0.27	$\int_{\mathcal{A}_r}$	integral over r-surface $\mathcal{A}_r$	84
0.28	$f^r_{\underset{\sim}{x}} = f^r(\underset{\sim}{x})$	differential r-form on $\mathcal{X}_m$	93
0.29	$f^r_{\underset{\sim}{x}} \wedge g^s_{\underset{\sim}{x}}$	exterior product of forms	96
0.30	d	exterior derivative operator	98
0.31	$C_{\underset{\sim}{v}}$	contraction operator w.r.t. $\underset{\sim}{v}$	101
0.32	$D_{\underset{\sim}{v}}$	covariant derivative of forms	103
0.33	$L_{\underset{\sim}{v}}$	Lie derivative of forms	104
0.34	y^*	the pull back mapping	106
0.35	$\nabla_{\underset{\sim}{x}}$	intrinsic gradient operator on $\mathcal{X}_m$	110
0.36	$[\ /\ ]$	intrinsic Lie bracket	111

Algebraic Identities

0.37 $\quad \underset{\sim}{a}\, A_r = \underset{\sim}{a}\cdot A_r + \underset{\sim}{a}\wedge A_r$

0.38 $\quad \underset{\sim}{a}\cdot(A_r \wedge B_s) = (\underset{\sim}{a}\cdot A_r)\wedge B_s + (-1)^r A_r \wedge(\underset{\sim}{a}\cdot B_s)$

0.39 $\quad \underset{\sim}{a}\cdot(\underset{\sim}{b}_1 \wedge \underset{\sim}{b}_2) = \underset{\sim}{a}\cdot\underset{\sim}{b}_1\, \underset{\sim}{b}_2 - \underset{\sim}{a}\cdot\underset{\sim}{b}_2\, \underset{\sim}{b}_1$

0.40 $\underset{\sim}{a}\cdot(\underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s) = \sum_{i=1}^{s}(-1)^{i+1}\,\underset{\sim}{a}\cdot\underset{\sim}{b}_i\;\underset{\sim}{b}_1\wedge\ldots\wedge\check{\underset{\sim}{b}}_i\wedge\ldots\wedge\underset{\sim}{b}_s$,

where $\check{\underset{\sim}{b}}_i$ means omit $\underset{\sim}{b}_i$ from product.

0.41 $(\underset{\sim}{a}_s\wedge\ldots\wedge\underset{\sim}{a}_1)\cdot(\underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s) = \begin{vmatrix} \underset{\sim}{a}_1\cdot\underset{\sim}{b}_1 , & \ldots , & \underset{\sim}{a}_1\cdot\underset{\sim}{b}_s \\ \vdots & & \vdots \\ \underset{\sim}{a}_s\cdot\underset{\sim}{b}_1 , & \ldots , & \underset{\sim}{a}_s\cdot\underset{\sim}{b}_s \end{vmatrix}$

0.42 $(A_r\wedge B_s)\cdot C_t = A_r\cdot(B_s\cdot C_t)$ for $r+s\leq t$

0.43 $(A_r\wedge B_s)\,\underset{\sim}{i}_x = A_r\cdot(B_s\,\underset{\sim}{i}_x)$, where $\underset{\sim}{i}_x$ is a pseudoscalar

0.44 $\underset{\sim}{a} = \underset{\sim}{i}_x^{-1}\,\underset{\sim}{i}_x\cdot\underset{\sim}{a} + \underset{\sim}{i}_x^{-1}\,\underset{\sim}{i}_x\wedge\underset{\sim}{a} \equiv \underset{\sim}{a}_{\parallel} + \underset{\sim}{a}_{\perp}$.

0.45 $A_r\wedge B_s = (-1)^{rs}\,B_s\wedge A_r$

0.46 $A_r^{\dagger} = (-1)^{\frac{r(r-1)}{2}}\,A_r$.

Properties of $y_{\dagger}$ and $y^{\dagger}$

		PAGE
0.47	$y_{\dagger}A_r \equiv A_r\cdot\nabla_{\bar{x}_r}\,\bar{y}_r$, $y^{\dagger}B^r \equiv \nabla_{\bar{x}_r}\,\bar{y}_r\cdot B^r$	12, 13
0.48	$y_{\dagger}(A\wedge B) = y_{\dagger}A\wedge y_{\dagger}B$ $y^{\dagger}(A\wedge B) = y^{\dagger}A\wedge y^{\dagger}B$	13

0.49 $\left\{\begin{array}{l} A = y^{\dagger}B \quad \text{iff} \quad i_{\underset{\sim}{y}} B = y_{\dagger} i_{\underset{\sim}{x}} A \\ B = y_{\dagger}A \quad \text{iff} \quad i_{\underset{\sim}{x}}^{-1} A = y^{\dagger} i_{\underset{\sim}{y}}^{-1} B \end{array}\right\}$ if $J_{\bar{y}_m} \neq 0$ 20

0.50 Let $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ and $\{\underset{\sim}{f}^i(\underset{\sim}{y})\}$ be frames on $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively, then:

$$y^{\dagger}\underset{\sim}{f}^i(\underset{\sim}{y}) = \underset{\sim}{e}^i(\underset{\sim}{x}) \quad \text{iff} \quad y_{\dagger}\underset{\sim}{e}_i = \underset{\sim}{f}_i$$ 53

0.51 $A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_{\dagger}A_r \, , \quad r \leq i \leq m$ 15

$\nabla_{\bar{x}_i} \bar{y}_i \cdot B^s = (y^{\dagger}B^s) \wedge \nabla_{\bar{x}_{i-s}} \bar{y}_{i-s} \, , \quad s \leq i \leq m$

0.52 $A_r \cdot y^{\dagger} B^r = (y_{\dagger}A_r) \cdot B^r$ 16

0.53 $(y_{\dagger}A_r) \cdot B^s = y_{\dagger}(A_r \cdot y^{\dagger} B^s) \, , \quad r \geq s$ 15

$A_r \cdot y^{\dagger} B^s = y^{\dagger}[(y_{\dagger}A_r) \cdot B^s] \, , \quad r \leq s$

0.54 $y_{\dagger}A = y^{\dagger}A$ if $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ 22

0.55 $\left\{\begin{array}{l} A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_i} \bar{y}_i \cdot A_r \\ A_r \wedge \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_i} \bar{y}_i \wedge A_r \end{array}\right\}$, if $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ 24

0.56 $y_{\dagger}A = A = y^{\dagger}A$, if $y(\underset{\sim}{x}) \equiv \underset{\sim}{x}$ 29

0.57 $A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = \left\{\begin{array}{ll} \binom{m-r}{i-r} A_r & \text{if} \quad r \leq i \\ \binom{r}{i} A_i & \text{if} \quad r \geq i \end{array}\right\} = \nabla_{\bar{x}_i} \bar{x}_i \cdot A_r$ 30

0.57 (cont.)

$$A_r \wedge \nabla_{\bar{x}_i} \bar{x}_i = \begin{cases} \binom{m-r}{i} A_r & \text{if } r+i \le m \\ 0 & \text{if } r+i > m \end{cases} = \nabla_{\bar{x}_i} \bar{x}_i \wedge A_r$$

0.58 $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} = m \, , \quad \nabla_{\bar{x}_i} \bar{x}_i = \binom{m}{i}$ 28, 32

Properties of ∇_x

0.59 $\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}} = 0$ 7

0.60 $y^{\dagger} \nabla_{\underset{\sim}{y}} = \nabla_{\underset{\sim}{x}}$ (chain rule) 7,36

0.61 $y_{\dagger}(A_r \cdot \nabla_{\underset{\sim}{x}}) = (y_{\dagger} A_r) \cdot \nabla_{\underset{\sim}{y}}$ 38

0.62 $y_{\dagger} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{y}} \nabla_{\underset{\sim}{y}}$ (dual chain rule) 39

0.63 $\nabla_{\underset{\sim}{x}} = \sum_i \underset{\sim}{e}^i \, \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}$ 52

Derivatives of Fields Under Mappings

0.64 $y^{\dagger}(\nabla_{\underset{\sim}{y}} \wedge B_{\underset{\sim}{y}}) = \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} B_{y(\underset{\sim}{x})}$ 37

0.65 $(y_{\dagger} A_r) \cdot \nabla_{\underset{\sim}{y}} B_{\underset{\sim}{y}} = y_{\dagger}(A_r \cdot \nabla_{\underset{\sim}{x}}) B_{y(\underset{\sim}{x})}$ 38

0.66 $y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger} \underset{\sim}{a} - \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} \left| J_{\bar{y}_m}(\underset{\sim}{x}) \right|$ 57

0.67 $y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot A_r = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger} A_r$ iff $\left| J_{\bar{y}_m}(\underset{\sim}{x}) \right|$ is constant. 58

0.68 $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{e}_i(\underset{\sim}{x}) = \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \sqrt{g_{\underset{\sim}{x}}}$ (for a coordinate frame

$\{\underset{\sim}{e}^i\}$.) 60

<u>Properties of Lie Brackets</u>

(The formulas below hold only for tangent multivector fields.)

		PAGE
0.69	$[\mathbf{a},\mathbf{b}] \equiv \mathbf{a}\cdot\nabla_{\mathbf{x}}\, \mathbf{b}(\mathbf{x}) - \mathbf{b}\cdot\nabla_{\mathbf{x}}\, \mathbf{a}(\mathbf{x})$	40
	$[\mathbf{a},B_s] \equiv \mathbf{a}\cdot\nabla_{\mathbf{x}}\, B_s(\mathbf{x}) - \mathbf{a}(\mathbf{x})\wedge[\nabla_{\mathbf{x}}^{\dagger}\cdot B_s]$	
	$[A_r,B_s] \equiv (A_r\cdot\nabla_{\mathbf{x}})\wedge B_s(\mathbf{x}) - A_r(\mathbf{x})\wedge[\nabla_{\mathbf{x}}^{\dagger}\cdot B_s]$	
0.70	$[A_r+B_s,\ C_t] = [A_r,C_t] + [B_s,C_t]$	41
	$[A_r,\ B_s+C_t] = [A_r,B_s] + [A_r,C_t]$	
0.71	$[A_r\wedge\mathbf{a},\ B_s] = A_r\wedge[\mathbf{a},B_s] + (-1)^r\mathbf{a}\wedge[A_r,B_s]$	41
	$[A_r,\ \mathbf{b}\wedge B_s] = [A_r,\mathbf{b}]\wedge B_s + (-1)^s[A_r,B_s]\wedge\mathbf{b}$	
0.72	$[\mathbf{a},\ \mathbf{b}_1\wedge\ldots\wedge\mathbf{b}_s] = \sum_{i=1}^{s} \mathbf{b}_1\wedge\ldots\wedge\mathbf{b}_{i-1}\wedge[\mathbf{a},\mathbf{b}_i]\wedge\mathbf{b}_{i+1}\wedge\ldots\wedge\mathbf{b}_s$	45
	$[A_r,\mathbf{b}_1\wedge\ldots\wedge\mathbf{b}_s] = \sum_{i=1}^{s} (-1)^{i+1}[A_r\wedge\mathbf{b}_i]\wedge[\mathbf{b}_1\wedge\ldots\wedge\check{\mathbf{b}}_i\wedge\ldots\wedge\mathbf{b}_s]$	
0.73	$[\mathbf{a},B_s] = -[B_s,\mathbf{a}]$	41
	$[A_r,B_s] = -(-1)^{(r-1)(s-1)}[B_s,A_r]$	
	$[A_r,B_s] = -[B_s^{\dagger},A_r^{\dagger}]^{\dagger}$	
0.74	$[\mathbf{a}\cdot\nabla_{\mathbf{x}},\ \mathbf{b}\cdot\nabla_{\mathbf{x}}] \equiv \mathbf{a}\cdot\nabla_{\mathbf{x}}\ \mathbf{b}\cdot\nabla_{\mathbf{x}} - \mathbf{b}\cdot\nabla_{\mathbf{x}}\ \mathbf{a}\cdot\nabla_{\mathbf{x}}$	43

PAGE

The Shape Operator

Linear Mappings

0.94 cont. $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} |dY_r|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} |dX_r \cdot \nabla_{\bar{X}_r}\, \bar{y}_r|\, F[y(\underset{\sim}{x})]$$

$$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} |dY_m|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} |dX_m|\, |J_{\bar{y}_m}|\, F[y(\underset{\sim}{x})]$$

0.95 $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} dY_r \cdot \nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} dX_r \cdot \nabla_{\bar{X}_r}\, \bar{y}_{r-1}\, F[y(\underset{\sim}{x}_r)]$$ 84

$$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} dY_m\, \nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} dX_m\, \nabla_{\bar{X}_m}\, \bar{y}_{m-1}\, F[y(\underset{\sim}{x}_m)]$$ 85

Examples of Mappings

0.96 If a mapping is of the kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\, \underset{\sim}{x}$, then

(i) $y_{\dagger} A_r = \psi^{r-1}[\psi A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi) \wedge \underset{\sim}{x}]$ 86

(ii) $y^{\dagger} B^r = \psi^{r-1}[\psi B^r + (\nabla_{\underset{\sim}{x}} \psi) \wedge (\underset{\sim}{x} \cdot B^r)]$

(iii) $J_{\bar{y}_m} = \psi^{m-1}[\psi + (\nabla_{\underset{\sim}{x}} \psi) \cdot \underset{\sim}{x}]$

(iv) $\nabla_{\underset{\sim}{y}} = \psi^{m-1}\, J_{\bar{y}_m}^{-1}\{\psi \nabla_{\underset{\sim}{x}} + \underset{\sim}{x} \cdot [(\nabla_{\underset{\sim}{x}} \psi) \wedge \nabla_{\underset{\sim}{x}}]$.

0.97 If a mapping is of the kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\, \underset{\sim}{p}$, then

(i) $y_{\dagger} A_r = A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi) \wedge \underset{\sim}{p}$ 88

1. Introduction

In references [9] and [10] D. Hestenes sets down the fundamentals of differential and integral calculus in terms of geometric algebra. Two greatly simplifying features of the resulting "geometric calculus" are that it is coordinate-free and uses only one differential operator.

The purpose of this paper is to apply geometric calculus to the study of smooth mappings between smooth surfaces in Euclidean space. A great simplification of this theory is made possible by the introduction of the following important concepts:

(i) The concept of an r-vector variable, and of differentiating with respect to an r-vector variable.

(ii) The concept of "characteristic multivectors" of a mapping as a generalization of well-known invariants of a mapping, such as the Jacobian, divergence, and curl.

(iii) The concept of the Lie bracket of multivector fields as a generalization of the Lie bracket of vector fields.

This paper is divided into two parts and a series of appendices.

Part I is a study of the differential and adjoint mappings. These linear mappings are induced between the tangent spaces of two surfaces when a point mapping is given between them.

In Part II the "field" properties of the differential and adjoint mappings are studied by considering them as mappings of tangent multivector fields on the two surfaces.

The appendices make up an important part of this paper. They complement the material in Parts I and II, and at the same time relate it to more usual formulations found in the literature.

In Appendix A the methods of Part I are used in the study of linear mappings on Euclidean n-space. By looking at the characteristic equation of a linear mapping, further insight is gained into the nature of the characteristic multivectors of a mapping.

Appendix B discusses the Jacobien, and shows how integral transformation formulas can be easily derived from properties of the differential and adjoint mappings.

Appendix C provides explicit calculations for two kinds of mappings which occur frequently in applications.

Appendix D shows that a one-to-one correspondence exists between differential r-forms and r-vector fields. This correspondence is then exploited to show how all the properties of forms, and operators on forms, follow easily and elegantly from algebraic properties of geometric algebra and the gradient operator.

Appendix E introduces the intrinsic gradient operator on a surface and relates it to the tangential gradient. In addition, the Gauss curvature equation for a surface is formulated in a new way.

Of the references cited in the bibliography, all but [9], [10], [11] and [18] are standard textbooks in advanced calculus and differential geometry. References [9], [10], [11] have already been mentioned in connection with Hestenes. Reference [18] is Whitney's <u>Geometric Integration Theory</u>. In Part I of this book, Whitney uses a geometric approach which is the closest to the one adopted here (with the exception of [9], [10] and [11]). However, in most cases, references to Whitney have been avoided since his approach is not as familiar to most readers as some of the others.

2. Preliminaries

This paper makes extensive use of the geometric algebra and calculus as developed by Hestenes in [9], [10], and [11]. A partial list of the algebraic identities that will be used repeatedly is included in the summary.

Let $\mathcal{E}_n$ denote Euclidean n-space. Points in $\mathcal{E}_n$ are named by vectors. These vectors, under the operations of geometric addition and multiplication, generate the geometric algebra $\mathcal{D}$ of 2^n-dimensions. At each point $\underset{\sim}{p} \in \mathcal{E}_n$ there is associated a geometric algebra $\mathcal{D}_{\underset{\sim}{p}}$, called the tangent algebra to $\mathcal{E}_n$ at $\underset{\sim}{p}$. Since $\mathcal{E}_n$ is flat, $\mathcal{D}_{\underset{\sim}{p}} = \mathcal{D}$, i.e., $\mathcal{D}_{\underset{\sim}{p}}$ is a copy of $\mathcal{D}$ at each point $\underset{\sim}{p} \in \mathcal{E}_n$.

Let $\mathcal{X}_m$ denote an m-surface in $\mathcal{E}_n$. At each point $\underset{\sim}{x} \in \mathcal{X}_m$ there is associated a geometric algebra $\mathcal{D}_{\underset{\sim}{x}}$, called the tangent algebra to $\mathcal{X}_m$ at $\underset{\sim}{x}$. Note that $\mathcal{D}_{\underset{\sim}{x}}$ is of 2^m-dimensions and that $\mathcal{D}_{\underset{\sim}{x}} \subset \mathcal{D}$, i.e., the tangent algebra of the m-surface $\mathcal{X}_m$ at each point $\underset{\sim}{x}$ is a 2^m-dimensional subalgebra of $\mathcal{D}$.

Formal definitions are now given.

Definition 2.1 Euclidian n-space is denoted by $\mathcal{E}_n$. The geometric algebra of $\mathcal{E}_n$ is denoted by $\mathcal{D}$. By $\mathcal{D}^r$ is meant the set of r-vectors $A_r \in \mathcal{D}$, where $0 \leq r \leq n$.

Definition 2.2 An m-surface in $\mathcal{E}_n$ is denoted by $\mathcal{X}_m$. The tangent algebra of $\mathcal{X}_m$ at a point $\underset{\sim}{x}$ is denoted by $\mathcal{D}_{\underset{\sim}{x}}$. By $\mathcal{D}^r_{\underset{\sim}{x}}$ is meant the set of tangent r-vectors $A_r \in \mathcal{D}_{\underset{\sim}{x}}$, where $0 \leq r \leq m$.

Note that 1-vectors will always be distinguished from other directed quantities by small underlined letters, such as $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{x}$, $\underset{\sim}{v}$, etc.

The vector $\underset{\sim}{x}$ is always used for the name of a point on the surface $\mathcal{X}_m$. Similarly, $\mathcal{D}_{\underset{\sim}{x}}$ always denotes the tangent algebra of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$. The general rule is: Anything subscripted with an $\underset{\sim}{x}$ refers to the surface $\mathcal{X}_m$.

Definition 2.3 A surface $\mathcal{X}_m$ is said to be flat, or a tangent m-plane if for any two points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, $\mathcal{D}_{\underset{\sim}{x}_1} = \mathcal{D}_{\underset{\sim}{x}_2}$.

Definition 2.4 A function $F(\underset{\sim}{x})$ is said to be a multivector field on $\mathcal{X}_m$ if $F(\underset{\sim}{x}) \in \mathcal{D}$ for each $\underset{\sim}{x} \in \mathcal{X}_m$. If $F(\underset{\sim}{x}) \in \mathcal{D}_{\underset{\sim}{x}}$ for each $\underset{\sim}{x} \in \mathcal{X}_m$, then $F(\underset{\sim}{x})$ is said to be a tangent multivector field on $\mathcal{X}_m$.

Often $F_{\underset{\sim}{x}}$, where $F_{\underset{\sim}{x}} \equiv F(\underset{\sim}{x})$, is used to denote the value of the function $F(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$.

Definition 2.5 The set of all multivector fields on $\mathcal{X}_m$ is denoted by $\{F(\underset{\sim}{x})\}$. The set of all tangent multivector fields on $\mathcal{X}_m$ is denoted by $\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

This paper is only concerned with surfaces, multivector fields, and mappings which are sufficiently smooth to allow all indicated operations of differentiation to be well defined and continuous.

Definition 2.6 The symbol $\nabla_{\underset{\sim}{x}}$ is called the gradient or tangential derivative operator on the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$.

The tangential derivative $\nabla_{\underset{\sim}{x}}$ differentiates multivector fields on $\mathcal{X}_m$, and behaves algebraically like a vector of $\mathcal{D}^1_{\underset{\sim}{x}}$. For a further discussion of $\nabla_{\underset{\sim}{x}}$, see [9] and [10].

"Dotting" the gradient $\nabla_{\underset{\sim}{x}}$ with a tangent vector $\underset{\sim}{v} \in \mathcal{D}^1_{\underset{\sim}{x}}$ gives $\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}$, the directional derivative operator. This can be shown to be equivalent to the following more usual definition:

Definition 2.7 $\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}F(\underset{\sim}{x}) \equiv |\underset{\sim}{v}| \lim_{\Delta\underset{\sim}{x}\to 0} \dfrac{F(\underset{\sim}{x}+\Delta\underset{\sim}{x}) - F(\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$, where

$F(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}$, and $\Delta\underset{\sim}{x} \to 0$ in such a way that:

(i) $\underset{\sim}{x} + \Delta\underset{\sim}{x}$ is always a point on $\mathcal{X}_m$

(ii) $\lim_{\Delta\underset{\sim}{x}\to 0} \dfrac{\Delta\underset{\sim}{x}}{|\Delta\underset{\sim}{x}|} = \hat{\underset{\sim}{v}}$, where $\hat{\underset{\sim}{v}} \equiv \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$.

Definition 2.8 $y\colon \mathcal{X}_m \to \mathcal{Y}_k$ is said to be a mapping from the m-surface $\mathcal{X}_m$ to the k-surface $\mathcal{Y}_k$, if $\underset{\sim}{y} = y(\underset{\sim}{x}) \in \mathcal{Y}_k$ for each $\underset{\sim}{x} \in \mathcal{X}_m$.

The smooth surfaces and mappings considered in this paper have the following properties:

Property 2.9 There exists a smooth pseudoscaler field $i_{\underset{\sim}{x}} = i(\underset{\sim}{x})$ on $\mathcal{X}_m$. I.e., there exists a multivector field $i(\underset{\sim}{x})$ on $\mathcal{X}_m$ such that $i_{\underset{\sim}{x}} = i(\underset{\sim}{x}) \in \mathcal{D}^m_{\underset{\sim}{x}}$ for each $\underset{\sim}{x} \in \mathcal{X}_m$. (This property is equivalent to saying $\mathcal{X}_m$ is orientable.)

Property 2.10 If $A_{r+1}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^{r+1}_{\underset{\sim}{x}}$ then there are multivector fields $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^{1}_{\underset{\sim}{x}}$, and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^{r}_{\underset{\sim}{x}}$ such that $A_{r+1} = \underset{\sim}{a}\wedge A_r$.

Property 2.11 $\nabla_{\underset{\sim}{x}} = i^{-1}_{\underset{\sim}{x}} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}}$. Property 2.11 guarantees that $\nabla_{\underset{\sim}{x}}$ behaves algebraically like a vector in $\mathcal{D}^{1}_{\underset{\sim}{x}}$.

Property 2.12 If $y: \mathcal{X}_m \to \mathcal{Y}_k$ is a mapping and $F(\underset{\sim}{y}) \in \{F(\underset{\sim}{y})\}$, then for each $\underset{\sim}{v} \in \mathcal{D}^{1}_{\underset{\sim}{x}}$,

$$\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = [\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})]\cdot\nabla_{\underset{\sim}{y}} F(\underset{\sim}{y}) .$$

(This is a statement of the chain rule for partial differentiation.)

Property 2.13 For any smooth multivector field $F(\underset{\sim}{x})$ on $\mathcal{X}_m$, $\nabla_{\underset{\sim}{x}}\wedge\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x}) = 0$.

(This is equivalent to the property that partial derivatives commute in a flat space. For a further discussion of the significance of this property see Appendix E.)

A "chain rule" for the gradient operator is derived from properties 2.11 and 2.12 in the following theorem.

Theorem 2.14 $\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})\cdot\nabla_{\underset{\sim}{y}} F(\underset{\sim}{y})$.

Proof $\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})] = i^{-1}_{\underset{\sim}{x}} i_{\underset{\sim}{x}}\cdot\nabla_{\underset{\sim}{x}} F[y(\underset{\sim}{x})]$ property 2.11

property 2.12 $= i^{-1}_{\underset{\sim}{x}} i_{\underset{\sim}{x}}\cdot\nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})\cdot\nabla_{\underset{\sim}{y}} F[y(\underset{\sim}{x})]$

property 2.11 $= \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x})\cdot\nabla_{\underset{\sim}{y}} F(\underset{\sim}{y})$.

XXXX

Let $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r$ be points on $\mathcal{X}_m$.

Definition 2.15 Call $\bar{x}_r \equiv \frac{1}{r!}(\underset{\sim}{x}_1 - \underset{\sim}{x}) \wedge \ldots \wedge (\underset{\sim}{x}_r - \underset{\sim}{x})$ the r-vector variable of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x} \in \mathcal{X}_m$.

The r-vector variable $\bar{x}_r$ is an oriented measure of the r-simplex with vertices at the points $\underset{\sim}{x}, \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r$. Note that $|\bar{x}_r|$ is volume of the simplex. See [18, p. 80].

Definition 2.16 Call $\bar{y}_r(\underset{\sim}{x}) \equiv \frac{1}{r!}[y(\underset{\sim}{x}_1) - y(\underset{\sim}{x})] \wedge \ldots \wedge [y(\underset{\sim}{x}_r) - y(\underset{\sim}{x})]$ the r-vector variable of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ at the point $\underset{\sim}{y} = y(\underset{\sim}{x}) \in \mathcal{Y}_k$.

Definition 2.17 Call $\nabla_{\bar{x}_r} \equiv \nabla_{\underset{\sim}{x}_r} \wedge \ldots \wedge \nabla_{\underset{\sim}{x}_1}$ the gradient operator with respect to the r-vector variable $\bar{x}_r$ at the point $\underset{\sim}{x} \in \mathcal{X}_m$. It is understood that $\nabla_{\underset{\sim}{x}_i}$ differentiates only with respect to $\underset{\sim}{x}_i$ and is to be evaluated at $\underset{\sim}{x}_i = \underset{\sim}{x}$.

Certain multivectors $J_{\bar{y}_r} = J_{\bar{y}_r}(\underset{\sim}{x})$, called the characteristic multivectors of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$, are now defined.

Definition 2.18 $J_{\bar{y}_r}(\underset{\sim}{x}) \equiv \nabla_{\bar{x}_r} \bar{y}_r$, for $r = 1, \ldots, m$.

The usual Jacobian $J_y(\underset{\sim}{x})$ of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is related to $J_{\bar{y}_m}(\underset{\sim}{x})$ by the following equation:

$$(2.19) \qquad J_y(\underset{\sim}{x}) = \pm |J_{\bar{y}_m}(\underset{\sim}{x})| ,$$

where the $\pm$ sign is chosen according to orientation. This justifies the following definition:

Definition 2.20 A mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be non-singular if $J_{\bar{y}_m}(\underset{\sim}{x}) \neq 0$ for each $\underset{\sim}{x} \in \mathcal{X}_m$.

The relationship of the Jacobian of a mapping to $J_{\bar{y}_m}$ is further discussed in Appendix B.

This section ends with the lemma given below. It is useful in the proofs of theorems in later sections.

Lemma 2.21

(i) $\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = 0 = \nabla_{\bar{x}_2} \underset{\sim}{y}_1 \cdot \underset{\sim}{y}_2$.

(ii) $\frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 \ldots \underset{\sim}{y}_r = \nabla_{\bar{x}_r} \bar{y}_r = \nabla_{\underset{\sim}{x}_r} \nabla_{\underset{\sim}{x}_{r-1}} \ldots \nabla_{\underset{\sim}{x}_1} \bar{y}_r$.

Proof

(i) $$\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = \frac{1}{2} \nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} [y(\underset{\sim}{x}_1) - y(\underset{\sim}{x})] \wedge [y(\underset{\sim}{x}_2) - y(\underset{\sim}{x})]$$

$$= \frac{-1}{2} \nabla_{\underset{\sim}{x}_1} \cdot \nabla_{\underset{\sim}{x}_2} [y(\underset{\sim}{x}_2) - y(\underset{\sim}{x})] \wedge [y(\underset{\sim}{x}_1) - y(\underset{\sim}{x})]$$

$$= -\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 .$$

Hence $\nabla_{\underset{\sim}{x}_2} \cdot \nabla_{\underset{\sim}{x}_1} \bar{y}_2 = 0$.

Similarly $\nabla_{\bar{x}_2} \underset{\sim}{y}_1 \cdot \underset{\sim}{y}_2 = 0$.

(ii) The proof of (ii) follows by repeated use of (i).

$$\nabla_{\bar{x}_r} \bar{y}_r = \frac{1}{r!} \nabla_{\bar{x}_r} (\underset{\sim}{y}_1 - \underset{\sim}{y}) \wedge \ldots \wedge (\underset{\sim}{y}_r - \underset{\sim}{y})$$

$$= \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \wedge \ldots \wedge \underset{\sim}{y}_r$$

identity 0.37 $$= \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \underset{\sim}{y}_2 \wedge \ldots \wedge \underset{\sim}{y}_r - \frac{1}{r!} \nabla_{\bar{x}_r} \underset{\sim}{y}_1 \cdot (\underset{\sim}{y}_2 \wedge \ldots \wedge \underset{\sim}{y}_r)$$

using (i) and 0.40 $\quad = \frac{1}{r!}\nabla_{\bar{x}_r}\, \underset{\sim}{y}_1\underset{\sim}{y}_2(\underset{\sim}{y}_3\wedge\ldots\wedge\underset{\sim}{y}_r) - \frac{1}{r!}\nabla_{\bar{x}_r}\underset{\sim}{y}_1\underset{\sim}{y}_2\cdot(\underset{\sim}{y}_3\wedge\ldots\wedge\underset{\sim}{y}_r)$

$\vdots$

$= \frac{1}{r!}\nabla_{\bar{x}_r}\, \underset{\sim}{y}_1\underset{\sim}{y}_2\ldots\underset{\sim}{y}_r$.

Similarly $\nabla_{\bar{x}_r}\bar{y}_r = \nabla_{\underset{\sim}{x}_r}\nabla_{\underset{\sim}{x}_{r-1}}\cdots\nabla_{\underset{\sim}{x}_1}\bar{y}_r$.

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PART I

THE DIFFERENTIAL AND ADJOINT MAPPINGS

3. Definitions and Basic Properties

For each point $\underset{\sim}{x} \in \mathcal{X}_m$ the mapping $y: \mathcal{X}_m \to \mathcal{Y}_k$ induces two linear mappings: (i) The differential mapping y_+ from the geometric algebra $\mathcal{D}$ of $\mathcal{E}_n$ at the point $\underset{\sim}{x}$, to the tangent algebra $\mathcal{D}_{\underset{\sim}{y}}$ of $\mathcal{Y}_k$ at the point $\underset{\sim}{y} \doteq y(\underset{\sim}{x})$. (ii) The adjoint mapping $y^{\dagger}$ from the geometric algebra $\mathcal{D}$ of $\mathcal{E}_n$ at the point $\underset{\sim}{y}$, to the tangent algebra $\mathcal{D}_{\underset{\sim}{x}}$ of $\mathcal{X}_m$ at the point $\underset{\sim}{x}$. These mappings are now defined.

Definition 3.1 $y_+: \mathcal{D} \to \mathcal{D}_y$ is given by:

(i) $y_+ A_0 \equiv A_0$, for $A_0 \in \mathcal{D}^0$.

(ii) $y_+ A_r = A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r$, for $A_r \in \mathcal{D}^r$ and $1 \leq r \leq n$

(iii) $y_+ A = \sum_{i=0}^{n} y_+ A_i$, where $A = \sum_{i=0}^{n} A_i \in \mathcal{D}$.

Note that the domain of y_+ is not restricted to $\mathcal{D}_{\underset{\sim}{x}}$ as might be expected, but is all of $\mathcal{D}$ the geometric algebra of $\mathcal{E}_n$.

The mapping y_+ is sometimes called the "push forward" mapping because it maps tangent vectors in the same "direction" as $y(\underset{\sim}{x})$ maps points.

Definition 3.2 $y^\dagger : \mathcal{D} \to \mathcal{D}_{\underset{\sim}{x}}$ is given by:

(i) $y^\dagger A^0 = A^0$, for $A^0 \in \mathcal{D}^0$.

(ii) $y^\dagger A^r = \nabla_{\bar{x}_r} \bar{y}_r \cdot A^r$, for $A^r \in \mathcal{D}^r$ and $1 \leq r \leq n$.

(iii) $y^\dagger A = \sum_{i=0}^{n} y^\dagger A^i$, where $A = \sum_{i=0}^{n} A_i \in \mathcal{D}$.

Just as for $y_\dagger$, the domain of $y^\dagger$ is not restricted to $\mathcal{D}_{\underset{\sim}{y}}$ the tangent algebra of the surface $\mathcal{V}_m$ at the point $\underset{\sim}{y}$, but is all of $\mathcal{D}$ the geometric algebra of $\mathcal{E}_n$.

The mapping $y^\dagger$ is sometimes called the "pull back" mapping because it maps tangent vectors in the opposite "direction" to the "direction" that $y(\underset{\sim}{x})$ maps points.

Finally note that upper and lower indices are used to distinguish between what is being "pushed forward" (lower indices), and what is being "pulled back" (upper indices).

Basic properties of the mappings $y_\dagger$ and $y^\dagger$ are now studied.

Theorem 3.3 (i) $y_\dagger(A \wedge B) = y_\dagger A \wedge y_\dagger B$, for $A, B \in \mathcal{D}$.

(ii) $y^\dagger(A \wedge B) = y^\dagger A \wedge y^\dagger B$, for $A, B \in \mathcal{D}$.

Proof Since $y_\dagger$ and $y^\dagger$ are linear, it is sufficient to show the theorem for r and s-vectors A_r and $B_s \in \mathcal{D}$.

(i) $y_\dagger(A_r \wedge \underset{\sim}{b}) = (A_r \wedge \underset{\sim}{b}) \cdot \nabla_{\bar{x}_{r+1}} \bar{y}_{r+1}$

identity 0.42 $\qquad = A_r \cdot (\underset{\sim}{b} \cdot \nabla_{\bar{x}_{r+1}}) \bar{y}_{r+1}$

identity 0.40 $$= (r+1)\,(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}_{r+1}}\nabla_{\bar{x}_r})\bar{y}_{r+1}$$

$$= (A_r\cdot\nabla_{\bar{x}_r}\,\bar{y}_r)\wedge(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}_{r+1}}\underset{\sim}{y}_{r+1})$$

$$= y_{\dagger}A_r \wedge y_{\dagger}\underset{\sim}{b}\ .$$

The proof of (i) is completed by induction on s .

(ii) $$y^{\dagger}(\underset{\sim}{a}\wedge B^s) = \nabla_{\bar{x}_{s+1}}\bar{y}_{s+1}\cdot(\underset{\sim}{a}\wedge B^s)$$

identity 0.42 $$= \nabla_{\bar{x}_{s+1}}(\bar{y}_{s+1}\cdot\underset{\sim}{a})\cdot B^s$$

identity 0.40 $$= \nabla_{\bar{x}_{s+1}}(\bar{y}_s\,\underset{\sim}{y}_{s+1}\cdot\underset{\sim}{a})\cdot B^s$$

$$= (\nabla_{\underset{\sim}{x}_{s+1}}\underset{\sim}{y}_{s+1}\cdot\underset{\sim}{a})\wedge(\nabla_{\bar{x}_s}\,\bar{y}_s\cdot B^s)$$

$$= y^{\dagger}\underset{\sim}{a}\wedge y^{\dagger}B^s\ .$$

The proof of (ii) is completed by induction on r .

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The statement of theorem 3.3 with "dots" replacing "wedges" does not hold, i.e.: $y_{\dagger}(A\cdot B) \neq y_{\dagger}A\cdot y_{\dagger}B$ and $y^{\dagger}(A\cdot B) \neq y^{\dagger}A\cdot y^{\dagger}B$, for an arbitrary mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. If $\underset{\sim}{y} = y(\underset{\sim}{x})$ is a linear mapping, the condition that $y_{\dagger}(\underset{\sim}{a}\cdot\underset{\sim}{b}) = y_{\dagger}\underset{\sim}{a}\cdot y_{\dagger}\underset{\sim}{b}$ for all $\underset{\sim}{a}$, $\underset{\sim}{b} \in \mathfrak{X}^1$ is equivalent to saying $\underset{\sim}{y} = y(\underset{\sim}{x})$ is an orthogonal mapping. Orthogonal linear mappings are further discussed in Appendix A.

Theorem 3.4 (i) $A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_\dagger A_r$, where $A_r \in \mathcal{D}$, and $r \leq i \leq m$.

(ii) $\nabla_{\bar{x}_i} \bar{y}_i \cdot B^s = (y^\dagger B^s) \wedge \nabla_{\bar{x}_{i-s}} \bar{y}_{i-s}$, where $B^s \in \mathcal{D}$, and $s \leq i \leq m$.

Proof

(i) $A_r \cdot \nabla_{\bar{x}_i} \bar{y}_i = \frac{(i-r)!r!}{i!} A_r \cdot (\nabla_{\bar{x}_r} \wedge \nabla_{\bar{x}_{i-r}})\bar{y}_{i-r} \wedge \bar{y}_r$

[11, p. 13, 3.12] $= \frac{(i-r)!r!}{i!} \binom{i}{r} A_r \cdot \nabla_{\bar{x}_r} \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge \bar{y}_r$

$= \nabla_{\bar{x}_{i-r}} \bar{y}_{i-r} \wedge y_\dagger A_r$.

(ii) The proof of (ii) is similar to (i) and is omitted.

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The following theorem relates the differential and adjoint mappings through the inner product.

Theorem 3.5 (i) $(y_\dagger A_r) \cdot B^s = y_\dagger (A_r \cdot y^\dagger B^s)$, where $r \geq s$, and A_r , $B^s \in \mathcal{D}$.

(ii) $A_r \cdot y^\dagger B^s = y^\dagger [(y_\dagger A_r) \cdot B^s]$, where $r \leq s$, and A_r , $B^s \in \mathcal{D}$.

Proof

(i) $$(y_{\dagger}A_r)\cdot B^s = A_r\cdot\nabla_{\bar{x}_r}\,\dot{\bar{y}}_r\cdot B^s$$

theorem 3.4(ii) $$= A_r\cdot[(y^{\dagger}B^s)\wedge\nabla_{\bar{x}_{r-s}}]\,\bar{y}_{r-s}$$

identity 0.42 $$= [A_r\cdot(y^{\dagger}B^s)]\cdot\nabla_{\bar{x}_{r-s}}\,\bar{y}_{r-s}$$

$$= y_{\dagger}[A_r\cdot(y^{\dagger}B^s)]\ .$$

(ii) $$A_r\cdot y^{\dagger}B^s = A_r\cdot\nabla_{\bar{x}_s}\,\bar{y}_s\cdot B^s$$

theorem 3.4(i) $$= \nabla_{\bar{x}_{s-r}}\,(\bar{y}_{s-r}\wedge y_{\dagger}A_r)\cdot B^s$$

summary 0.42 $$= \nabla_{\bar{x}_{s-r}}\,\bar{y}_{s-r}\cdot[(y_{\dagger}A_r)\cdot B^s$$

$$= y^{\dagger}[(y_{\dagger}A_r)\cdot B^s]\ .$$

XXXX

Corollary 3.6 $A_r\cdot y^{\dagger}B^r = (y_{\dagger}A_r)\cdot B^r$.

Proof Set $r = s$ in part (i) or (ii) of theorem 3.5

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In Appendix D theorems 3.3 and 3.5 are used to prove related properties of differential forms.

4. Composed Mappings

Let $\mathcal{X}_m$, $\mathcal{Y}_k$, and $\mathcal{Z}_l$ be surfaces in $\mathcal{E}_n$, and suppose $y: \mathcal{X}_m \to \mathcal{Y}_k$, and $z: \mathcal{Y}_k \to \mathcal{Z}_l$. Then the composed mapping $z \circ y: \mathcal{X}_m \to \mathcal{Z}_l$.

Lemma 4.1 $\nabla_{\bar{x}_r} \overline{z \circ y}_r = \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r$

Proof: $\nabla_{\bar{x}_r} \overline{z \circ y}_r = \nabla_{\bar{x}_r} \frac{1}{r!} z[y(\underset{\sim}{x}_1)] \wedge \ldots \wedge z[y(\underset{\sim}{x}_r)]$

theorem 2.14 $= \nabla_{\bar{x}_r} \frac{1}{r!} [y(\underset{\sim}{x}_1) \cdot \nabla_{\underset{\sim}{y}_1} z(\underset{\sim}{y}_1)] \wedge \ldots \wedge [y(\underset{\sim}{x}_r) \cdot \nabla_{\underset{\sim}{y}_r} z(\underset{\sim}{y}_r)]$

theorem 3.3(i) $= \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r$.

Theorem 4.2 (i) $(z \circ y)_\dagger A = z_\dagger(y_\dagger A)$, where $A \in \mathcal{D}$.

(ii) $(z \circ y)^\dagger A = y^\dagger(z^\dagger A)$, where $A \in \mathcal{D}$.

Proof Since $y_\dagger$ and $y^\dagger$ are linear, it is sufficient to show the theorem for r-vectors $A_r \in \mathcal{D}$.

(i) $(z \circ y)_\dagger A_r = A_r \cdot \nabla_{\bar{x}_r} \overline{z \circ y}_r$

lemma 4.1 $= A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r \cdot \nabla_{\bar{y}_r} \bar{z}_r$

$$= z_\dagger(y_\dagger A_r) \quad .$$

(ii) $(z\circ y)^\dagger A^r = \nabla_{\bar{x}_r}\, \overline{z\circ y}_r \cdot A^r$

Lemma 4.1 $\qquad = \nabla_{\bar{x}_r}\, \bar{y}_r \cdot \nabla_{\bar{y}_r}\, \bar{z}_r \cdot A^r$

$$= y^\dagger(z^\dagger A^r) \quad .$$

The following theorem gives the characteristic multivectors of mappings composed by addition or multiplication.

<u>Theorem 4.3</u> (i) $J_{\overline{z\circ y}_r} = (y^\dagger\, \nabla_{\bar{y}_r})\, \bar{z}_r$.

(ii) If $y(\underset{\sim}{x}) = g(\underset{\sim}{x}) + h(\underset{\sim}{x})$, where $g\colon \mathcal{X}_m \to \mathcal{E}_n$ and $h\colon \mathcal{X}_m \to \mathcal{E}_n$, then $J_{\bar{y}_r} = \sum_{i=0}^{r} \nabla_{\bar{x}_i} \wedge \nabla_{\bar{x}_{r-i}}\, \bar{g}_{r-i} \wedge \bar{h}_i$.

<u>Proof:</u>

(i) is a restatement of lemma 4.1 .

(ii) $J_{\bar{y}_r} = \nabla_{\bar{x}_r}\, \bar{y}_r(\underset{\sim}{x})$

$$= \frac{1}{r!} \nabla_{\bar{x}_r} [g(\underset{\sim}{x}_1) + h(\underset{\sim}{x}_1)] \wedge \ldots \wedge [g(\underset{\sim}{x}_r) + h(\underset{\sim}{x}_r)]$$

$$= \nabla_{\bar{x}_r}\, \bar{g}_r + \nabla_{\underset{\sim}{x}_r} \wedge \nabla_{\bar{x}_{r-1}}\, \bar{g}_{r-1} \wedge h(\underset{\sim}{x}_r) + \ldots + \nabla_{\bar{x}_r}\, \bar{h}_r$$

$$= \sum_{i=1}^{r} \nabla_{\bar{x}_i} \wedge \nabla_{\bar{x}_{r-i}}\, \bar{g}_{r-i} \wedge \bar{h}_i \quad .$$

XXXX

In Appendix A, theorem 3.9(ii) is used in calculating the characteristic polynomial of a linear mapping.

5. Non-singular mappings

When $y: \mathcal{X}_m \to \mathcal{Y}_m$ is an invertible mapping (non-singular one-to-one) between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$, the differential and adjoint mappings are also invertible, provided their domains are restricted to $\mathcal{D}_{\underset{\sim}{x}}$ and $\mathcal{D}_{\underset{\sim}{y}}$ respectively. This is now shown.

Let $i_{\underset{\sim}{x}} \in \mathcal{D}^m_{\underset{\sim}{x}}$ be a non-zero pseudoscaler on $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, and let $i_{\underset{\sim}{y}} = y_{\dagger} i_{\underset{\sim}{x}}$ be the corresponding pseudoscaler on $\mathcal{Y}_m$ at the point $\underset{\sim}{y} = y(\underset{\sim}{x})$. (Note that $J_{\bar{y}_m} \neq 0$ implies $i_{\underset{\sim}{y}} \neq 0$, since $i_{\underset{\sim}{y}} = y_{\dagger} i_{\underset{\sim}{x}} = i_{\underset{\sim}{x}} \cdot \nabla_{\bar{x}_m} \bar{y}_m = i_{\underset{\sim}{x}} J_{\bar{y}_m}$.)

<u>Theorem 5.1</u> If $A \in \mathcal{D}_{\underset{\sim}{x}}$ and $B \in \mathcal{D}_{\underset{\sim}{y}}$, and $J_{\bar{y}_m}(\underset{\sim}{x}) \neq 0$,

then: (i) $A = y^{\dagger} B$ iff $i_{\underset{\sim}{y}} B = y_{\dagger}(i_{\underset{\sim}{x}} A)$.

(ii) $B = y_{\dagger} A$ iff $i^{-1}_{\underset{\sim}{x}} A = y^{\dagger}(i^{-1}_{\underset{\sim}{y}} B)$.

<u>Proof</u> Since $y_{\dagger}$ and $y^{\dagger}$ are linear, it is sufficient to show the theorem for all r-vectors $A^r \in \mathcal{D}_{\underset{\sim}{x}}$ and $B^r \in \mathcal{D}_{\underset{\sim}{y}}$.

(i) $A^r = y^{\dagger} B^r$ is equivalent to $i_{\underset{\sim}{x}} A^r = i_{\underset{\sim}{x}} y^{\dagger} B^r$. Since $y_{\dagger}$ is a non-singular linear operator, this is also equivalent to

3.5 (i)

$$y_\dagger i_{\underset{\sim}{x}} A^r = y_\dagger (i_{\underset{\sim}{x}} y^\dagger B^r) \;\; = (y_\dagger i_x)\cdot B^r$$

$$= (i_{\underset{\sim}{x}} \nabla_{\bar{x}_r} \bar{y}_r \cdot B^r) \cdot \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$$

identity 0.42 $\quad = i_{\underset{\sim}{x}} (\nabla_{\bar{x}_r} \bar{y}_r \cdot B^r) \wedge \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$

$$= i_{\underset{\sim}{x}} (y^\dagger B^r) \wedge \nabla_{\bar{x}_{m-r}} \bar{y}_{m-r}$$

theorem 3.4(ii) $\quad = i_{\underset{\sim}{x}} \nabla_{\bar{x}_m} \bar{y}_m \cdot B^r$

$$= i_{\underset{\sim}{x}} \cdot \nabla_{\bar{x}_m} \bar{y}_m \, B^r = i_{\underset{\sim}{y}} B^r \, .$$

(ii) Let $A^r = i_{\underset{\sim}{x}}^{-1} A_r$ and $B^r = i_{\underset{\sim}{y}}^{-1} B_r$ in part (i) which has just been proved. Then $i_{\underset{\sim}{x}}^{-1} A_r = y^\dagger i_{\underset{\sim}{y}}^{-1} B_r$ iff $i_{\underset{\sim}{y}} i_{\underset{\sim}{y}}^{-1} B_r = y_\dagger i_{\underset{\sim}{x}} i_{\underset{\sim}{x}}^{-1} A_r$, or $i_{\underset{\sim}{x}}^{-1} A_r = y^\dagger i_{\underset{\sim}{y}}^{-1} B_r$ iff $B_r = y_\dagger A_r$, and the proof is complete.

XXXX

6. Curl Free Mappings

Let $y\colon \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the surface $\mathcal{X}_m$ into the surface $\mathcal{Y}_k$.

Definition 6.1 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be curl free at the point $\underset{\sim}{x} \in \mathcal{X}_m$, if $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$. The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be curl free, if it is curl free at each point $\underset{\sim}{x} \in \mathcal{X}_m$.

The following theorem shows that at points $\underset{\sim}{x} \in \mathcal{X}_m$ where the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is curl free, the differential and adjoint mappings are identical.

Theorem 6.2 If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, then $y_{\dagger}A = y^{\dagger}A$ for each $A \in \mathcal{D}$.

Proof It is sufficient to show the theorem is true for r-vectors $A_r \in \mathcal{D}$.

The proof is by induction on $r \le n$. For $r = 1$,

$\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ implies $\underset{\sim}{a}\cdot[\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x})] \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) - \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})\cdot\underset{\sim}{a} = 0$, or $y_\dagger \mathbf{a} - y^\dagger \mathbf{a} = 0$. For $r = i$ suppose $y_\dagger A_i = y^\dagger A_i$, and for $r = i + 1$ write $A_{i+1} = \mathbf{a} \wedge A_i$. Then

$$y_\dagger A_{i+1} = y_\dagger \mathbf{a} \wedge A_i$$

theorem 3.3(i) $\qquad = y_\dagger \mathbf{a} \wedge y_\dagger A_i$

$\qquad\qquad\qquad\qquad = y^\dagger \mathbf{a} \wedge y^\dagger A_i$

theorem 3.3(ii) $\quad = y^\dagger \mathbf{a} \wedge A_i = y^\dagger A_{i+1}$.

Hence the theorem is proved.

XXXX

<u>Corollary 6.3</u> $\quad y_\dagger \equiv y^\dagger : \mathcal{D} \to \mathcal{D}_{\mathbf{x}} \cap \mathcal{D}_{\mathbf{y}}$.

<u>Proof</u> The proof follows immediately from theorem 6.2 and the facts that $y_\dagger : \mathcal{D} \to \mathcal{D}_{\mathbf{y}}$, and $y^\dagger : \mathcal{D} \to \mathcal{D}_{\mathbf{x}}$.

XXXX

<u>Corollary 6.4</u> $\quad \mathbf{a} \wedge \nabla_{\mathbf{x}}\, y(\mathbf{x}) = \nabla_{\mathbf{x}}\, y(\mathbf{x}) \wedge \mathbf{a}$, for $\mathbf{a} \in \mathcal{D}'$.

<u>Proof</u> $\quad \mathbf{a} \wedge \nabla_{\mathbf{x}} y(\mathbf{x}) = \mathbf{a}\, \nabla_{\mathbf{x}} y(\mathbf{x}) - \mathbf{a} \cdot \nabla_{\mathbf{x}} y(\mathbf{x})$ $\qquad$ identity 0.37

theorem 6.2 $\qquad\qquad = \nabla_{\mathbf{x}} y(\mathbf{x})\mathbf{a} - \nabla_{\mathbf{x}} y(\mathbf{x}) \cdot \mathbf{a}$

identity 0.37 $\qquad\qquad = \nabla_{\mathbf{x}} y(\mathbf{x}) \wedge \mathbf{a}$

XXXX

Theorem 6.2 and corollary 6.4 have the following generalization:

<u>Theorem 6.5</u> If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ and $A_r \in \mathfrak{D}$, then

(i) $A_r \cdot \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \cdot A_r$.

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} y(\underset{\sim}{x}) \wedge A_r$.

<u>Proof</u> (i) $A_r \cdot \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = A_r\, \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) - A_r \wedge \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

identity 0.37 $\qquad = A_r\, \nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x}) - (\dot{A}_r \wedge \nabla_{\underset{\sim}{x}}) \cdot y(\underset{\sim}{x})$

identity 0.38 $\qquad = [A_r \cdot y(\underset{\sim}{x})] \wedge \nabla^{\dagger}_{\underset{\sim}{x}} = \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) \cdot A_r$.

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = A_r\, \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y} - A_r \cdot \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y}$

using part (i) $\qquad = \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y}\, A_r - \nabla_{\underset{\sim}{x}}\, \underset{\sim}{y} \cdot A_r$

identity 0.37 $\qquad = \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) \wedge A_r$.

XXXX

Theorem 6.5 is generalized by the next theorem.

<u>Theorem 6.6</u> If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, then for all $i,\ r \leq n$, and $A_r \in \mathfrak{D}$, (i) $A_r \cdot \nabla_{\bar{x}_i}\, \bar{y}_i = \nabla_{\bar{x}_i}\, \bar{y}_i \cdot A_r$.

(ii) $A_r \wedge \nabla_{\bar{x}_i}\, \bar{y}_i = \nabla_{\bar{x}_i}\, \bar{y}_i \wedge A_r$.

<u>Proof</u> Part (ii) is proved first, since it is used in the proof of (i) .

(ii) $A_r \wedge \nabla_{\bar{x}_i}\, \bar{y}_i = \frac{1}{i!} [(A_r \wedge \nabla_{\underset{\sim}{x}_i} \wedge \ldots \wedge \nabla_{\underset{\sim}{x}_2}) \wedge \nabla_{\underset{\sim}{x}_1}\, \underset{\sim}{y}_1] \underset{\sim}{y}_2 \cdots \underset{\sim}{y}_i$

by lemma 2.21(ii)

theorem 6.5(ii) $= \frac{1}{i!} \nabla_{\underset{\sim}{x}_1} [\underset{\sim}{y}_1 \wedge A_r \wedge \nabla_{\underset{\sim}{x}_i} \wedge \ldots \wedge \nabla_{\underset{\sim}{x}_2}\ \underset{\sim}{y}_2] \underset{\sim}{y}_3 \ldots \underset{\sim}{y}_i$

$\vdots$

theorem 6.5(ii) $= \frac{1}{i!} \nabla_{\underset{\sim}{x}_1} \ldots \nabla_{\underset{\sim}{x}_i}\ \underset{\sim}{y}_i \wedge \ldots \wedge \underset{\sim}{y}_1 \wedge A_r$

lemma 2.21(ii) $= \nabla_{\bar{x}_i}\ \bar{y}_i \wedge A_r$.

(i) Let $I \in \mathfrak{D}^n$ be a pseudoscaler of $\mathfrak{D}$. Then for all i , $r \leq n$,

$$A_r \cdot \nabla_{\bar{x}_i}\ \bar{y}_i = I^{-1}(I\ A_r) \wedge \nabla_{\bar{x}_i}\ \bar{y}_i \qquad \text{identity 0.43}$$

using part (ii) $= I^{-1}\ \nabla_{\bar{x}_i}\ \bar{y}_i \wedge (I\ A_r)$

identity 0.43 $= \nabla_{\bar{x}_i}\ \bar{y}_i \cdot A_r$.

The proof of the theorem is complete.

XXXX

The characteristic multivectors of a curl free mapping are particularly simple, as is shown by the final theorem of this section.

<u>Theorem 6.7</u> If $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, then $J_{\bar{y}_r} = \nabla_{\bar{x}_r} \cdot \bar{y}_r$, for $1 \leq r \leq m$.

<u>Proof</u> The proof is by induction on $r \leq m$. Corollary A.7(ii) and theorem A.6, which are proved in Appendix A, will be used repeatedly. And since the actual value of $\nabla_{\bar{x}_r}\ \bar{y}_r$ is not the point of interest, scalar constants α_i will be used freely

to simplify expressions.

For $r = 1$ there is nothing to prove. Suppose now for all $r < i$ that $\nabla_{\bar{x}_r} \bar{y}_r = \nabla_{\bar{x}_r} \cdot \bar{y}_r$. Then for $r = i$,

$$\nabla_{\bar{x}_i} \bar{y}_i = \frac{1}{i} [\nabla_{\bar{x}_{i-1}} \nabla_{\underset{\sim}{x}} \underset{\sim}{y} \wedge \bar{y}_{i-1}] \qquad \text{lemma 2.21(ii)}$$

identity 0.37 $$= \frac{1}{i} [\nabla_{\bar{x}_{i-1}} \nabla_{\underset{\sim}{x}} \cdot (\underset{\sim}{y} \wedge \bar{y}_{i-1})]$$

identity 0.38 $$= \frac{1}{i} [\nabla_{\underset{\sim}{x}} \cdot y(\underset{\sim}{x})\, \nabla_{\bar{x}_{i-1}} \bar{y}_{i-1} - \nabla_{\bar{x}_{i-1}} y(\dot{\underset{\sim}{x}}) \wedge (\nabla^{\dagger}_{\underset{\sim}{x}} \cdot \bar{y}_{i-1})]$$

identity 0.40 $$= \frac{1}{i} [\alpha_1 - \nabla_{\bar{x}_{i-2}} \nabla_{\underset{\sim}{x}}\, y^2_{\dagger}(\underset{\sim}{x}) \wedge \bar{y}_{i-2}]$$

cor. A.7(ii) and theorem A.6 $$= \frac{1}{i} \{\alpha_1 - \nabla_{\bar{x}_{i-2}} \nabla_{\underset{\sim}{x}} \cdot [y^2_{\dagger}(\underset{\sim}{x}) \wedge \bar{y}_{i-2}]\}$$

theorem A.6 $\vdots$

cor. A.7(ii) $\vdots$

$$= \frac{1}{i} [\alpha_s \pm \nabla_{\underset{\sim}{x}} y^i_{\dagger} \underset{\sim}{x}]$$

cor. A.7(ii) and theorem A.6 $$= \frac{1}{i} [\alpha_s \pm \nabla_{\underset{\sim}{x}} \cdot y^i_{\dagger} \underset{\sim}{x}]$$

XXXX

A proof similar to that of the last theorem is given in Appendix A (theorem A.17).

7. The Identity Mapping

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$, and let $y: \mathcal{X}_m \to \mathcal{X}_m$ be the identity mapping $y(\underset{\sim}{x}) \equiv \underset{\sim}{x}$.

Theorem 7.1 For $A_r \in \mathcal{D}_{\underset{\sim}{x}}$, (i) $A_r \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = r\, A_r$

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = (m-r)\, A_r$ (iii) $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} \cdot A_r = r\, A_r$

(iv) $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} \wedge A_r = (m-r)\, A_r$.

Proof: (i) The proof is by induction on r . The case $r = 1$ follows immediately from definition 2.7 with $F(\underset{\sim}{x}) \equiv \underset{\sim}{x}$. Now assume for $r = i$, that $A_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = i\, A_i$, and for $r = i + 1$ write $A_{i+1} = \underset{\sim}{a} \wedge A_i$. Then:

$$A_{i+1} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = (\underset{\sim}{a} \wedge A_i) \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}$$

identity 0.38 $\qquad = \underset{\sim}{a} \wedge (A_i \cdot \nabla_{\underset{\sim}{x}}) \underset{\sim}{x} + (-1)^i A_i\, \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}$

identity 0.37 $\qquad = \underset{\sim}{a}\, A_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - \underset{\sim}{a} \cdot (A_i \cdot \nabla_{\underset{\sim}{x}}) \underset{\sim}{x} + (-1)^i A_i\, \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}$

identity 0.42 $\qquad = i\, \underset{\sim}{a}\, A_i - (a \cdot A_i) \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} + (-1)^i A_i\, \underset{\sim}{a}$

identity $\qquad = i\, \underset{\sim}{a}\, A_i - (i-1) \underset{\sim}{a} \cdot A_i + (-1)^i A_i\, \underset{\sim}{a}$

identity 0.37 $\qquad = (i+1)\, \underset{\sim}{a} \wedge A_i = (i+1)\, A_{i+1}$.

(ii) $A_r \wedge \nabla_{\underset{\sim}{x}} \underset{\sim}{x} = A_r \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - A_r \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}$

using (i) $\quad = A_r i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - rA_r$

property 2.11 $\quad = A_r i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x} - rA_r$

using (i) $\quad = A_r i_{\underset{\sim}{x}}^{-1} m \, i_{\underset{\sim}{x}} - rA_r$.

$\quad = (m-r) A_r$.

(iii) and (iv) follow from (i) and (ii), using theorem 6.5, if it can be shown that $\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x} = 0$. This is shown below.

$$\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x} = [i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}} \underset{\sim}{x}]_2$$

property 2.11 $\quad = [i_{\underset{\sim}{x}}^{-1} i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{x}]_2$

using (i) $\quad = [\, m \,]_2 = 0$.

XXXX

Corollary 7.2 $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} = m$, or equivalently $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{x} = m$ and $\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x} = 0$.

Proof The proof is in the proof of theorem 7.1(iii) .

XXXX

<u>Lemma 7.3</u> (i) For the mapping $y(x) \equiv \underset{\sim}{x}$ and $A_r \in \mathcal{D}_{\underset{\sim}{x}}$, $y_\dagger A_r = A_r = y^\dagger A_r$.

(ii) More generally for $A_r \in \mathcal{D}$, $y_\dagger A_r = A_{r_\parallel} = y^\dagger A_r$ where $A_{r_\parallel} \in \mathcal{D}_{\underset{\sim}{x}}$ is the tangential part of A_r to the surface $\mathcal{X}_m$ in the decomposition $A_r = A_{r_\parallel} + A_{r_\perp}$.

<u>Proof</u> (i) The proof is by induction on r . For $r = 1$, the lemma follows from theorem 7.1(i). Now suppose for $r = i$, $y_\dagger A_i = A_i$, and for $r = i + 1$ write $A_{i+1} = \underset{\sim}{a} \wedge A_i$. Then:

$$y_\dagger A_{i+1} = y_\dagger \underset{\sim}{a} \wedge A_i$$

theorem 3.3(i)
$$= y_\dagger \underset{\sim}{a} \wedge y_\dagger A_i$$

$$= \underset{\sim}{a} \wedge A_i = A_{i+1} .$$

(ii) The proof of (ii) follows from the decomposition $A_r = A_{r_\parallel} + A_{r_\perp}$ and part (i). I.e.:

$$y_\dagger A_r = y_\dagger (A_{r_\parallel} + A_{r_\perp})$$

$$= y_\dagger A_{r_\parallel} + y_\dagger A_{r_\perp}$$

using (i)
$$= A_{r_\parallel} + y_\dagger A_{r_\perp} . \quad \text{But}$$

$$y_{\dagger}A_{r_\perp} = A_{r_\perp}\cdot\nabla_{\bar{x}_r}\,\bar{x}_r$$

property 2.11 $\quad = 0$.

Thus $\quad y_{\dagger}A_r = A_{r_\parallel}$ for any $A_r \in \mathfrak{D}$.

XXXX

Lemma 7.3(i) is used in the proof of the next theorem. Part (ii) of this lemma is later used in section 10.

Theorem 7.4 Let $A_r \in \mathfrak{D}^r_{\underset{\sim}{x}}$. Then

$$\text{(i)}\quad A_r\cdot\nabla_{\bar{x}_i}\,\bar{x}_i = \begin{Bmatrix} \binom{m-r}{i-r} A_r & \text{for} & r \le i \\ \binom{r}{i} A_r & \text{for} & r \ge i \end{Bmatrix} = \nabla_{\bar{x}_i}\,\bar{x}_i\cdot A_r \ .$$

$$\text{(ii)}\quad A_r\wedge\nabla_{\bar{x}_i}\,\bar{x}_i = \begin{Bmatrix} \binom{m-r}{i} A_r & \text{for} & r + i \le m \\ 0 & \text{for} & r + i > m \end{Bmatrix} = \nabla_{\bar{x}_i}\,\bar{x}_i\wedge A_r \ .$$

Proof (ii) is proved first since it is used in the proof of (i).

(ii) The proof is by induction on i . For $i = 1$, the theorem is theorem 7.1. Now assume that (ii) is true for $i = s$. Then for $i = s + 1$,

$$A_r \Lambda \nabla_{\bar{x}_{s+1}} \bar{x}_{s+1} = \frac{1}{s+1} \{[A_r \Lambda \nabla_{\bar{x}_s}] \Lambda \nabla_{\underset{\sim}{x}_1} \underset{\sim}{x}_1\} \bar{x}_s$$

theorem 7.1(ii) $$= \frac{m-(r+s)}{s+1} A_r \Lambda \nabla_{\bar{x}_s} \bar{x}_s$$

induction hypothesis $$= \frac{m-(r+s)}{s+1} \binom{m-r}{s} A_r$$

$$= \binom{m-r}{s+1} A_r \quad .$$

The second equality of theorem 7.4(ii) follows from theorem 6.8(ii).

(i) For $r \leq i$,

$$A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = \nabla_{\bar{x}_{i-r}} \bar{x}_{i-r} \Lambda y_+ A_r \qquad \text{theorem 3.4(i)}$$

lemma 7.3(i) $$= \nabla_{\bar{x}_{i-r}} \bar{x}_{i-r} \Lambda A_r$$

theorem 7.4(ii) $$= \binom{m-r}{i-r} A_r \quad .$$

For $r \geq i$,

$$A_r \cdot \nabla_{\bar{x}_i} \bar{x}_i = i_{\underset{\sim}{x}}^{-1}(i_{\underset{\sim}{x}} A_r) \Lambda \nabla_{\bar{x}_i}$$

theorem 7.4(ii) $$= \binom{r}{i} A_r \quad .$$

XXXX

<u>Corollary 7.5</u> $\nabla_{\bar{x}_r} \bar{x}_r = \binom{m}{r}$.

<u>Proof</u> $\nabla_{\bar{x}_r} \bar{x}_r = i_{\underset{\sim}{x}}^{-1} \, i_{\underset{\sim}{x}} \cdot \nabla_{\bar{x}_r} \bar{x}_r$ property 2.11

theorem 7.4(i) $= i_{\underset{\sim}{x}}^{-1} \binom{m}{r} i_{\underset{\sim}{x}}$

$= \binom{m}{r}$.

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PART II

MULTIVECTOR FIELDS ON SURFACES

Whereas Part I of this paper only studies the mappings $y_{\dagger}$ and $y^{\dagger}$ between the tangent algebras $\mathcal{D}_{\underset{\sim}{x}}$ and $\mathcal{D}_{y(\underset{\sim}{x})}$ for a fixed point $\underset{\sim}{x} \in \mathcal{X}_m$, Part II studies their "field" properties by considering them as mappings of tangent multivector fields on $\mathcal{X}_m$ and $\mathcal{Y}_k$.

8. The Differential and Adjoint Mappings of Multivector Fields

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the m-surface $\mathcal{X}_m$ into the k-surface $\mathcal{Y}_k$.

The adjoint mapping $y^{\dagger}$, defined and studied in Part I, can be extended pointwise to a mapping of multivector fields on $\mathcal{Y}_k$ into tangent multivector fields on $\mathcal{X}_m$. This is done in the definition below.

Definition 8.1 $y^{\dagger}: \{G(\underset{\sim}{y})\} \to \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ is given by

$F(\underset{\sim}{x}) \equiv y^{\dagger}G\,[y(\underset{\sim}{x})]$, for each $\underset{\sim}{x} \in \mathcal{X}_m$ and $G(\underset{\sim}{y}) \in \{G(\underset{\sim}{y})\}_{\underset{\sim}{y}}$.

The field $F(\underset{\sim}{x})$ is said to be the "pull back" of the field $G(\underset{\sim}{y})$.

Since the mapping $y^{\dagger}$ is extended pointwise to a mapping of multivector fields, all properties proved in Part I for $y^{\dagger}$ remain valid.

If the mapping $y: \mathcal{X}_m \to \mathcal{Y}_m$ is invertible (one-to-one and non-singular), then the differential mapping $y_{\dagger}$ can also be extended pointwise to a mapping of multivector fields on $\mathcal{X}_m$ into tangent multivector fields on $\mathcal{Y}_m$. This is done in the following definition.

Let $x: \mathcal{Y}_m \to \mathcal{X}_m$ denote the inverse mapping of $y(\underset{\sim}{x})$,

i.e.: $\underset{\sim}{x} = x(\underset{\sim}{y})$ iff $\underset{\sim}{y} = y(\underset{\sim}{x})$.

<u>Definition 8.2</u> Let $y(\underset{\sim}{x})$ and $x(\underset{\sim}{y})$ be given as above. Then $y_{\dagger}: \{F(\underset{\sim}{x})\} \to \{G(\underset{\sim}{y})\}_{\underset{\sim}{y}}$ is given by $G(\underset{\sim}{y}) \equiv y_{\dagger}F[x(\underset{\sim}{y})]$, for each $F \in \{F(\underset{\sim}{x})\}$, and $\underset{\sim}{y} \in \mathcal{Y}_m$.

The field $G(\underset{\sim}{y})$, where $G(\underset{\sim}{y}) \equiv y_{\dagger}F[x(\underset{\sim}{y})]$ is said to be the "push forward" of the field $F(\underset{\sim}{x})$.

Since the mapping $y_{\dagger}$ is extended pointwise, all properties of $y_{\dagger}$ proved in Part I remain valid.

9. Mapping the Gradient Operator

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$ be a mapping of the m-surface $\mathcal{X}_m$ into the k-surface $\mathcal{Y}_k$.

Since by property 2.11 the gradient $\nabla_{\underset{\sim}{y}}$ behaves like an ordinary vector of $\mathcal{D}^1_{\underset{\sim}{y}}$, the chain rule for the gradient operator (theorem 2.14) can be written in the following instructive way:

$$(9.1) \qquad \nabla_{\underset{\sim}{x}} = y^{\dagger} \nabla_{\underset{\sim}{y}} \ .$$

Equation (9.1) shows that the gradient $\nabla_{\underset{\sim}{x}}$ on the surface $\mathcal{X}_m$ is the gradient $\nabla_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_k$ "pulled back" to the surface $\mathcal{X}_m$.

The next result is theorem 3.3(ii) applied to the gradient $\nabla_{\underset{\sim}{y}}$. It is valid because $\nabla_{\underset{\sim}{y}}$ is a vector operator.

<u>Theorem 9.2</u> $\quad y^{\dagger}[\nabla_{\underset{\sim}{y}_1} \wedge B(\underset{\sim}{y}_1)] = y^{\dagger}\nabla_{\underset{\sim}{y}_1} \wedge y^{\dagger}B(\underset{\sim}{y}_1)$, where $B(\underset{\sim}{y})$ is a multivector field on $\mathcal{Y}_k$.

Note that on the right side of the equality in theorem 9.2 that the gradient $\nabla_{\underset{\sim}{y}_1}$ only differentiates $B(\underset{\sim}{y}_1)$ and not $y^{\dagger}$. This restriction is now shown to be unnecessary by the use of property 2.13.

<u>Theorem 9.3</u> $\quad y^{\dagger}[\nabla_{\underset{\sim}{y}} \wedge B(\underset{\sim}{y})] = \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} B[y(\underset{\sim}{x})]$

<u>Proof</u> $\quad y^{\dagger}[\nabla_{\underset{\sim}{y}_{s+1}} \wedge B^{s}(\underset{\sim}{y}_{s+1})] = y^{\dagger}\nabla_{\underset{\sim}{y}_{s+1}} \wedge \; y^{\dagger}B^{s}(\underset{\sim}{y}_{s+1})$ theorem 9.2

equation (9.1) $\qquad = \nabla_{\underset{\sim}{x}_{s+1}} \wedge \; \nabla_{\bar{x}_s} \; \bar{y}_s \cdot B^{s}[y(\underset{\sim}{x}_{s+1})]$

property 2.13 $\qquad = \nabla_{\underset{\sim}{x}} \wedge \nabla_{\bar{x}_s} \; \bar{y}_s \cdot B^{s}[y(\underset{\sim}{x})]$

$$= \nabla_{\underset{\sim}{x}} \wedge y^{\dagger} B^{s}$$

XXXX

The corresponding statement of theorem 9.3 for dots is false, i.e.: $y^{\dagger}\nabla_{\underset{\sim}{y}} \cdot B(\underset{\sim}{y}) \neq \nabla_{\underset{\sim}{x}} \cdot y^{\dagger} B(\underset{\sim}{y})$. However, in section 13 of this paper it is shown that under certain conditions

$y_{\dagger} \nabla_{\underset{\sim}{x}} \cdot A(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger} A(\underset{\sim}{x})$, where $A(\underset{\sim}{x})$ is a multivector field on $\mathcal{X}_m$.

Theorem 9.3 is used in Appendix D to show that the d - operator on differential forms commutes with the pull back mapping of forms.

The result below is theorem 3.5(i) applied to the gradient operator. It too is valid because of the vector-like properties of the gradient.

<u>Theorem 9.4</u> $(y_\dagger A_r)\cdot\nabla_{\underset{\sim}{y}_r} B^s(\underset{\sim}{y}_r) = y_\dagger(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x}_r)]$,

where $A_r \in \mathcal{D}$, and $B^s(\underset{\sim}{y}) \in \{F(\underset{\sim}{y})\}$.

Note that $\nabla_{\underset{\sim}{x}_r}$ does not differentiate $y_\dagger$ in theorem 9.4 above. The following theorem allows $\nabla_{\underset{\sim}{x}}$ to differentiate $y_\dagger$. Its proof depends upon property 2.13.

<u>Theorem 9.5</u> $(y_\dagger A_r)\cdot\nabla_{\underset{\sim}{y}} B^s(\underset{\sim}{y}) = y_\dagger(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x})]$.

<u>Proof</u> $(y_\dagger A_r)\cdot\nabla_{\underset{\sim}{y}} B^s(\underset{\sim}{y}) = y_\dagger(A_r\cdot\nabla_{\underset{\sim}{x}_r}) B^s[y(\underset{\sim}{x}_r)]$ lemma 9.4

$$= (A_r\cdot\nabla_{\underset{\sim}{x}_r})\cdot\nabla_{\bar{x}_{r-1}} \bar{y}_{r-1} B^s[y(\underset{\sim}{x}_r)]$$

property 2.13 $$= A_r\cdot(\nabla_{\underset{\sim}{x}}\wedge\nabla_{\bar{x}_{r-1}})\bar{y}_{r-1} B^s[y(\underset{\sim}{x})]$$

identity 0.42 $$= (A_r\cdot\nabla_{\underset{\sim}{x}})\cdot\nabla_{\bar{x}_{r-1}} \bar{y}_{r-1} B^s[y(\underset{\sim}{x})]$$

$$= y_\dagger(A_r\cdot\nabla_{\underset{\sim}{x}}) B^s[y(\underset{\sim}{x})]$$

XXXX

Theorem 9.5 is a generalization of the chain rule for the gradient operator.

When $y(\underset{\sim}{x})$ is an invertible mapping between m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$, theorem 5.1(i) can be applied to equation (9.1) to give a "dual" chain rule for the gradient operator:

$$(9.6) \quad y_{\dagger}\, i_{\underset{\sim}{x}}\, \nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{y}}\, \nabla_{\underset{\sim}{y}} ,$$

where $i_{\underset{\sim}{x}}$ is a pseudoscalar field on $\mathcal{X}_m$, and $i_{\underset{\sim}{y}} \equiv y_{\dagger}\, i_{\underset{\sim}{x}}$ is the corresponding pseudoscalar field on $\mathcal{Y}_m$.

Equation 9.6 shows that the operator $i_{\underset{\sim}{x}} \nabla_{\underset{\sim}{x}}$ on the surface $\mathcal{X}_m$ is "pushed forward" by $y_{\dagger}$ into the operator $i_{\underset{\sim}{y}} \nabla_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_m$.

Equation 9.6 can also be immediately derived from theorem 9.5 by letting $A_i = i_{\underset{\sim}{x}}$ in that theorem.

10. Lie Brackets

Fundamental to the study of multivector fields on surfaces is the Lie bracket, or bracket operation. In section 10a the definition of the Lie bracket operation of tangent multivector fields is given and its basic properties are studied. Most importantly, it is shown that the Lie bracket of tangent multivector fields is a tangent multivector field, and that the divergence of a tangent multivector field is a tangent multivector field. In section 10b it is shown that the Lie bracket of tangent multivector fields is preserved under the differential mapping.

a) Definition and Basic Properties

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

<u>Definition 10.1</u> $[A_r,B_s] \equiv (A_r\cdot\nabla_x) \wedge B_s(x) - A_r(x)\wedge(\nabla_x^\dagger\cdot B_s)$,

where $A_r(x),B_s(x) \in \{F(x)\}_x$, and $\nabla_x^\dagger$ is understood to differentiate <u>only</u> to the left.

For $r = 1$, definition 10.1 reduces to $[a,B_s] = a\cdot\nabla_x B_s(x) - a(x) \wedge (\nabla_x^\dagger\cdot B_s)$, and for both $r = 1 = s$, $[a,b] = a\cdot\nabla_x b(x) - b\cdot\nabla_x a(x)$.

The following theorem gives fundamental properties of the bracket operation. Let $\underset{\sim}{a}(\underset{\sim}{x})$, $A_r(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x})$, $B_s(\underset{\sim}{x})$, $C_t(\underset{\sim}{x})$, $\varepsilon\ \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Theorem 10.2</u> (i) $[A_r+B_s,\ C_t] = [A_r,C_t] + [B_s,C_t]$, and $[A_r,\ B_s+C_t] = [A_r,B_s] + [A_r,C_t]$.

(ii) $[A_r\wedge\underset{\sim}{a},\ B_s] = A_r\wedge[\underset{\sim}{a},B_s] + (-1)^r\underset{\sim}{a}\wedge[A_r,B_s]$, and $[A_r,\ \underset{\sim}{b}\wedge B_s] = [A_r,\underset{\sim}{b}]\wedge B_s + (-1)^s\ [A_r,B_s]\wedge\underset{\sim}{b}$.

(iii) $[A_r,B_s] = -\,[B_s^\dagger,\ A_r^\dagger]^\dagger$.

<u>Proof</u> (i) The proof is trivial and is omitted.

(ii) $[A_r\wedge\underset{\sim}{a},\ B_s] = [(A_r\wedge\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{x}}]\wedge B_s(x) - [A_r(\underset{\sim}{x})\wedge\underset{\sim}{a}(\underset{\sim}{x})]\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)$

summary 0.38 $$= A_r\wedge[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\ B_s(\underset{\sim}{x})] + (-1)^r\underset{\sim}{a}\wedge(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge B_s(\underset{\sim}{x})$$

$$- A_r\wedge[\underset{\sim}{a}(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)] - (-1)^r\underset{\sim}{a}\wedge[A_r(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)]$$

$$= A_r\wedge[\underset{\sim}{a},B_s] + (-1)^r\underset{\sim}{a}\wedge[A_r,B_s]\ .$$

The other part of (ii) is proved in a similar way.

(iii) $[A_r,B_s] = \{[A_r,B_s]^\dagger\}^\dagger$

$$= \{[(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge B_s(\underset{\sim}{x}) - A_r(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot B_s)]^\dagger\}^\dagger$$

$$= \{B_s^\dagger(\underset{\sim}{x})\wedge(\nabla_{\underset{\sim}{x}}^\dagger\cdot A_r^\dagger) - (B_s^\dagger\cdot\nabla_{\underset{\sim}{x}})\wedge A_r(\underset{\sim}{x})\}^\dagger$$

$$= -[B_s^\dagger, A_r^\dagger]^\dagger .$$

Similarly, the other part of (iii) is proved.

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<u>Corollary 10.3</u> $[A_r, B_s] = -(-1)^{(r-1)(s-1)}[B_s, A_r]$

<u>Proof</u> The proof follows immediately by substituting

$$A_r^\dagger = (-1)^{\frac{r(r-1)}{2}} A_r ,$$

$$B_s^\dagger = (-1)^{\frac{s(s-1)}{2}} B_s \text{ , and}$$

$$[B_s, A_r] = (-1)^{\frac{(r+s-1)(r+s-1)}{2}} [B_s^\dagger, A_r^\dagger] ,$$

into the right side of theorem 10.3(iii).

XXXX

A special case of corollary 10.3 is:

<u>Corollary 10.4</u> $[\underset{\sim}{a}, B_s] = -[B_s, \underset{\sim}{a}]$.

The bracket operation is also defined for directional derivatives. The definition is now given.

<u>Definition 10.5</u> $[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}] \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}} - \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}$,

for $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}^1$.

The theorem below gives the important relationship between the bracket operation of vector fields, and the bracket operation of directional derivatives. It is further discussed in Appendix E in connection with the "curvature" of a surface.

Theorem 10.6 For any $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \mathcal{D}'_{\underset{\sim}{x}}$,

(i) $(a \wedge b) \cdot (\nabla_x \wedge \nabla_x) \equiv [\underset{\sim}{a}, \underset{\sim}{b}] \cdot \nabla_{\underset{\sim}{x}} - [\underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}, \underset{\sim}{b} \cdot \dot{\nabla}_{\underset{\sim}{x}}]$

(ii) $[\underset{\sim}{a}, \underset{\sim}{b}] \cdot \nabla_{\underset{\sim}{x}} - [\underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}, \underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}}] \equiv 0$.

Proof (i) The proof of (i) is direct.

$$(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot (\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}}) = \underset{\sim}{a} \cdot [\underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}}\ \nabla_{\underset{\sim}{x}}] - \underset{\sim}{b} \cdot [\underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}\ \nabla_{\underset{\sim}{x}}]$$

$$= \underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} - (\underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}}\ \underset{\sim}{a}) \cdot \nabla_{\underset{\sim}{x}} - \underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}} + (\underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}\ \underset{\sim}{b}) \cdot \nabla_{\underset{\sim}{x}}$$

$$= [\underset{\sim}{a}, \underset{\sim}{b}] \cdot \nabla_{\underset{\sim}{x}} - [\underset{\sim}{a} \cdot \nabla_{\underset{\sim}{x}}, \underset{\sim}{b} \cdot \nabla_{\underset{\sim}{x}}] \ .$$

(ii) The proof of (ii) follows from (i) since by property 2.13.

$$(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot (\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}}) \equiv 0 \ .$$

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Parts (i) and (ii) of theorem 10.6 are kept separate because it is important to see part (i) as an identity by itself.

It is now shown that if A_r and B_s are tangent multivector fields on $\mathcal{X}_m$, then so is $[A_r, B_s]$. First two lemmas asserting special cases are proved.

<u>Lemma 10.7</u> $[\underset{\sim}{a},\underset{\sim}{b}] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ if $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> The lemma is readily proved by operating on $\underset{\sim}{x}$ by theorem 10.6(ii) and noting that

$$[\underset{\sim}{a},\underset{\sim}{b}]\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x} = [\underset{\sim}{a},\underset{\sim}{b}]_{\|}$$

by lemma 7.3(ii), and

$$[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}},\ \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}]\underset{\sim}{x} = [\underset{\sim}{a},\underset{\sim}{b}]$$

by using definition 10.5 and lemma 7.3(i).

XXXX

<u>Lemma 10.8</u> If $\underset{\sim}{a}$, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, then $[\underset{\sim}{a},B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> The proof is by induction on s . For $s = 1$, lemma 10.8 reduces to lemma 10.7. Now suppose for $s = i$ that $[\underset{\sim}{a},B_i] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ and for $s = i + 1$, write $B_{i+1} = \underset{\sim}{b} \wedge B_i$, where $\underset{\sim}{b} \in \{F(\underset{\sim}{x})\}^1_{\underset{\sim}{x}}$. Then

$$[\underset{\sim}{a},\ B_{i+1}] = [\underset{\sim}{a},\ \underset{\sim}{b}\wedge B_i]$$

theorem 10.2(ii) $$= [\underset{\sim}{a},\underset{\sim}{b}]\wedge B_i + (-1)^i[\underset{\sim}{a},B_i]\wedge\underset{\sim}{b} .$$

Since both terms after the last equality are in $\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, $[\underset{\sim}{a},\ B_{i+1}] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ and the proof is complete.

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<u>Theorem 10.9</u> If $A_r, B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, then $[A_r, B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> The proof is by induction on r, for a fixed s. For $r = 1$, $[\underset{\sim}{a}, B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ by the previous lemma. Now suppose for $r = i$, $[A_i, B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and for $r = i + 1$ write $A_{i+1} = A_i \wedge \underset{\sim}{a}$, where $\underset{\sim}{a} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$. Then

$$[A_{i+1}, B_s] = [A_i \wedge \underset{\sim}{a}, B_s]$$

theorem 10.2(ii)
$$= A_i \wedge [\underset{\sim}{a}, B_s] + (-1)^i \underset{\sim}{a} \wedge [A_i, B_s] \ .$$

Since both terms after the last equality are in $\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, $[A_{i+1}, B_s] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and the proof is complete.

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An alternative proof of lemma 10.8 and theorem 10.9 can be obtained by using lemma 10.7 and the following decomposition theorem.

<u>Theorem 10.10</u> (i) $[\underset{\sim}{a}, B_s] = \sum_{i=1}^{s} \underset{\sim}{b}_1 \wedge \ldots \wedge \underset{\sim}{b}_{i-1} \wedge [\underset{\sim}{a}, \underset{\sim}{b}_i] \wedge \underset{\sim}{b}_{i+1} \wedge \ldots \wedge \underset{\sim}{b}_s$, for a, $B_s = \underset{\sim}{b}_1 \wedge \ldots \wedge \underset{\sim}{b}_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

(ii) $[A_r, B_s] = \sum_{i=1}^{s} (-1)^{i+1} [A_r, \underset{\sim}{b}_i] \wedge [\underset{\sim}{b}_1 \wedge \overset{\vee}{\underset{i}{\cdot}} \wedge \underset{\sim}{b}_s]$, for A_r, $B_s = \underset{\sim}{b}_1 \wedge \ldots \wedge \underset{\sim}{b}_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> Since (i) is a special case of (ii), only (ii) is proved. The proof is direct.

$$[A_r,B_s] = (A_r\cdot\nabla_{\underset{\sim}{x}})\wedge\underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s - A_r\wedge[\nabla^{\dagger}_{\underset{\sim}{x}}\cdot(\underset{\sim}{b}_1\wedge\ldots\wedge\underset{\sim}{b}_s)]$$

identity 0.40

$$= \sum_{i=1}^{s}(-1)^{i+1}[(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge\underset{\sim}{b}_i(\underset{\sim}{x})]\wedge\underset{\sim}{b}_1\wedge\ldots\wedge\check{\underset{\sim}{b}}_i\wedge\ldots\wedge\underset{\sim}{b}_s$$

$$-\sum_{i=1}^{s}(-1)^{i+1}[A_r(\underset{\sim}{x})\,\nabla^{\dagger}_{\underset{\sim}{x}}\cdot\underset{\sim}{b}_i]\wedge\underset{\sim}{b}_1\wedge\ldots\wedge\check{\underset{\sim}{b}}_i\wedge\ldots\wedge\underset{\sim}{b}_s$$

$$= \sum_{i=1}^{s}(-1)^{i+1}[A_r,\underset{\sim}{b}_i]\wedge[\underset{\sim}{b}_1\wedge\ldots\wedge\check{\underset{\sim}{b}}_i\wedge\ldots\wedge\underset{\sim}{b}_s]\ .$$

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Theorem 10.10 is used in Appendix D on differential forms.

An important consequence of theorem 10.9 is that the divergence of a tangent multivector field on $\mathcal{X}_m$ is itself a tangent multivector field on $\mathcal{X}_m$. This is proved in the next theorem using the following lemma.

<u>Lemma 10.11</u> (i) $\nabla_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\wedge A_i) = [\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x})]A_i - \underset{\sim}{a}\wedge[\nabla_{\underset{\sim}{x}}\cdot A_i(\underset{\sim}{x})] + [\underset{\sim}{a},A_i]$, for $\underset{\sim}{a},A_i \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

(ii) $\nabla_{\underset{\sim}{x}}\cdot(A_r\wedge B_s) = (\nabla_{\underset{\sim}{x}}\cdot A_r)\wedge B_s + (-1)^r A_r\wedge(\nabla_{\underset{\sim}{x}}\cdot B_s) + (-1)^{r+1}[A_r,B_s]$, for A_r , $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> Since (i) is a special case of (ii), only (ii) is proved.

$$\text{(ii)}\quad \nabla_{\underset{\sim}{x}}\cdot(A_r \wedge B_s) = [\nabla_{\underset{\sim}{x}}\cdot A_r(\underset{\sim}{x})]\wedge B_s(\underset{\sim}{x}) + (-1)^r A_r(\underset{\sim}{x})\wedge[\dot{\nabla}_{\underset{\sim}{x}}\cdot B_s(\underset{\sim}{x})]$$

identity 0.38

$$= (\nabla_{\underset{\sim}{x}}\cdot A_r)\wedge B_s + (-1)^{r+1}(A_r\cdot\nabla_{\underset{\sim}{x}})\wedge B_s + (-1)^r A_r \wedge (\nabla^{\dagger}_{\underset{\sim}{x}}\cdot B_s) + (-1)^r A_r \wedge(\nabla_{\underset{\sim}{x}}\cdot B_s)$$

$$= (\nabla_{\underset{\sim}{x}}\cdot A_r)\wedge B_s + (-1)^r A_r\wedge(\nabla_{\underset{\sim}{x}}\cdot B_s) + (-1)^{r+1}[A_r, B_s]$$

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In the proof above $\dot{\nabla}_{\underset{\sim}{x}}$ means that the gradient operator differentiates both ways, and $\nabla^{\dagger}_{\underset{\sim}{x}}$ means that it differentiates only to the left.

<u>Theorem 10.12</u> If $A_r \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, then $\nabla_{\underset{\sim}{x}}\cdot A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> The proof is by induction on r. For $r = 1$ the theorem is true, since $\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}$ is a scaler. Suppose now for $r = i$ that $\nabla_{\underset{\sim}{x}}\cdot A_i \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and for A_{i+1} write $A_{i+1} = \underset{\sim}{a}\wedge A_i$. Then

$$\nabla_{\underset{\sim}{x}}\cdot A_{i+1} = \nabla_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\wedge A_i)$$

lemma 10.11(i) $\qquad = [\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}]A_i - a\,[\nabla_{\underset{\sim}{x}}\cdot A_i] + [\underset{\sim}{a}, A_i]$.

Since all terms of the last sum are in $\{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, it follows that $\nabla_{\underset{\sim}{x}} \cdot A_{i+1} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

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b) The Lie Bracket Under the Differential Mapping

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$. Only tangent multivector fields on $\mathcal{X}_m$ and $\mathcal{Y}_m$ are considered here.

The following lemmas are needed to prove that the Lie bracket is preserved under the differential mapping.

<u>Lemma 10.13</u> $[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}] = [(y_\dagger\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{y}}, (y_\dagger\underset{\sim}{b})\cdot\nabla_{\underset{\sim}{y}}]$

<u>Proof</u> $[\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}] = [\underset{\sim}{a}\cdot y^\dagger\nabla_{\underset{\sim}{y}}, \underset{\sim}{b}\cdot y^\dagger\nabla_{\underset{\sim}{y}}]$ using eqn. (9.1)

theorem 9.4 $\quad = [(y_\dagger\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{y}}, (y_\dagger\underset{\sim}{b})\cdot\nabla_{\underset{\sim}{y}}]$

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<u>Lemma 10.14</u> $y_\dagger[\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}} = [y_\dagger\underset{\sim}{a}, y_\dagger\underset{\sim}{b}]_{\underset{\sim}{y}}$, for $\underset{\sim}{a}$, $\underset{\sim}{b} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

<u>Proof</u> $y_\dagger[\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}} = [\underset{\sim}{a},\underset{\sim}{b}]_{\underset{\sim}{x}}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

theorem 10.6 $\quad = [\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}, \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}]y(\underset{\sim}{x})$

lemma 10.13 $\quad = [(y_\dagger\underset{\sim}{a})\cdot\nabla_{\underset{\sim}{y}}, (y_\dagger\underset{\sim}{b})\cdot\nabla_{\underset{\sim}{y}}]\, y(\underset{\sim}{x})$

theorem 10.6 $\quad = [y_\dagger\underset{\sim}{a}, y_\dagger\underset{\sim}{b}]_{\underset{\sim}{y}}$.

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Lemma 10.15 $y_+[\underset{\sim}{a}, B_s]_{\underset{\sim}{x}} = [y_+\underset{\sim}{a}, y_+B_s]_{\underset{\sim}{y}}$ for $\underset{\sim}{a}$, $B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$

Proof The proof is by induction on s. For $s = 1$, lemma 10.15 is lemma 10.14. Now suppose for $s = i$ that $y_+[\underset{\sim}{a}, B_i]_{\underset{\sim}{x}} = [y_+\underset{\sim}{a}, y_+B_i]_{\underset{\sim}{y}}$ and for $s = i + 1$, write $B_{i+1} = \underset{\sim}{b} \wedge B_i$. Then

$$y_+[\underset{\sim}{a}, B_{i+1}]_{\underset{\sim}{x}} = y_+[\underset{\sim}{a}, \underset{\sim}{b} \wedge B_i]_{\underset{\sim}{x}}$$

theorem 10.9(ii)	$= y_+\{[\underset{\sim}{a}, \underset{\sim}{b}] \wedge B_i + (-1)^i [\underset{\sim}{a}, B_i] \wedge \underset{\sim}{b}\}$
theorem 3.3(i) and lemma 10.14	$= [y_+\underset{\sim}{a}, y_+\underset{\sim}{b}] \wedge y_+B_i + (-1)^i [y_+\underset{\sim}{a}, y_+B_i] \wedge y_+\underset{\sim}{b}$
theorem 10.9(ii)	$= [y_+\underset{\sim}{a}, y_+\underset{\sim}{b} \wedge y_+B_i]_{\underset{\sim}{y}}$
theorem 3.3(i)	$= [y_+\underset{\sim}{a}, y_+B_{i+1}]_{\underset{\sim}{y}}$

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It now can be shown that the Lie bracket is preserved under the differential mapping.

Theorem 10.19 $y_+[A_r, B_s]_{\underset{\sim}{x}} = [y_+A_r, y_+B_s]_{\underset{\sim}{y}}$, for $A_r, B_s \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Proof The proof is by induction on r for a fixed s. It is omitted because it closely parallels the proof of lemma 10.15

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11. Frames

a) Definitions and Basic Properties

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

Definition 11.1 A set $\{\underset{\sim}{e}_i(\underset{\sim}{x}) \mid i = 1, \dots, m\}$ of m-linearly independent tangent vector fields on $\mathcal{X}_m$ is called a frame on $\mathcal{X}_m$.

Once a frame $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ is chosen on $\mathcal{X}_m$, it is convenient to construct a reciprocal frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ on $\mathcal{X}_m$. It is defined below.

Definition 11.2 The reciprocal frame to $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ is the unique frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ on $\mathcal{X}_m$ satisfying the relations:

$$\underset{\sim}{e}_i(\underset{\sim}{x}) \cdot \underset{\sim}{e}^j(\underset{\sim}{x}) = \delta_i^j \quad \text{for all } i,j \leq m .$$

A particularly important frame on a surface is a "coordinate" frame. Its definition is given below.

Definition 11.3 A coordinate frame on $\mathcal{X}_m$ is a frame

$\{e^i(x)\}$ with the property that for each $e^i(x) \in \{e^i(x)\}$, there is a scalar field $\psi^i(x)$ such that $e^i(x) = \nabla_x \psi^i(x)$.

For a discussion of the above definitions, and a construction of the reciprocal frames, see [1], p. 83].

<u>Theorem 11.4</u> If $\{e^i(x)\}$ is a frame and $\{e_i(x)\}$ its reciprocal frame, then $[e_i \cdot \nabla_x e_j(x)] \cdot e^t = - e_j \cdot [e_i \cdot \nabla_x e^t(x)]$.

<u>Proof</u> The proof follows immediately from differentiating the relationship $e_j(x) \cdot e^t(x) = \delta_j^t$ by $e_i \cdot \nabla_x$.

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<u>Theorem 11.5</u> If $\{e^i(x)\}$ is a coordinate frame and $\{e_i(x)\}$ its reciprocal frame, then: (i) $\nabla_x \wedge e^t(x) = 0$

(ii) $[e_i, e_j] = 0$

(iii) $[e_i \cdot \nabla_x, e_j \cdot \nabla_x] = 0$.

<u>Proof</u> (i) Since $e^t(x)$ is a coordinate vector, there exists a scalar field $\psi^t(x)$ such that $\nabla_x \psi^t(x) = e^t(x)$. Thus $\nabla_x \wedge e^t(x) = \nabla_x \wedge \nabla_x \psi^t(x) = 0$ by property 2.13.

(ii) By (i), $\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{e}^t(\underset{\sim}{x}) = 0$. This implies:

$$0 = (\underset{\sim}{e}_i \wedge \underset{\sim}{e}_j)\cdot(\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{e}^t)$$

$$= \underset{\sim}{e}_i\cdot(\underset{\sim}{e}_j\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{e}^t) - \underset{\sim}{e}_j\cdot(\underset{\sim}{e}_i\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{e}^t)$$

theorem 11.5
$$= -(\underset{\sim}{e}_j\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{e}_i)\cdot\underset{\sim}{e}^t + (\underset{\sim}{e}_i\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{e}_j)\cdot\underset{\sim}{e}^t$$

$$= [\underset{\sim}{e}_i,\underset{\sim}{e}_j]\cdot\underset{\sim}{e}^t \; .$$

Since this is true for each $\underset{\sim}{e}^t(\underset{\sim}{x}) \in \{\underset{\sim}{e}^i(\underset{\sim}{x})\}$, and $[\underset{\sim}{e}_i,\underset{\sim}{e}_j] \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ by lemma 10.7 , $[\underset{\sim}{e}_i,\underset{\sim}{e}_j] = 0$ for each $i,j \le m$.

(iii) By theorem 10.6, $[\underset{\sim}{e}_i\cdot\nabla_{\underset{\sim}{x}},\, \underset{\sim}{e}_j\cdot\nabla_{\underset{\sim}{x}}] = [\underset{\sim}{e}_i,\underset{\sim}{e}_j]\cdot\nabla_{\underset{\sim}{x}}$. But by (ii) , $[\underset{\sim}{e}_i,\underset{\sim}{e}_j] = 0$. Hence $[\underset{\sim}{e}_i\cdot\nabla_{\underset{\sim}{x}},\, \underset{\sim}{e}_j\cdot\nabla_{\underset{\sim}{x}}] = 0$

b) Representation of the Gradient Operator

The gradient operator is now expressed in terms of a frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ and its reciprocal frame $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$. Let $i_{\underset{\sim}{x}} = \underset{\sim}{e}_1(\underset{\sim}{x}) \wedge\ldots\wedge \underset{\sim}{e}_m(\underset{\sim}{x})$, and $i_{\underset{\sim}{x}}^{-1} = \underset{\sim}{e}^m(\underset{\sim}{x}) \wedge\ldots\wedge \underset{\sim}{e}^1(\underset{\sim}{x})$. By using the identity 0.41, it is easily seen that $i_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}}^{-1} = 1$.

<u>Theorem 11.6</u> $\nabla_{\underset{\sim}{x}} \equiv \sum_{i=1}^{m} \underset{\sim}{e}^i(\underset{\sim}{x})\, \underset{\sim}{e}_i\cdot\nabla_{\underset{\sim}{x}}$.

<u>Proof</u> The proof is direct .

$$\nabla_{\underset{\sim}{x}} = i_{\underset{\sim}{x}}^{-1}\, i_{\underset{\sim}{x}} \cdot \nabla_{\underset{\sim}{x}}$$

$$= \underset{\sim}{e}^m \wedge \ldots \wedge \underset{\sim}{e}^1 (\underset{\sim}{e}_1 \wedge \ldots \wedge \underset{\sim}{e}_m) \cdot \nabla_{\underset{\sim}{x}}$$

identity 0.40 $$= \sum_{i=1}^{m} (-1)^i (\underset{\sim}{e}^1 \wedge \ldots \wedge \underset{\sim}{e}^m) \cdot (\underset{\sim}{e}_m \wedge \ldots \wedge \check{\underset{\sim}{e}}_i \wedge \ldots \wedge \underset{\sim}{e}_1) \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}$$

identity 0.42 $$= \sum_{i=1}^{m} \underset{\sim}{e}^i\, \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \quad .$$

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c) Frames Under Mappings

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping (non-singular and one-to-one) between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$.

Let $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$, and $\{\underset{\sim}{f}^i(\underset{\sim}{y})\}$ and $\{\underset{\sim}{f}_i(\underset{\sim}{y})\}$, be frames and their reciprocals on $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

<u>Theorem 11.7</u> (i) $\{\underset{\sim}{f}_i(\underset{\sim}{x}) = y_{\dagger}\underset{\sim}{e}_i(\underset{\sim}{y})\}$ iff $\{\underset{\sim}{e}^i(\underset{\sim}{x}) = y^{\dagger}\underset{\sim}{f}^i(\underset{\sim}{y})\}$

(ii) $\{\underset{\sim}{e}^i(\underset{\sim}{x}) \equiv y^{\dagger}\underset{\sim}{f}^i(\underset{\sim}{y})\}$ is a coordinate frame on $\mathcal{X}_m$ iff $\{f^i(y)\}$ is a coordinate frame on $\mathcal{Y}_m$.

<u>Proof</u> (i) Suppose for every i, $\underset{\sim}{f}_i = y_{\dagger}\, \underset{\sim}{e}_i$. Then for every i and j,

$$\delta_i^j = \underset{\sim}{f}^j \cdot \underset{\sim}{f}_i$$

$$= \underset{\sim}{f}^j \cdot y_{\dagger}\, \underset{\sim}{e}_i$$

cor. 3.6 $$= (y^{\dagger}\, \underset{\sim}{f}^j) \cdot \underset{\sim}{e}_i \quad .$$

This implies that $\underset{\sim}{e}^j = y^{\dagger}\, \underset{\sim}{f}^j$, since reciprocal frames are unique. By reversing the above steps the second half of (i) is proved.

(ii) Suppose that $\{\underset{\sim}{f}^i(\underset{\sim}{y})\}$ is a coordinate frame on $\mathcal{Y}_m$. Then for each i there is a $\psi^i(\underset{\sim}{y})$ such that $\underset{\sim}{f}^i(\underset{\sim}{y}) = \nabla_{\underset{\sim}{y}}\, \psi(\underset{\sim}{y})$. But then,

$$\underset{\sim}{e}^i \equiv y^{\dagger}\, \underset{\sim}{f}^i$$

$$= y^{\dagger}\, \nabla_{\underset{\sim}{y}}\, \psi^i(\underset{\sim}{y})$$

eqn. (9.1) $$= \nabla_{\underset{\sim}{x}}\, \psi^i[y(\underset{\sim}{x})] \quad .$$

Thus $\{\underset{\sim}{e}^i \equiv y^{\dagger}\underset{\sim}{f}^i\}$ is a coordinate frame on $\mathcal{X}_m$. The above steps can be reversed to show the second half of (ii).

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Theorem 11.7 shows that frames and their reciprocals "map" in different directions, and further that coordinate frames when pulled back remain coordinate frames. The first part of the theorem is analogous to theorem 5.1 of section 5.

12. The Divergence of a Field

This section relates the divergences of tangent vector fields on different surfaces. In addition it gives the necessary and sufficient condition for the differential mapping to commute with the divergence operation. Finally properties particular to a coordinate frame are studied.

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between $\mathcal{X}_m$ and $\mathcal{Y}_m$. Let $i_{\underset{\sim}{x}} = i(\underset{\sim}{x})$ be a pseudoscalar field on $\mathcal{X}_m$, and $i_{\underset{\sim}{y}} \equiv y_{\dagger} i_{\underset{\sim}{x}}$ the corresponding pseudoscalar field on $\mathcal{Y}_m$.

<u>Definition 12.1</u> $g_{\underset{\sim}{x}} \equiv |i_{\underset{\sim}{x}}|^2$, or $\sqrt{g_{\underset{\sim}{x}}} \equiv |i_{\underset{\sim}{x}}|$.

The $\sqrt{g_{\underset{\sim}{x}}}$ is called the "density" or volume element of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, with respect to the pseudoscalar field $i_{\underset{\sim}{x}}$.

Related versions of the following lemma are needed in several proofs of this section.

<u>Lemma 12.2</u> (i) $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} = 2\, i_{\underset{\sim}{x}}^{\dagger}\cdot(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}})$

(ii) $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} = 2\,(i_{\underset{\sim}{x}}^{\dagger}\, \underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}})$.

<u>Proof</u> The proofs of (i) and (ii) are found in the string of equalities below.

$$\begin{aligned}
\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} &= \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, |i_{\underset{\sim}{x}}|^2 \\
&\equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}}^{\dagger}\cdot i_{\underset{\sim}{x}} \\
&= 2\, i_{\underset{\sim}{x}}^{\dagger}\cdot(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}}) \\
\text{identity } 0.43 \qquad &= 2\,[(i_{\underset{\sim}{x}}^{\dagger}\underset{\sim}{a})\wedge\nabla_{\underset{\sim}{x}}]\cdot i_{\underset{\sim}{x}} \\
\text{identity } 0.42 \qquad &= 2\,(i_{\underset{\sim}{x}}^{\dagger}\underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}})\ .
\end{aligned}$$

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Let $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and define $\underset{\sim}{b}(\underset{\sim}{y}) \equiv y_{\dagger}\, \underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{y})\}_{\underset{\sim}{y}}$.

The following theorem gives the divergence of $\underset{\sim}{a}(\underset{\sim}{x})$ in terms of the divergence of $\underset{\sim}{b}(\underset{\sim}{y})$.

<u>Theorem 12.3</u> $\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}}\cdot\underset{\sim}{b}(\underset{\sim}{y}) + \frac{1}{2g_{\underset{\sim}{x}}}\,\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, g_{\underset{\sim}{x}} - \frac{1}{2g_{\underset{\sim}{y}}}\,\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\, g_{\underset{\sim}{y}}$.

<u>Proof</u> $\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}}\cdot(i_{\underset{\sim}{x}}\, i_{\underset{\sim}{x}}^{-1}\, \underset{\sim}{a})$

$$= (i_{\underset{\sim}{x}}^{-1}\underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot i_{\underset{\sim}{x}}) + (i_{\underset{\sim}{x}}\, \nabla_{\underset{\sim}{x}})\cdot(\underset{\sim}{a}\, i_{\underset{\sim}{x}}^{-1})\ .$$

But by a version of lemma 12.2(ii),

$$(\underset{\sim}{i}_{\underset{\sim}{x}}^{-1}\ \underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\cdot \underset{\sim}{i}_{\underset{\sim}{x}}) = \frac{1}{2g_{\underset{\sim}{x}}}\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\ g_{\underset{\sim}{x}}\ .$$

On the other hand,

$$(i_{\underset{\sim}{x}}\ \nabla_{\underset{\sim}{x}})\cdot(\underset{\sim}{a}\ i_{\underset{\sim}{x}}^{-1}) = i_{\underset{\sim}{x}}\cdot[\nabla_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot y^{\dagger}\ i_{\underset{\sim}{y}}^{-1})]$$

theorem 3.5(ii), eqn. (9.1) $\quad = i_{\underset{\sim}{x}}\cdot\{y^{\dagger}\nabla_{\underset{\sim}{y}} \wedge y^{\dagger}[(y_{+}\underset{\sim}{a})\cdot i_{\underset{\sim}{y}}^{-1}]\}$

theorem 9.2, cor. 3.6 $\quad = i_{\underset{\sim}{y}}\cdot[\nabla_{\underset{\sim}{y}} \wedge (\underset{\sim}{b}\cdot i_{\underset{\sim}{y}}^{-1})]$

identity 0.43 $\quad = \nabla_{\underset{\sim}{y}}\cdot b(\underset{\sim}{y}) + i_{\underset{\sim}{y}}\cdot(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\ i_{\underset{\sim}{y}}^{-1})$

version of lemma 12.2(i) $\quad = \nabla_{\underset{\sim}{y}}\cdot\underset{\sim}{b}(\underset{\sim}{y}) - \frac{1}{2g_{\underset{\sim}{y}}}\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{y}}\ g_{\underset{\sim}{y}}\ .$

Taking the sum of the last two expressions completes the proof of the theorem.

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By an easy computation, using the chain rule and the identity

$$|J_{\bar{y}_m}(\underset{\sim}{x})| \equiv |i_{\underset{\sim}{x}}^{-1}|\ |i_{\underset{\sim}{y}}| = \frac{\sqrt{g_{\underset{\sim}{y}}}}{\sqrt{g_{\underset{\sim}{x}}}}\quad ,$$

theorem 12.3 is equivalent to:

<u>Theorem 12.4</u> $\quad \nabla_{\underset{\sim}{x}}\cdot a(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}}\cdot b(\underset{\sim}{y}) - \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\ |J_{\bar{y}_m}(\underset{\sim}{x})|$, where $\underset{\sim}{b}(\underset{\sim}{y}) = y_{+}\ \underset{\sim}{a}(\underset{\sim}{x})$.

A trivial but important consequence of theorem 12.4 is:

<u>Corollary 12.5</u> $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$ for each $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ iff $|J_{\bar{y}_m}(\underset{\sim}{x})|$ is constant.

Note that $\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$ in corollary 12.5 can be written as $(y^{\dagger}\ \nabla_{\underset{\sim}{y}}) \cdot \underset{\sim}{a} = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}$. This is a statement of corollary 3.6 for the gradient operator. It is now shown that under the same conditions as in corollary 12.5, theorem 3.5(i) can be applied to the gradient operator to get:

<u>Theorem 12.6</u> $y_{\dagger}[\nabla_{\underset{\sim}{x}} \cdot A_r] = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\ A_r$ for each $r \geq 1$ and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^r_{\underset{\sim}{x}}$, iff $|J_{\bar{y}_m}|$ is constant.

<u>Proof</u> Because of corollary 12.5 it is sufficient to show that $y_{\dagger}\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{a}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\underset{\sim}{a}(\underset{\sim}{x})$ for each $\underset{\sim}{a}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$ implies that $y_{\dagger}\nabla_{\underset{\sim}{x}} \cdot A_r = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}A_r$ for each $r \geq 1$, and $A_r(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}^r_{\underset{\sim}{x}}$.

The proof of this is by induction on $r \geq m$. For $r = 1$ there is nothing to prove. Now suppose for $r = i$ that $y_{\dagger}\ \nabla_{\underset{\sim}{x}} \cdot A_i = \nabla_{\underset{\sim}{y}} \cdot y_{\dagger}\ A_i$, and for $r = i + 1$ write $A_{i+1} = \underset{\sim}{a} \wedge A_i$, where $\underset{\sim}{a}, A_i \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$. Then:

$$\nabla_{\underset{\sim}{y}}\cdot y_{\dagger} A_{i+1} = \nabla_{\underset{\sim}{y}}\cdot y_{\dagger}(\underset{\sim}{a}\wedge A_i)$$

theorem 3.3(i) $\quad = \nabla_{\underset{\sim}{y}}\cdot(y_{\dagger}\underset{\sim}{a} \wedge y_{\dagger}A_i)$

lemma 10.11 $\quad = [\nabla_{\underset{\sim}{y}}\cdot y_{\dagger}\underset{\sim}{a}]y_{\dagger}A_i - (y_{\dagger}\underset{\sim}{a})\wedge(\nabla_{\underset{\sim}{y}}\cdot y_{\dagger}A_i) + [y_{\dagger}\underset{\sim}{a}, y_{\dagger}A_i]_{\underset{\sim}{y}}$

lemma 10.15 $\quad = (\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a})y_{\dagger}A_i - y_{\dagger}\underset{\sim}{a} \wedge y_{\dagger}\nabla_{\underset{\sim}{x}}\cdot A_i + y_{\dagger}[\underset{\sim}{a},A_i]_{\underset{\sim}{x}}$

theorem 3.3(i) $\quad = y_{\dagger}\{(\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{a})A_i - \underset{\sim}{a}\wedge(\nabla_{\underset{\sim}{x}}\cdot A_i) + [\underset{\sim}{a},A_i]\}$

lemma 10.11 $\quad = y_{\dagger}\ \nabla_{\underset{\sim}{x}}\cdot(\underset{\sim}{a}\wedge A_i) = y_{\dagger}\ \nabla_{\underset{\sim}{x}}\cdot A_{i+1}$.

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Note that any linear mapping $y\ (\underset{\sim}{x})$ satisfies the conditions of the last theorem.

Now let $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$ be a coordinate frame on $\mathcal{X}_m$ and $\{\underset{\sim}{e}_i(\underset{\sim}{x})\}$ its reciprocal frame. For the remainder of this section let $i(\underset{\sim}{x}) = \underset{\sim}{e}_1(\underset{\sim}{x}) \wedge\ldots\wedge \underset{\sim}{e}_m(\underset{\sim}{x})$ be the pseudoscalar field under consideration.

The pseudoscalar field $i^{-1}(\underset{\sim}{x}) = \underset{\sim}{e}^m(\underset{\sim}{x}) \wedge\ldots\wedge \underset{\sim}{e}^1(\underset{\sim}{x})$ is called the coordinate pseudoscalar field on $\mathcal{X}_m$ with respect to the coordinate frame $\{\underset{\sim}{e}^i(\underset{\sim}{x})\}$.

The following theorem shows that <u>any</u> pseudoscalar field $h(\underset{\sim}{x})$ on $\mathcal{X}_m$ is curl free.

<u>Theorem 12.7</u> $\quad \nabla_{\underset{\sim}{x}}\ h(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}}\cdot h(\underset{\sim}{x})$, where $h(\underset{\sim}{x})$ is any pseudoscalar field on $\mathcal{X}_m$.

<u>Proof</u> The proof hinges on $\nabla_{\underset{\sim}{x}} \wedge i_{\underset{\sim}{x}}^{-1} = 0$, and

$\nabla_{\underset{\sim}{x}} \psi(\underset{\sim}{x}) \in \mathcal{D}_x$ when $\psi(\underset{\sim}{x})$ is a scalar field.

Since $h(\underset{\sim}{x})$ and $i_{\underset{\sim}{x}}^{-1}$ are both pseudoscalar fields on $\mathcal{X}_m$, $h(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\, i_{\underset{\sim}{x}}^{-1}$ for some scalar valued $\psi(\underset{\sim}{x})$. But then

$$\nabla_{\underset{\sim}{x}} \wedge h(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \wedge \psi(\underset{\sim}{x}) i_{\underset{\sim}{x}}^{-1}(\underset{\sim}{x})$$

$$= [\nabla_{\underset{\sim}{x}} \psi(\underset{\sim}{x})] \wedge i_{\underset{\sim}{x}}^{-1} + \psi(\underset{\sim}{x})\, \dot{\nabla}_{\underset{\sim}{x}} \wedge i_{\underset{\sim}{x}}^{-1}$$

theorem 11.5(i) $\qquad = 0$.

Hence $\nabla_{\underset{\sim}{x}} h(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \cdot h(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}} \wedge h(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \cdot h(\underset{\sim}{x})$.

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The theorem given below is well known. For a more usual formulation and proof see [14, p. 130].

<u>Theorem 12.8</u> $\quad \nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{e}_i(\underset{\sim}{x}) = \dfrac{1}{2g_{\underset{\sim}{x}}} \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} g_{\underset{\sim}{x}} = \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \ln \sqrt{g_{\underset{\sim}{x}}}$.

<u>Proof</u> $\quad \underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}\, g = 2g\, (\underset{\sim}{e}_i \nabla_{\underset{\sim}{x}} i_{\underset{\sim}{x}}) \cdot i_{\underset{\sim}{x}}^{-1}$ $\quad$ version of lemma 12.2(i)

$$= 2g\, [\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{e}_1 \wedge \ldots \wedge \underset{\sim}{e}_m] \cdot [\underset{\sim}{e}^m \wedge \ldots \wedge \underset{\sim}{e}^1]$$

$$= 2g \sum_{j=1}^{m} [\underset{\sim}{e}_1 \wedge \ldots \wedge (\underset{\sim}{e}_i \cdot \nabla_x \underset{\sim}{e}_j) \wedge \ldots \wedge \underset{\sim}{e}_m] \cdot [\underset{\sim}{e}^m \wedge \ldots \wedge \underset{\sim}{e}^1]$$

identity 0.42 $\qquad = 2g \sum_{j=1}^{m} (\underset{\sim}{e}_i \cdot \nabla_x \underset{\sim}{e}_j) \cdot \underset{\sim}{e}^j$

theorem 11.5(ii) $= 2g \sum_{j=1}^{m} e^j \cdot (e_j \cdot \nabla_x e_i)$

theorem 11.6 $= 2g \nabla_x \cdot e_i(x)$.

Thus $e_i \cdot \nabla_x g = 2g_x \nabla_x \cdot e_i(x)$. The remainder of the proof is trivial.

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Theorem 12.9 The following statements are equivalent:

(i) g_x is constant.

(ii) $\nabla_x \cdot e_i(x) = 0$ for each $i \leq m$.

(iii) $\nabla_x \cdot i_x \equiv 0$.

Proof That (i) $\leftrightarrows$ (ii) is a direct consequence of theorem 12.8.

That (i) $\leftrightarrows$ (iii) is a direct consequence of a version of lemma 12.2(ii).

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13. The Shape Operator

The results of the preceding sections require only that the surfaces under consideration be sufficiently smooth. The shapes of the surfaces have not been a point of interest. In this final section, a "shape operator" is defined, which provides a measure of the shape of a surface.

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$, and let $p_{\underset{\sim}{x}} \equiv p(\underset{\sim}{x})$ be a unit pseudoscalar field on $\mathcal{X}_m$.

Definition 13.1 Call $S(\underset{\sim}{a}) \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, p(\underset{\sim}{x})$ for each $\underset{\sim}{a} \in \mathcal{D}^1_{\underset{\sim}{x}}$, the shape operator of the surface $\mathcal{X}_m$ with respect to the pseudoscalar field $p_{\underset{\sim}{x}}$.

(Note that a unit pseudoscalar field on a surface is unique up to an orientation.)

A few basic properties of the shape operator are given in the following theorem.

Let $\underset{\sim}{a}(\underset{\sim}{x})$, $\underset{\sim}{b}(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$.

Theorem 13.2

(i) $S(\underset{\sim}{a}) \wedge \underset{\sim}{b} = -\, p_{\underset{\sim}{x}} \wedge [\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}]$.

(ii) $S(\underset{\sim}{a}) \wedge \underset{\sim}{b} = S(\underset{\sim}{b}) \wedge \underset{\sim}{a}$.

(iii) $S(\underset{\sim}{a})\cdot p_{\underset{\sim}{x}} = 0$.

(iv) $\nabla_{\bar{x}_2}\, p(\underset{\sim}{x}_1)\cdot p(\underset{\sim}{x}_2) = 0$.

Proof (i) Since $\underset{\sim}{b} \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, $p_{\underset{\sim}{x}} \wedge b = 0$. Thus,

$$0 = \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, p_{\underset{\sim}{x}}\wedge\underset{\sim}{b}$$

$$= (\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, p_{\underset{\sim}{x}})\wedge\underset{\sim}{b} + p_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b})$$

$$= S(\underset{\sim}{a})\wedge\underset{\sim}{b} + p_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b})$$

Hence $S(\underset{\sim}{a}) \wedge \underset{\sim}{b} = -\, p_{\underset{\sim}{x}} \wedge [\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}]$.

(ii) $$S(\underset{\sim}{a}) \wedge \underset{\sim}{b} - S(\underset{\sim}{b}) \wedge \underset{\sim}{a} = -\, p_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}) + p_{\underset{\sim}{x}}\wedge(\underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{a})$$

$$= -\, p_{\underset{\sim}{x}}\wedge(\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\underset{\sim}{b} - \underset{\sim}{b}\cdot\nabla_{\underset{\sim}{x}}\underset{\sim}{a})$$

$$= -\, p_{\underset{\sim}{x}}\wedge[\underset{\sim}{a},\underset{\sim}{b}] = 0 ,$$

since by lemma 10.7, $[a,b] \in \mathcal{D}^{1}_{\underset{\sim}{x}}$. This is sufficient to prove (ii)

(iii) follows easily by differentiating the equation

$p_{\underset{\sim}{x}}\cdot p_{\underset{\sim}{x}} = \pm 1$ by $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}$.

(iv) The proof of (iv) is similar to the proof of lemma 2.21(i), and is omitted.

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The shape operator $S(\underset{\sim}{a})$ is a generalization of the Weingarten mapping to surfaces which are not necessarily hypersurfaces. See [12, p. 21, 77]. The Weingarten mapping is further discussed in Appendix E.