APPENDICES

Appendix A. Linear Mappings

In this appendix basic ideas of linear algebra are reformulated in terms of the geometric language developed in this paper. The use of geometric algebra in the study of linear mappings makes the introduction of matrix algebra largely unnecessary.

Let $y\colon \mathcal{E}_n \to \mathcal{E}_n$ be a linear mapping from $\mathcal{E}_n$ into $\mathcal{E}_n$. Since the mapping $y(\underset{\sim}{x})$ is from $\mathcal{E}_n$ into $\mathcal{E}_n$, $\mathcal{D}_{\underset{\sim}{x}} = \mathcal{D} = \mathcal{D}_{y(\underset{\sim}{x})}$ for all $\underset{\sim}{x} \in \mathcal{E}_n$. Also $\mathcal{E}_n = \mathcal{D}^1$, i.e., vectors which are names for points in $\mathcal{E}_n$ are identified with tangent vectors of $\mathcal{D}^1$.

In this appendix the mapping $y(\underset{\sim}{x})$ is always taken to be linear.

a) Basic Definitions and Properties

Definition A.1 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be linear, provided for all scalars α, β, and points $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$,

$y(\alpha\underset{\sim}{x}_1+\beta\underset{\sim}{x}_2) = \alpha y(\underset{\sim}{x}_1) + \beta y(\underset{\sim}{x}_2)$.

Definition A.2 The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be symmetric if for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$, $y(\underset{\sim}{x}_1)\cdot\underset{\sim}{x}_2 = \underset{\sim}{x}_1\cdot y(\underset{\sim}{x}_2)$.

<u>Definition A.3</u> The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be skew-symmetric if for all $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2 \in \mathcal{E}_n$, $y(\underset{\sim}{x}_1)\cdot\underset{\sim}{x}_2 = -\underset{\sim}{x}_1\cdot y(\underset{\sim}{x}_2)$.

The following theorem shows that a linear mapping is equivalent to its differential mapping at each point $\underset{\sim}{x} \in \mathcal{E}_n$.

<u>Theorem A.4</u> If $y_\dagger$ is the differential mapping of $y(\underset{\sim}{x})$ at any point $\underset{\sim}{x} \in \mathcal{E}_n$, then $y_\dagger(\underset{\sim}{a}) = y(\underset{\sim}{a})$ for all $\underset{\sim}{a} \in \mathcal{D}^1_{\underset{\sim}{x}} = \mathcal{E}_n$

<u>Proof</u> $y_\dagger\underset{\sim}{a} = \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$

def. 2.7 $$= |\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \frac{y(\underset{\sim}{x}+\Delta\underset{\sim}{x}) - y(\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$$

$$= |\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \frac{y(\Delta\underset{\sim}{x})}{|\Delta\underset{\sim}{x}|}$$

$$= y\left(|\underset{\sim}{a}| \lim_{\Delta\underset{\sim}{x}\to 0} \frac{\Delta\underset{\sim}{x}}{|\Delta\underset{\sim}{x}|} \right)$$

def. 2.7 $$= y\,[|\underset{\sim}{a}|\,\hat{\underset{\sim}{a}}] = y(\underset{\sim}{a})$$

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<u>Lemma A.5</u>

(i) $\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x}\cdot y(\underset{\sim}{x}) = y_\dagger\underset{\sim}{x} + y^\dagger\underset{\sim}{x}$

(ii) $\underset{\sim}{x}\cdot[\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)] = y_\dagger\underset{\sim}{x} - y^\dagger\underset{\sim}{x}$.

<u>Proof</u> (i) $\nabla_{\underset{\sim}{x}} \underset{\sim}{x} \cdot y(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}_1} \underset{\sim}{x}_1 \cdot y(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1} y(\underset{\sim}{x}_1) \cdot \underset{\sim}{x}$

$$= y_\dagger \underset{\sim}{x} + y^\dagger \underset{\sim}{x}$$

(ii) $\underset{\sim}{x} \cdot [\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)] = \underset{\sim}{x} \cdot \nabla_{\underset{\sim}{x}_1} y(\underset{\sim}{x}_1) - \nabla_{\underset{\sim}{x}_1} y(\underset{\sim}{x}_1) \cdot \underset{\sim}{x}$ identity 0.39

$$= y_\dagger \underset{\sim}{x} - y^\dagger \underset{\sim}{x}$$

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<u>Theorem A.6</u> The following statements are equivalent:

(i) $y(\underset{\sim}{x})$ is symmetric

(ii) $y_\dagger \equiv y^\dagger$

(iii) $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

<u>Proof</u> It is shown that (i) $\rightleftarrows$ (ii) $\rightleftarrows$ (iii) .

(i) $\rightleftarrows$ (ii). If $y(\underset{\sim}{x})$ is symmetric, then for all $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2 \in \mathcal{E}_n$,

$$y(\underset{\sim}{x}_1) \cdot \underset{\sim}{x}_2 = \underset{\sim}{x}_1 \cdot y(\underset{\sim}{x}_2)$$

theorem A.4 $= \underset{\sim}{x}_1 \cdot y_\dagger \underset{\sim}{x}_2$

cor. 3.6 $= (y^\dagger \underset{\sim}{x}_1) \cdot \underset{\sim}{x}_2$

This implies $y_\dagger \underset{\sim}{x}_1 = y^\dagger \underset{\sim}{x}_1$ for all $\underset{\sim}{x}_1 \in \mathcal{E}_n$.

Since the above argument can be reversed, the proof of this part is complete.

(ii) $\Leftrightarrow$ (iii) follows trivially from lemma A.5(ii).

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Corollary A.7 (i) If $y(\underset{\sim}{x})$ and $w(\underset{\sim}{x})$ are symmetric linear mappings, and $y \cdot w \equiv w \cdot y$, then $y \circ w$ is symmetric.

(ii) If $y(\underset{\sim}{x})$ is symmetric, then $y^{i}(\underset{\sim}{x}) \equiv y \overset{1}{\cdot} \ldots \overset{i-1}{\circ} y(\underset{\sim}{x})$ is symmetric.

Proof (i) Because of theorem A.6(ii) it is sufficient to show that $(y \circ w)_{\dagger} = (y \circ w)^{\dagger}$.

$$(y \circ w)_{\dagger} = y_{\dagger} \circ w_{\dagger} \qquad \text{theorem 4.2(i)}$$

theorem A.4 $\qquad = w_{\dagger} \circ y_{\dagger}$

theorem A.6(ii) $\qquad = w^{\dagger} \circ y^{\dagger}$

theorem 4.2(ii) $\qquad = (y \circ w)^{\dagger}$

(ii) is a trivial consequence of (i).

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The analogous theorem to A.6 for skew-symmetric mappings is:

Theorem A.8 The following statements are equivalent:

(i) $y(\underset{\sim}{x})$ is skew-symmetric

(ii) $y_{\dagger} \equiv -y^{\dagger}$

(iii) $\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x} \cdot y(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

The next theorem decomposes $y(\underset{\sim}{x})$ into the sum of a symmetric mapping and a skew-symmetric mapping. Its proof is an easy consequence of the preceding theorems, and is omitted.

<u>Theorem A.9</u> $y(\underset{\sim}{x}) = \frac{1}{2}\nabla_{\underset{\sim}{x}}\, \underset{\sim}{x}\cdot y(\underset{\sim}{x}) + \frac{1}{2}\underset{\sim}{x}\cdot[\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]$, where the first term on the right is symmetric and the second term skew-symmetric.

<u>Definition A.10</u> Call $\nabla_{\underset{\sim}{x}}\cdot y(\underset{\sim}{x})$ the trace of $y(\underset{\sim}{x})$.

This is equivalent to the definition of trace in matrix theory.

<u>Theorem A.11</u> If $y(\underset{\sim}{x})$ is skew-symmetric, then $\nabla_{\underset{\sim}{x}}\cdot y(\underset{\sim}{x}) = 0$.

<u>Proof</u> By theorem A.9, $y(\underset{\sim}{x}) = \frac{1}{2}\underset{\sim}{x}\cdot[\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]$. Thus,

$$\nabla_{\underset{\sim}{x}}\cdot y(\underset{\sim}{x}) = \frac{1}{2}\ \nabla_{\underset{\sim}{x}}\cdot\{\underset{\sim}{x}\cdot[\nabla_{\underset{\sim}{x}_1} \wedge y(\underset{\sim}{x}_1)]\}$$

identity 0.42 $\qquad = \frac{1}{2}[\nabla_{\underset{\sim}{x}} \wedge \underset{\sim}{x}]\cdot[\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x})]$

cor. 7.2 $\qquad = 0.$

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<u>Definition A.12</u> The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ is said to be orthogonal if for each $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$, $y(\underset{\sim}{x}_1)\cdot y(\underset{\sim}{x}_2) = \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2$.

<u>Theorem A.13</u> The following statements are equivalent:

(i) $y(\underset{\sim}{x})$ is orthogonal.

(ii) $y^{\dagger}(y_{\dagger}\underset{\sim}{x}) = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in \mathcal{E}_n^1$.

(iii) $(y_{\dagger}\underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

<u>Proof</u> It will be shown that (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (i) .

(i) $\rightarrow$ (ii) If $y(\underset{\sim}{x}_1)\cdot y(\underset{\sim}{x}_2) = \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2$ for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$, then by using theorem A.4 and corollary 3.6, $\underset{\sim}{x}_1\cdot y^{\dagger}y_{\dagger}\underset{\sim}{x}_2 = \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2$ for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$. This implies $y^{\dagger}(y_{\dagger}\underset{\sim}{x}_2) = \underset{\sim}{x}_2$ for all $\underset{\sim}{x}_2 \in \mathcal{E}_n$.

(ii) $\rightarrow$ (iii) If $y^{\dagger}y_{\dagger}\underset{\sim}{x} = \underset{\sim}{x}$ for all $\underset{\sim}{x} \in \mathcal{E}_n$, then

$$\underset{\sim}{x}^2 = \underset{\sim}{x}\cdot y^{\dagger}y_{\dagger}\underset{\sim}{x}$$

cor. 3.6 $$= y_{\dagger}\underset{\sim}{x}\cdot y_{\dagger}\underset{\sim}{x} \; .$$

Hence, $(y_{\dagger}\underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$.

(iii) $\rightarrow$ (i) If $(y_{\dagger}\underset{\sim}{x})^2 = \underset{\sim}{x}^2$ for all $\underset{\sim}{x} \in \mathcal{E}_n$, then

$$\begin{aligned} y_{\dagger}\underset{\sim}{x}_1\cdot y_{\dagger}\underset{\sim}{x}_2 &\equiv \frac{1}{2}\{[y_{\dagger}(\underset{\sim}{x}_1+\underset{\sim}{x}_2)]^2 - (y_{\dagger}\underset{\sim}{x}_1)^2 - (y_{\dagger}\underset{\sim}{x}_2)^2\} \\ &= \frac{1}{2}\{(\underset{\sim}{x}_1+\underset{\sim}{x}_2)^2 - \underset{\sim}{x}_1^2 - \underset{\sim}{x}_2^2\} \\ &= \underset{\sim}{x}_1\cdot\underset{\sim}{x}_2 \; , \end{aligned}$$

for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{E}_n$.

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b) The Characteristic Polynomial

Definition A.14 For the linear mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$, let $c(\underset{\sim}{x}) = y(\underset{\sim}{x}) - \lambda\underset{\sim}{x}$ for each $\underset{\sim}{x} \in \mathcal{E}_n$, and where λ is a scalar. Then the characteristic polynomial of $\underset{\sim}{y} = y(\underset{\sim}{x})$ in the variable λ is $\Psi(\lambda) \equiv J_{\bar{c}_n}$, and its characteristic equation in the variable λ is $\Psi(\lambda) = 0$.

Theorem A.15 $\Psi(\lambda) = \sum_{i=0}^{n} (-1)^i [J_{\bar{y}_{n-i}}]_0 \lambda^i$.

Proof In theorem 4.3(ii) let $g(\underset{\sim}{x}) = y(\underset{\sim}{x})$, and $h(\underset{\sim}{x}) = -\lambda\underset{\sim}{x}$. Then $\Psi(\lambda) \equiv J_{\bar{c}_n}$

theorem 4.3(ii) $= \sum_{i=0}^{n} \nabla_{\bar{x}_i} \Lambda \nabla_{\bar{x}_{n-i}} \bar{y}_{n-i} \Lambda \overline{(-\lambda x)}_i$

$= \sum_{i=0}^{n} (-1)^i \lambda^i \nabla_{\bar{x}_{n-i}} \Lambda \nabla_{\bar{x}_i} \bar{x}_i \Lambda \bar{y}_{n-i}$

identity 0.42 $= \sum_{i=0}^{n} (-1)^i \lambda^i \nabla_{\bar{x}_{n-i}} \cdot [\nabla_{\bar{x}_i} \bar{x}_i \Lambda \bar{y}_{n-i}]$

theorem 7.4(ii) $= \sum_{i=0}^{n} (-1)^i \lambda^i \nabla_{\bar{x}_{n-i}} \cdot \bar{y}_{n-i}$.

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Theorem A.15 identifies the scalar parts of the characteristic multivectors of a mapping as being the coefficients or scalar invariants of its characteristic equation. By theorem 6.9, these are the only parts when $y(\underset{\sim}{x})$ is symmetric.

<u>Theorem A.16</u> A linear mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ satisfies its characteristic equation.

<u>Proof</u> It must be shown that $\nabla_{\bar{x}_n} \cdot \bar{y}_n \underset{\sim}{x} - \nabla_{\bar{x}_{n-1}} \cdot \bar{y}_{n-1} y(\underset{\sim}{x}) + \ldots + (-1)^n y^n(\underset{\sim}{x}) \equiv 0$ for all $\underset{\sim}{x} \in \mathcal{E}_n$. The first term on the left will be decomposed into the negative of the remaining terms.

$$\nabla_{\bar{x}_n} \cdot \bar{y}_n \underset{\sim}{x} = \underset{\sim}{x}\, \nabla_{\bar{x}_n} \cdot \bar{y}_n$$

$$= \underset{\sim}{x} \cdot \nabla_{\bar{x}_n} \bar{y}_n$$

theorem 3.4(i) $$= \nabla_{\bar{x}_{n-1}} \ddot{y}_{n-1} \wedge y_{+}\underset{\sim}{x}$$

theorem A.4 $$= \nabla_{\bar{x}_{n-1}} \cdot [\bar{y}_{n-1} \wedge y(\underset{\sim}{x})]$$

[11, p.13, 3.12] $$= \nabla_{\bar{x}_{n-1}} \cdot \bar{y}_{n-1}\, y(\underset{\sim}{x}) - \frac{1}{(n-1)!} \nabla_{\bar{x}_{n-1}} \cdot (y_2 \wedge \ldots \wedge y_{n-1} \wedge y)\, y_1$$

$$\vdots$$

$$= \nabla_{\bar{x}_{n-1}} \cdot \bar{y}_{n-1} y(\underset{\sim}{x}) - \nabla_{\bar{x}_{n-2}} \cdot \bar{y}_{n-2} y^2(\underset{\sim}{x}) + \ldots + (-1)^{n-1} y^n(\underset{\sim}{x}) \,.$$

The last steps follow by expanding $[\nabla_{\bar{x}_{n-1}} \cdot (\underset{\sim}{y}_2 \wedge \ldots \wedge \underset{\sim}{y}_{n-1} \wedge \underset{\sim}{y})]\underset{\sim}{y}_1$, and making repeated use of the fact that $y^i(\underset{\sim}{x}) \cdot \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x}) = y^{i+1}(\underset{\sim}{x})$.

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A known result in matrix theory is that the scalar invariants of the characteristic polynomial of a matrix can be expressed in terms of traces of powers of the matrix. The final theorem of this section shows the equivalent by a recursive decomposition of $[J_{\bar{y}_r}]_0$.

<u>Theorem A.17</u> For the mapping $y(\underset{\sim}{x})$, $[J_{\bar{y}_1}]_0 \equiv \nabla_{\underset{\sim}{x}} \cdot y$, and $[J_{\bar{y}_r}]_0 = \frac{1}{r}\{\nabla_{\underset{\sim}{x}} \cdot y[J_{\bar{y}_{r-1}}]_0 - \nabla_{\underset{\sim}{x}} \cdot y^2[J_{\bar{y}_{r-2}}]_0 + \ldots + (-1)^{r+1}\nabla_{\underset{\sim}{x}} \cdot y^r\}$, for $r \leq n$.

<u>Proof</u> $[J_{\bar{y}_r}]_0 = \nabla_{\bar{x}_r} \cdot \bar{y}_r$

$$= \frac{1}{r}(\nabla_{\bar{x}_{r-1}} \wedge \nabla_{\underset{\sim}{x}}) \cdot (y \wedge \bar{y}_{r-1})$$

identity 0.38
$$= \frac{1}{r}\{\nabla_{\underset{\sim}{x}} \cdot y\ \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - [(\nabla_{\bar{x}_{r-1}} \cdot y) \wedge \nabla_{\underset{\sim}{x}}^{\dagger}] \cdot \bar{y}_{r-1}\}$$

identity 0.40
$$= \frac{1}{r}\{\nabla_{\underset{\sim}{x}} \cdot y\ \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - (\nabla_{\bar{x}_{r-2}} \wedge \nabla_{\underset{\sim}{x}}) \cdot (y^2 \wedge \bar{y}_{r-2})\}$$

$$\vdots$$

$$= \frac{1}{r}\{\nabla_{\underset{\sim}{x}} \cdot y\ \nabla_{\bar{x}_{r-1}} \cdot \bar{y}_{r-1} - \nabla_{x} \cdot y^2\ \nabla_{\bar{x}_{r-2}} \cdot \bar{y}_{r-2} + \ldots + (-1)^{r+1}\ \nabla_{\underset{\sim}{x}} \cdot y^r\} \quad .$$

The proof is now complete.

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The proof of this theorem is almost identical to that of theorem 6.7.

c) Invariant Linear Subspaces

<u>Definition A.18</u> A point set $\mathcal{J}_r \subset \mathcal{E}_n$ is called an r-plane of $\mathcal{E}_n$ through the origin if there is a simple r-vector $I_r \varepsilon \mathcal{D}^r$ such that,

$$\mathcal{J}_r = \mathcal{J}_r(I_r) \equiv \{\underset{\sim}{x} \varepsilon \mathcal{E}_n \mid \underset{\sim}{x} \wedge I_r = 0\} .$$

The simple r-vector I_r is said to define the r-plane $\mathcal{J}(I_r)$.

As a "flat" r-surface in $\mathcal{E}_n$, as defined in section 2, $\mathcal{J}_r$ has a geometric algebra $\mathcal{D}_{\mathcal{J}_r}$ of 2^r-dimensions. The r-vector I_r is a pseudoscalar of $\mathcal{D}_{\mathcal{J}_r}$, and as such is unique up to a scalar multiple. The r-plane $\mathcal{J}_r$ can also be regarded as a linear subspace of $\mathcal{E}_n$.

<u>Definition A.19</u> Let I_r be a simple r-vector, and λ a scalar. If $\underset{\sim}{y}_+ I_r = \lambda I_r$, then I_r is said to be an invariant r-vector, and λ an r-value of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. If in

addition $\lambda \neq 0$, then λ is called a proper r-value and I_r is called a proper invariant r-vector.

Invariant 1-vectors and 1-values are also called eigenvectors and eigenvalues.

The following theorem gives the relationships between invariant r-vectors, and invariant linear subspaces of a linear mapping $y(\underset{\sim}{x})$.

Theorem A.20

(i) If $\mathcal{J}_r$ is an invariant linear subspace, and $\mathcal{J}_r = \mathcal{J}(I_r)$, then I_r is an invariant r-vector.

(ii) If I_r is a proper invariant r-vector, then $\mathcal{J}_r \equiv \mathcal{J}_r(I_r)$ is an invariant subspace, and the mapping $y(\underset{\sim}{x})$ when restricted to $\mathcal{J}_r$ is non-singular.

Proof (i) Since I_r is a pseudoscalar element of $\mathcal{D}_{\mathcal{J}_r}$, there are vectors $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r \in \mathcal{J}_r$ such that $I_r = \underset{\sim}{x}_1 \wedge \ldots \wedge \underset{\sim}{x}_r$. Thus,

$$y_{\dagger} I_r = y_{\dagger}(\underset{\sim}{x}_1 \wedge \ldots \wedge \underset{\sim}{x}_r)$$

theorem 3.3(i) $$= y_{\dagger}\underset{\sim}{x}_1 \wedge \ldots \wedge y_{\dagger}\underset{\sim}{x}_r$$

theorem A.4 $\qquad = y(\underset{\sim}{x}_1) \wedge \ldots \wedge y(\underset{\sim}{x}_r)$

$$= \lambda I_r ,$$

for some scalar λ, since $y(\underset{\sim}{x}_i) \in \mathcal{J}_r$ for each i, and pseudo-scalar elements are unique up to a scalar multiple.

(ii) Let $\underset{\sim}{x} \in \mathcal{J}_r$. It must be shown that $y(\underset{\sim}{x}) \in \mathcal{J}_r$, or equivalently that $y(\underset{\sim}{x}) \wedge I_r = 0$. Since $\underset{\sim}{x} \in \mathcal{J}_r$, $\underset{\sim}{x} \wedge I_r = 0$, and thus

$$0 = y_{\dagger} (\underset{\sim}{x} \wedge I_r)$$

theorem 3.3(i) $\qquad = y_{\dagger}\underset{\sim}{x} \wedge y_{\dagger} I_r$

theorem A.4 $\qquad = y(\underset{\sim}{x}) \wedge y_{\dagger} I_r$.

But since $y_{\dagger} I_r = \lambda I_r$ where $\lambda \neq 0$, it follows that $y(\underset{\sim}{x}) \wedge I_r = 0$.

Finally the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to $\mathcal{J}_r$ is non-singular because,

$$J_{\bar{y}_{\mathcal{J}_r}} \equiv I_r^{-1} \, I_r \cdot \nabla_{\bar{x}_r} \, \bar{y}_r$$

$$= I_r^{-1} \, y_{\dagger} \, I_r$$

$$= I_r^{-1} \, \lambda I_r$$

$$= \lambda \neq 0 ,$$

since I_r is a proper invariant r-vector. XXXX

The final theorem of this appendix factors the characteristic polynomial of $\underset{\sim}{y} = y(\underset{\sim}{x})$ into the product of characteristic polynomials of $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to invariant linear subspaces.

Let I be a pseudoscalar element of $\mathcal{G}$, the geometric algebra of $\mathcal{E}_n$, and suppose $I = I_{r_1} \wedge \ldots \wedge I_{r_k}$, where I_{r_i} are invariant r_i-vectors of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. Let $\Psi(\lambda)$ be the characteristic polynomial of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$, and let $\Psi_{r_i}(\lambda)$ be the characteristic polynomials of the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ when restricted to the invariant subspaces $\mathcal{L}(I_{r_i})$.

<u>Theorem A.21</u> $\Psi(\lambda) = \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda)$

<u>Proof</u> Let $c(\underset{\sim}{x}) = y(\underset{\sim}{x}) - \lambda\underset{\sim}{x}$. Then by definition,

$$\Psi(\lambda) = \nabla_{\bar{x}_n} \bar{c}_n$$

$$= I^{-1}\, I \cdot \nabla_{\bar{x}_n} \bar{c}_n$$

$$= I^{-1}(I_{r_1} \wedge \ldots \wedge I_{r_k}) \cdot \nabla_{\bar{x}_n} \bar{c}_n$$

theorem 3.3(i) $$= I^{-1}[(I_{r_1} \cdot \nabla_{\bar{x}_{r_1}} \bar{c}_{r}) \wedge \ldots \wedge (I_{r_k} \cdot \nabla_{\bar{x}_{r_k}} \bar{c}_{r_k})]$$

$$= I^{-1}[(I_{r_1} I_{r_1}^{-1} I_{r_1} \cdot \nabla_{\bar{x}_{r_1}} \bar{c}_{r_1}) \wedge$$

$$\ldots \wedge (I_{r_k} I_{r_k}^{-1} I_{r_k} \cdot \nabla_{\bar{x}_{r_k}} \bar{c}_{r_k})]$$

$$= I^{-1}[I_{r_1} \Psi_{r_1}(\lambda) \wedge \ldots \wedge I_{r_k} \Psi_{r_k}(\lambda)]$$

$$= I^{-1} I \, \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda)$$

$$= \Psi_{r_1}(\lambda) \ldots \Psi_{r_k}(\lambda) .$$

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Appendix B. Jacobians and Transformations of Integrals

The purpose of this appendix is to show how the methods of this paper can be used to derive formulas from advanced calculus relating to Jacobians and transformations of integrals. In part (a) the relation of the characteristic multivector $\mathcal{J}_{\bar{y}_m}$ to the Jacobian is discussed. In part (b) differential statements proved in Part II are rewritten as transformation formulas for integrals.

a) The Jacobian of a Mapping

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be a mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ in $\mathcal{E}_n$. The following is a more general definition of the Jacobian than is given in advanced calculus books.

<u>Definition B.1</u> Call $J_{\bar{y}_m}(\underset{\sim}{x})$ the Jacobian of the mapping $y(\underset{\sim}{x})$ at the point $\underset{\sim}{x}$.

It will be shown below that this definition is equivalent to the usual definition of the Jacobian when $m = n$, i.e., when $\mathcal{X}_m$ and $\mathcal{Y}_m$ are n-surfaces in $\mathcal{E}_n$.

Suppose now that $\mathcal{X}_m$ and $\mathcal{Y}_m$ are n-surfaces in $\mathcal{E}_n$, and let $\{\underset{\sim}{e}_i \mid i = 1, \ldots, n\}$ be a constant orthonormal frame on

$\mathcal{E}_n$. In terms of this frame the mapping $y(\underset{\sim}{x})$ can be written as

$$y(\underset{\sim}{x}) = \sum_{i=1}^{n} y_i(\underset{\sim}{x})\, \underset{\sim}{e}_i \text{ , where } \underset{\sim}{x} = \sum_{i=1}^{n} x_i\, \underset{\sim}{e}_i \ .$$

The following is the usual definition of the Jacobian in terms of the partial derivatives of its components $y_i(\underset{\sim}{x})$. See for example [3, p.139].

<u>Definition B.2</u> The Jacobian of the mapping $y(\underset{\sim}{x})$ is:

$$J_y(\underset{\sim}{x}) \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1}, & \dots, & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1}, & \dots, & \frac{\partial y_n}{\partial x_n} \end{vmatrix} .$$

<u>Theorem B.3</u> When $\mathcal{X}_m$ and $\mathcal{Y}_m$ are n-surfaces in $\mathcal{E}_n$, $J_y(\underset{\sim}{x}) \equiv J_{\bar{y}_n}(\underset{\sim}{x})$.

<u>Proof</u> The proof follows directly from the definition of $J_{\bar{y}_n}(\underset{\sim}{x})$:

$$J_{\bar{y}_n}(\underset{\sim}{x}) \equiv \nabla_{\bar{x}_n} \bar{y}_n$$

property 2.11 $\qquad = (\underset{\sim}{e}_n \wedge \dots \wedge \underset{\sim}{e}_1)\,(\underset{\sim}{e}_1 \wedge \dots \wedge \underset{\sim}{e}_n)\cdot\nabla_{\bar{x}_n} \bar{y}_n$

theorem 3.3(i) $= [\underset{\sim}{e}_n \wedge \ldots \wedge \underset{\sim}{e}_1] \cdot [(\underset{\sim}{e}_1 \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y}) \wedge \ldots \wedge (\underset{\sim}{e}_n \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y})]$

identity 0.41
$$= \begin{vmatrix} \underset{\sim}{e}_1 \cdot (\underset{\sim}{e}_1 \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y}) , & \ldots , & \underset{\sim}{e}_n \cdot (\underset{\sim}{e}_1 \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y}) \\ \vdots & & \vdots \\ \underset{\sim}{e}_1 \cdot (\underset{\sim}{e}_n \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y}) , & \ldots , & \underset{\sim}{e}_n \cdot (\underset{\sim}{e}_n \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{y}) \end{vmatrix}$$

$$= J_y(\underset{\sim}{x}) ,$$

since $\underset{\sim}{e}_j \cdot [\underset{\sim}{e}_i \cdot \nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})] = \underset{\sim}{e}_j \cdot [\sum_k \frac{\partial y_k}{\partial x_i} \underset{\sim}{e}_k] = \sum_k \frac{\partial y_k}{\partial x_i} \delta_{jk} = \frac{\partial y_j}{\partial x_i}$

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The key to the interpretation of $J_{\bar{y}_m}(\underset{\sim}{x})$ is the following identity: (for $m \leq n$)

$$\text{(B.4)} \quad J_{\bar{y}_m}(\underset{\sim}{x}) = i_{\underset{\sim}{x}}^{-1}\, i_{\underset{\sim}{x}} \cdot \nabla_{\bar{x}_m} \bar{y}_m = i_{\underset{\sim}{x}}^{-1}\, i_{\underset{\sim}{y}} ,$$

where $i_{\underset{\sim}{x}}$ is a directed volume element of the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, and $i_{\underset{\sim}{y}} = y_+ i_{\underset{\sim}{x}}$ is the corresponding directed volume element of the surface $\mathcal{Y}_m$ at the point $\underset{\sim}{y} = y(\underset{\sim}{x})$.

In words (B.4) says that the Jacobian of the mapping $y(\underset{\sim}{x})$ is the ratio of corresponding directed volume elements on the surfaces.

Finally note that

$$\text{(B.5)} \qquad |J_{\bar{y}_m}(\underset{\sim}{x})| = |i_{\underset{\sim}{x}}^{-1}\, i_{\underset{\sim}{y}}| = \frac{|i_{\underset{\sim}{y}}|}{|i_{\underset{\sim}{x}}|} = \frac{\sqrt{g_{\underset{\sim}{y}}}}{\sqrt{g_{\underset{\sim}{x}}}} ,$$

is the ratio of the magnitudes of corresponding volume elements on the surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

Considering the surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ to be embedded in $\mathcal{E}_n$ allows not only the comparison in magnitudes (B.5), but a comparison in directions as well (B.4).

b) Integral Transformations

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be an invertible mapping between the m-surfaces $\mathcal{X}_m$ and $\mathcal{Y}_m$ in $\mathcal{E}_n$, and let $F(\underset{\sim}{y})$ be a multivector field on $\mathcal{Y}_m$. (Note that it is not required that $F(\underset{\sim}{y})$ be a tangent multivector field on $\mathcal{Y}_m$.)

Property 2.12 is a differential statement of the chain rule. It can also be represented in the following integral form: Let $\mathcal{C}_{\underset{\sim}{x}}$ be any (smooth) curve in $\mathcal{X}_m$, and $\mathcal{C}_{\underset{\sim}{y}}$ the (smooth) curve in $\mathcal{Y}_m$ which is the image of $\mathcal{C}_{\underset{\sim}{x}}$ under the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$.

$$\text{(B.6)} \qquad \int_{\mathcal{C}_{\underset{\sim}{x}}} d\underset{\sim}{x}\cdot\nabla_{\underset{\sim}{x}}\, F[y(\underset{\sim}{x})] = \int_{\mathcal{C}_{\underset{\sim}{y}}} d\underset{\sim}{y}\cdot\nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) \quad ,$$

where $d\underset{\sim}{y} = d\underset{\sim}{x}\cdot\nabla_{\underset{\sim}{x}}\, y(\underset{\sim}{x})$ is the differential vector of arc on the curve $\mathcal{C}_{\underset{\sim}{y}}$ corresponding to $d\underset{\sim}{x}$, the differential vector of arc on the curve $\mathcal{C}_{\underset{\sim}{x}}$. (As a reference, see [5, p.367].)

Now let $\mathcal{A}^r_{\underset{\sim}{x}}$ be an r-subsurface of $\mathcal{X}_m$, and $\mathcal{A}^r_{\underset{\sim}{y}}$ the r-subsurface of $\mathcal{Y}_m$ which is the image of $\mathcal{A}^r_{\underset{\sim}{x}}$ under the mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$. The following integral transformation formulas are integral formulations of differential statements proved in this paper.

<u>Theorem B.7</u> $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} dY_r\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_r\, F[y(\underset{\sim}{x})],$$

where $dY_r \equiv y_{\dagger} dX_r = dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_r$ is the differential r-vector of directed area on the surface $\mathcal{A}^r_{\underset{\sim}{y}}$ corresponding to dX_r, the differential r-vector of directed area on the surface $\mathcal{A}^r_{\underset{\sim}{x}}$.

<u>Corollary B.8</u> $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} |dY_r|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} |dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_r|\, F[y(\underset{\sim}{x})]$$

<u>Corollary B.9</u> $$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} |dY_m|\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} |dX_m|\, |J_{\bar{y}_m}(\underset{\sim}{x})|\, F[y(\underset{\sim}{x})],$$

where $\mathcal{A}^m_{\underset{\sim}{x}}$ and $\mathcal{A}^m_{\underset{\sim}{y}}$ are m-surfaces in $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

Corollary B.9 is a statement of the change of variables formula for integrals found in advanced calculus books. See for example [3, p.273].

<u>Theorem B.10</u> $$\int_{\mathcal{A}^r_{\underset{\sim}{y}}} dY_r \cdot \nabla_{\underset{\sim}{y}}\, F(\underset{\sim}{y}) = \int_{\mathcal{A}^r_{\underset{\sim}{x}}} dX_r \cdot \nabla_{\bar{x}_r}\, \bar{y}_{r-1}\, F[y(\underset{\sim}{x}_r)].$$

Theorem B.10 is an integral statement of theorem 9.4.

<u>Corollary B.11</u> $$\int_{\mathcal{A}^m_{\underset{\sim}{y}}} dY_m \, \nabla_{\underset{\sim}{y}} F(\underset{\sim}{y}) = \int_{\mathcal{A}^m_{\underset{\sim}{x}}} dX_m \, \nabla_{\bar{x}_m} \, \bar{y}_{m-1} \, F[y(\underset{\sim}{x}_m)] \, ,$$

where $\mathcal{A}^m_{\underset{\sim}{x}}$ and $\mathcal{A}^m_{\underset{\sim}{y}}$ are m-surfaces in $\mathcal{X}_m$ and $\mathcal{Y}_m$ respectively.

Corollary B.11 is the integral statement of equation (9.6), the "dual" chain rule for the gradient operator.

Appendix C. Examples of Mappings

This appendix provides explicit calculations for two kinds of mappings. In part (a), mappings are studied which are of the kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$, where $\psi(\underset{\sim}{x})$ is a scalar valued function. In part (b), mappings are studied which are of the kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\underset{\sim}{p}$, where $\psi(\underset{\sim}{x})$ is a scalar valued function, and $\underset{\sim}{p}$ is a constant vector.

a) Mappings of the Kind $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$.

Let $y: \mathcal{X}_m \to \mathcal{Y}_m$ be given by $y(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\underset{\sim}{x}$, where $\psi = \psi(\underset{\sim}{x})$ is a scalar valued function.

<u>Theorem C.1</u> For the mapping above, and tangent multivectors $A_r \in \mathcal{D}^r_{\underset{\sim}{x}}$, and $B^r \in \mathcal{D}^r_{\underset{\sim}{y}}$,

$$\text{(i)}\quad y_{\dagger}A_r = \psi^{r-1}\,[\psi A_r + (A_r\cdot\nabla_{\underset{\sim}{x}}\psi)\wedge\underset{\sim}{x}]$$

$$\text{(ii)}\quad y^{\dagger}B^r = \psi^{r-1}\,[\psi B^r + (\nabla_{\underset{\sim}{x}}\psi)\wedge(\underset{\sim}{x}\cdot B^r)]$$

$$\text{(iii)}\quad J_{\bar{y}_m} = \psi^{m-1}\,[\psi + (\nabla_{\underset{\sim}{x}}\psi)\cdot\underset{\sim}{x}]$$

$$\text{(iv)}\quad \nabla_{\underset{\sim}{y}} = \psi^{m-1}\,J^{-1}_{\bar{y}_m}\,\{\psi\nabla_{\underset{\sim}{x}} + \underset{\sim}{x}\cdot[(\nabla_{\underset{\sim}{x}}\psi)\wedge\nabla_{\underset{\sim}{x}}]\}$$

<u>Proof</u> (i) $y_{\dagger}A_r \equiv A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r$

$$= A_r \cdot \nabla_{\bar{x}_r} \frac{1}{r!} \psi_1 \underset{\sim}{x}_1 \wedge \ldots \wedge \psi_r \underset{\sim}{x}_r$$

$$= A_r \cdot [(\nabla_{\underset{\sim}{x}_r} \psi_r + \psi \nabla_{\underset{\sim}{x}_r}) \wedge \ldots \wedge (\nabla_{\underset{\sim}{x}_1} \psi + \psi \nabla_{\underset{\sim}{x}_1})]\bar{x}_r$$

$$= \psi^r A_r \cdot \nabla_{\bar{x}_r} \bar{x}_r + \psi^{r-1} A_r \cdot [(\nabla_{\underset{\sim}{x}} \psi) \wedge \nabla_{\bar{x}_{r-1}}] \bar{x}_{r-1} \wedge \underset{\sim}{x}$$

theorem 7.4(i) $$= \psi^r A_r + \psi^{r-1} [A_r \cdot (\nabla_{\underset{\sim}{x}} \psi)] \cdot \nabla_{\bar{x}_{r-1}} \bar{x}_{r-1} \wedge \underset{\sim}{x}$$

theorem 7.4(i) $$= \psi^{r-1} [\psi A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi) \wedge \underset{\sim}{x}].$$

(ii) is proved in a similar way to (i).

(iii) is proved by using (i) in the identity

$$J_{\bar{y}_m} \equiv i_{\underset{\sim}{x}}^{-1} y_{\dagger} i_{\underset{\sim}{x}} .$$

(iv) is proved by using (i) in (9.6), and the fact that

$$J_{\bar{y}_m}^{-1} = i_{\underset{\sim}{y}}^{-1} i_{\underset{\sim}{x}} ,$$

which follows from (B.4).

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An example of this kind of mapping is the following: Let $y\colon \mathcal{E}_3 - \{0\} \to \mathcal{E}_3$ be given by $y(\underset{\sim}{x}) = \frac{1}{\underset{\sim}{x}^2} \underset{\sim}{x}$. (The mapping $y(\underset{\sim}{x})$ is an inversion of $\mathcal{E}_3$ through the 2-sphere of radius one centered at the origin.)

Corollary C.2 For the mapping $y(\underset{\sim}{x})$ given above,

(i) $y_{\dagger}A_r = \left(\frac{1}{\underset{\sim}{x}^2}\right)^3 [A_r - \frac{2}{\underset{\sim}{x}^2}(A_r\cdot\underset{\sim}{x})\wedge\underset{\sim}{x}] = y^{\dagger}A_r$.

(ii) $J_{\bar{y}_3} = -\left(\frac{1}{\underset{\sim}{x}^2}\right)^3$

(iii) $\nabla_{\underset{\sim}{y}} = \underset{\sim}{x}^2\nabla_{\underset{\sim}{x}} - 2\underset{\sim}{x}\,\underset{\sim}{x}\cdot\nabla_{\underset{\sim}{x}} = -\underset{\sim}{x}\,\nabla_{\underset{\sim}{x}}\overset{\frown}{\underset{\sim}{x}}$, where $\overset{\frown}{\underset{\sim}{x}}$ indicates that the gradient operator is not to differentiate the $\underset{\sim}{x}$.

Proof The proof is a straight forward calculation using theorem C.1. It is helpful to note that since $\nabla_{\underset{\sim}{x}} \wedge y(\underset{\sim}{x}) = 0$, $y_{\dagger}A_r = y^{\dagger}A_r$ by theorem 6.2.

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b) Mappings of the Kind $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\,\underset{\sim}{p}$.

Let $y\colon \mathcal{X}_m \to \mathcal{Y}_m$ be given by $y(\underset{\sim}{x}) = \underset{\sim}{x} + \psi(\underset{\sim}{x})\,\underset{\sim}{p}$, where $\psi(x)$ is a scalar valued function, and $\underset{\sim}{p}$ is a constant vector in $\mathcal{D}$.

Theorem C.3 For the mapping given above, and tangent multivectors $A_r \in \mathcal{D}^r_{\underset{\sim}{x}}$, and $B^r \in \mathcal{D}^r_{\underset{\sim}{y}}$,

$$\text{(i)} \quad y_{\dagger}A_r = A_r + (A_r \cdot \nabla_{\underset{\sim}{x}} \psi) \wedge \underset{\sim}{p}$$

$$\text{(ii)} \quad y^{\dagger}B^r = B^r + (\nabla_{\underset{\sim}{x}}\psi) \wedge (\underset{\sim}{p} \cdot B^r)$$

$$\text{(iii)} \quad J_{\bar{y}_m} = 1 - \underset{\sim}{p}_{\perp} \nabla_{\underset{\sim}{x}}\psi + \underset{\sim}{p}_{\parallel} \cdot \nabla_{\underset{\sim}{x}}\psi \quad , \quad \text{where}$$

$p_{\parallel} \in \mathcal{D}^1_{\underset{\sim}{x}}$, is the tangential component of $\underset{\sim}{p}$ to the surface $\mathcal{X}_m$, $\underset{\sim}{p}_{\perp} = \underset{\sim}{p} - \underset{\sim}{p}_{\parallel}$ is the normal component of $\underset{\sim}{p}$ to the surface $\mathcal{X}_m$.

$$\text{(iv)} \quad \nabla_{\underset{\sim}{y}} = J_{\bar{y}_m}^{-1} \{\nabla_{\underset{\sim}{x}} - \underset{\sim}{p}_{\perp}(\nabla_{\underset{\sim}{x}}\psi) \wedge \nabla_{\underset{\sim}{x}} + \underset{\sim}{p}_{\parallel} \cdot [(\nabla_{\underset{\sim}{x}}\psi) \wedge \nabla_{\underset{\sim}{x}}]\},$$

where $\underset{\sim}{p}_{\parallel}$ and $\underset{\sim}{p}_{\perp}$ are given as in (iii).

<u>Proof</u> (i) $y_{\dagger}A_r \equiv A_r \cdot \nabla_{\bar{x}_r} \bar{y}_r$

$$= A_r \cdot \nabla_{\bar{x}_r} \frac{1}{r!} (\underset{\sim}{x}_1 + \psi_1 \underset{\sim}{p}) \wedge \ldots \wedge (\underset{\sim}{x}_r + \psi_r \underset{\sim}{p})$$

$$= A_r \cdot \nabla_{\bar{x}_r} \bar{x}_r + A_r \cdot [(\nabla_{\underset{\sim}{x}}\psi) \wedge \nabla_{\bar{x}_{r-1}}] \bar{x}_{r-1} \wedge \underset{\sim}{p}$$

theorem 7.4(i) $\quad = A_r + [A_r \cdot (\nabla_{\underset{\sim}{x}}\psi)] \cdot \nabla_{\bar{x}_{r-1}} \bar{x}_{r-1} \wedge \underset{\sim}{p}$

theorem 7.4(i) $\quad = A_r + [A_r \cdot (\nabla_{\underset{\sim}{x}}\psi)] \wedge \underset{\sim}{p}$

(ii) The proof of (ii) is similar to (i).

(iii) The proof of (iii) follows by using (i) in the identity $J_{\bar{y}_m} \equiv i_{\underset{\sim}{x}}^{-1} \, y_{\dagger} i_{\underset{\sim}{x}}$, and algebraically simplifying the resulting expression.

(iv) The proof of (iv) follows by using (i) in (9.6), the fact that $J_{\bar{y}_m}^{-1} = i_{\underset{\sim}{y}}^{-1}\, i_{\underset{\sim}{x}}$, and an algebraic simplification.

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An example of this kind of mapping is the following: Let $y: \mathcal{X}_2 \subset \mathcal{E}_3 \to \mathcal{Y}_2 \subset \mathcal{E}_3$ be given by $y(\underset{\sim}{x}) = \underset{\sim}{x} + \sqrt{1 + \underset{\sim}{x}^2}\, \underset{\sim}{p}$, where:

(i) $\mathcal{X}_2$ is a unit disc centered at the origin,

(ii) $\underset{\sim}{p}$ is a normal unit vector to the disc $\mathcal{X}_2$,

(iii) $\mathcal{Y}_2$ is the hemisphere having $\mathcal{X}_2$ as its base.

<u>Corollary C.4</u> For the mapping given above, and tangent multivectors $A_r \in \mathcal{D}_{\underset{\sim}{x}}^r$, and $B^r \in \mathcal{D}_{\underset{\sim}{y}}^r$,

(i) $y_{+}A_r = A_r - \dfrac{1}{\sqrt{1-\underset{\sim}{x}^2}}\,(A_r \cdot \underset{\sim}{x})\, \underset{\sim}{p}$

(ii) $y^{\dagger}B_r = B^r - \dfrac{1}{\sqrt{1-\underset{\sim}{x}^2}}\, \underset{\sim}{x} \wedge (\underset{\sim}{p} \cdot B^r)$

(iii) $J_{\bar{y}_2} = 1 + \dfrac{1}{\sqrt{1-\underset{\sim}{x}^2}}\, \underset{\sim}{p}\, \underset{\sim}{x}$

(iv) $\nabla_{\underset{\sim}{y}} = J_{\bar{y}}^{-1}\, [\nabla_{\underset{\sim}{x}} + \dfrac{1}{\sqrt{1-\underset{\sim}{x}^2}}\, \underset{\sim}{p}\, \underset{\sim}{x} \wedge \nabla_{\underset{\sim}{x}}]$

<u>Proof</u> The proof is a straight forward calculation using theorem C.3 . Note that since $\underset{\sim}{p} = \underset{\sim}{p}_{\parallel} + \underset{\sim}{p}_{\perp}$ is perpendicular to $\mathcal{X}_2$, $\underset{\sim}{p}_{\parallel} = 0$.

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As a final example of the kind of mapping in theorem C.3, let $y: \mathcal{X}_2 \subset \mathcal{E}_3 \rightarrow \mathcal{Y}_2 \subset \mathcal{E}_3$ be given by $y(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{x}^2 \underset{\sim}{p}$, where:

(i) $\mathcal{X}_2$ is a plane through the origin,

(ii) $\underset{\sim}{p}$ is a normal unit vector to the plane $\mathcal{X}_2$,

(iii) $\mathcal{Y}_2$ is a paraboloid having $\mathcal{X}_2$ as a tangent plane at the origin.

<u>Corollary C.5</u> For the mapping given above, and tangent multivectors $A_r \in \mathcal{D}^r_{\underset{\sim}{x}}$ and $B^r \in \mathcal{D}^r_{\underset{\sim}{y}}$,

(i) $y_{\dagger} A_r = A_r + 2\, A_r \cdot \underset{\sim}{x}\, \underset{\sim}{p}$

(ii) $y^{\dagger} B^r = B^r + 2\, \underset{\sim}{x} \wedge (\underset{\sim}{p} \cdot B^r)$

(iii) $J_{\overline{y}_2} = 1 + 2\underset{\sim}{x}\underset{\sim}{p}$

(iv) $\nabla_{\underset{\sim}{y}} = \nabla_{\underset{\sim}{x}} + \frac{2}{1+4\underset{\sim}{x}^2} (\underset{\sim}{p} - 2\underset{\sim}{x})\, \underset{\sim}{x} \cdot \nabla_{\underset{\sim}{x}}$.

<u>Proof</u> The proof is a straight forward calculation using theorem C.3. Again note that since $\underset{\sim}{p} = \underset{\sim}{p}_{\parallel} + \underset{\sim}{p}_{\perp}$ is normal to $\mathcal{X}_2$, $\underset{\sim}{p}_{\parallel} = 0$.

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Appendix D. Differential Forms

In this appendix the exact relationship between differential forms and geometric algebra is revealed. The algebraic and differential operators which are piecewise introduced on differential forms to enrich their algebraic character, are all simply and directly expressed in terms of geometric algebra together with its one vector differential operator.

A table of the relationships between the two algebraic systems is given in the summary of this paper.

a) Definitions and Basic Properties

The following formal definition of an r-form is used. It is equivalent to that given in [12, p.50] or [4, p.62] .

Definition D.1 A differential r-form on a surface $\mathcal{X}_m$ is a function $f^r(\underset{\sim}{x})$ which assigns to each point $\underset{\sim}{x} \in \mathcal{X}_m$ the real valued function $f^r_{\underset{\sim}{x}} = f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ of the r vector variables $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r \in \mathcal{D}_{\underset{\sim}{x}}$, with the following properties:

(i) $f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_i, \ldots, \underset{\sim}{w}_j, \ldots, \underset{\sim}{w}_r) =$

$$-f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_j, \ldots, \underset{\sim}{w}_i, \ldots, \underset{\sim}{w}_r),$$

i.e.: $f_{\underset{\sim}{x}}^r$ is antisymmetric over any interchange of its vector variables.

(ii) $f_{\underset{\sim}{x}}^r(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ is linear in each of its vector variables.

Since $f_{\underset{\sim}{x}}^r(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ is a function of the r vector variables $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r$, it can be differentiated by $\nabla_{\bar{W}_r}$, the gradient operator with respect to the r-vector variable $\bar{W}_r$ of the tangent m-plane to the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$. Note that $\nabla_{\bar{X}_r} \neq \nabla_{\bar{W}_r}$ unless the surface $\mathcal{X}_m$ is flat at the point $\underset{\sim}{x}$.

Differentiating $f_{\underset{\sim}{x}}^r(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$ by $\nabla_{\bar{W}_r}$ is the key idea to the following theorem which gives the one-to-one correspondence that exists between r-forms on $\mathcal{X}_m$, and tangent r-vector fields on $\mathcal{X}_m$.

<u>Theorem D.2</u> (i) To each r-form $f^r(\underset{\sim}{x})$, there is an r-vector field $F^r(\underset{\sim}{x})$ with the property that $f_{\underset{\sim}{x}}^r(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x})\cdot V_r^\dagger$, where $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_r$. It is given by

$$F^r(\underset{\sim}{x}) \equiv \frac{1}{r!} \nabla_{\bar{W}_r}^\dagger f_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r) .$$

(ii) Conversely, if an r-vector field $F^r(\underset{\sim}{x})$ is given, then $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv F^r(\underset{\sim}{x}) \cdot V^\dagger_r$ is a differential r-form.

<u>Proof</u> (i) The proof is a direct verification. Let $F^r(\underset{\sim}{x})$, and $V_r \in \mathcal{D}^r_{\underset{\sim}{x}}$ be given as in the theorem. Then:

$$F^r(\underset{\sim}{x}) \cdot V^\dagger_r = V^\dagger_r \cdot F^r(\underset{\sim}{x})$$

$$= \frac{1}{r!} (\underset{\sim}{v}_r \wedge \ldots \wedge \underset{\sim}{v}_1) \cdot (\nabla_{\underset{\sim}{w}_1} \wedge \ldots \wedge \nabla_{\underset{\sim}{w}_r}) f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)$$

identity 0.41
def. D.1(i)
$$= \frac{1}{r!} [r!\, \underset{\sim}{v}_1 \cdot \nabla_{\underset{\sim}{w}_1} \cdots \underset{\sim}{v}_r \cdot \nabla_{\underset{\sim}{w}_r} f^r_{\underset{\sim}{x}}(\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r)]$$

theorem A.4
$$= f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) .$$

(ii) It is easy to check that $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv F^r(\underset{\sim}{x}) \cdot V^\dagger_r$ is a differential r-form.

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The following are helpful definitions for giving a geometric interpretation to an r-form.

<u>Definition D.3</u> An r-form $f^r(\underset{\sim}{x})$ is said to be simple if its corresponding vector field $F^r(\underset{\sim}{x})$ is simple. (See [11, p.11] and [18, p.44].)

<u>Definition D.4</u> If A_r and B_r are simple r-vectors, then $\cos\theta \equiv \hat{A}_r \cdot \hat{B}_r^{\dagger} = \dfrac{A_r \cdot B_r^{\dagger}}{|A_r|\ |B_r|}$ defines the angle θ between them. (See [18, p.56].)

Theorem D.2 along with these definitions make the geometric interpretation of a simple r-form evident: A simple r-form $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r)$ is a scalar measure of the relative directions of the simple r-vector $F^r = F^r(\underset{\sim}{x})$ and the r-vector variable $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_r$. In particular, when $V_r = F^r$, $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = |F^r|^2$.

The Grassmann, or exterior product $f^r_{\underset{\sim}{x}} \wedge g^s_{\underset{\sim}{x}}$ of forms $f^r_{\underset{\sim}{x}}$ and $g^s_{\underset{\sim}{x}}$ is now defined in the conventional way. (See for example [12, p.51] or [1, p.55].)

<u>Definition D.5</u> $f^r_{\underset{\sim}{x}} \wedge g^s_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r, \underset{\sim}{v}_{r+1}, \ldots, \underset{\sim}{v}_{r+s})$

$$= \binom{r+s}{s} \sum_{\pi} (-1)^{\pi}\, f^r_{\underset{\sim}{x}} \otimes g^s_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r, \underset{\sim}{v}_{r+1}, \ldots, \underset{\sim}{v}_{r+s})^{\pi},$$

where $\otimes$ is the tensor product, and π is a permutation of the set $\{1, 2, \ldots, r+s\}$.

The theorem below gives the simple relationship between the exterior product of forms, and the outer product of multivector fields.

<u>Theorem D.6</u> If $f^r_{\underset{\sim}{X}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r_{\underset{\sim}{X}} \cdot V^\dagger_r$, and $g^s_{\underset{\sim}{X}}(\underset{\sim}{v}_{r+1}, \ldots, \underset{\sim}{v}_{r+s}) = G^s_{\underset{\sim}{X}} \cdot W^\dagger_s$, where $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_s$, and $W_s = \underset{\sim}{v}_{r+1} \wedge \ldots \wedge \underset{\sim}{v}_{r+s}$, then $f^r_{\underset{\sim}{X}} \wedge g^s_{\underset{\sim}{X}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r+s}) =$ $(F^r_{\underset{\sim}{X}} \wedge G^s_{\underset{\sim}{X}}) \cdot (V_r \wedge W_s)^\dagger$.

<u>Proof</u> The proof is an algebraic identity and is omitted.

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Using theorem D.6, the properties of the exterior product of forms follow easily from the properties of the outer product of multivectors in geometric algebra. Some of these properties are now given.

<u>Theorem D.7</u> (i) The exterior product of forms is bilinear.

(ii) $f^r_{\underset{\sim}{X}} \wedge g^s_{\underset{\sim}{X}} = (-1)^{rs}\, g^s_{\underset{\sim}{X}} \wedge f^r_{\underset{\sim}{X}}$.

(iii) $f^r_{\underset{\sim}{X}} \wedge (g^s_{\underset{\sim}{X}} \wedge h^t_{\underset{\sim}{X}}) = (f^r_{\underset{\sim}{X}} \wedge g^s_{\underset{\sim}{X}}) \wedge h^t_{\underset{\sim}{X}}$.

<u>Proof</u> Let $F^r_{\underset{\sim}{x}}$, $G^s_{\underset{\sim}{x}}$, and $H^t_{\underset{\sim}{x}}$ be the multivectors corresponding to the differential forms $f^r_{\underset{\sim}{x}}$, $g^s_{\underset{\sim}{x}}$, and $h^t_{\underset{\sim}{x}}$ respectively, at the point $\underset{\sim}{x} \varepsilon \mathcal{X}_m$. The proof of the theorem follows from the following algebraic properties of geometric algebra, and theorem D.6.

(i) The $\wedge$-product of multivectors is bilinear.

(ii) $F^r_{\underset{\sim}{x}} \wedge G^s_{\underset{\sim}{x}} = (-1)^{rs} G^s_{\underset{\sim}{x}} \wedge F^r_{\underset{\sim}{x}}$. (identity 0.45)

(iii) $F^r_{\underset{\sim}{x}} \wedge (G^s_{\underset{\sim}{x}} \wedge H^t_{\underset{\sim}{x}}) = (F^r_{\underset{\sim}{x}} \wedge G^s_{\underset{\sim}{x}}) \wedge H^t_{\underset{\sim}{x}}$.

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b) The Exterior Derivative

The exterior derivative d-operator of forms is often defined in the following way. See for example [12, p.89] or [4, p.65].

<u>Definition D.8</u> Let $f^r_{\underset{\sim}{x}}$ be an r-form. Then

$$df^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r+1}) \equiv \sum_{i=1}^{r+1} (-1)^{i+1} \underset{\sim}{v}_i \cdot \nabla_{\underset{\sim}{x}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \check{\underset{\sim}{v}}_i, \ldots, \underset{\sim}{v}_{r+1})$$

$$+ \sum_{i<j} (-1)^{i+j} f^r_{\underset{\sim}{x}} ([\underset{\sim}{v}_i, \underset{\sim}{v}_j], \underset{\sim}{v}_1, \ldots, \check{\underset{\sim}{v}}_i, \ldots, \check{\underset{\sim}{v}}_j, \ldots, \underset{\sim}{v}_{r+1})$$

where $[\underset{\sim}{v}_i, \underset{\sim}{v}_j]$ is Lie bracket of vectors defined in section 10, and

$\check{\underset{\sim}{v}}_j$, means the j^{th} vector $\underset{\sim}{v}_j$ is omitted.

The theorem below shows that if $F^r(\underset{\sim}{x})$ is the r-vector field corresponding to $f^r_{\underset{\sim}{x}}$, then $\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})$ is the (r+1)-vector field corresponding to $df^r_{\underset{\sim}{x}}$.

Theorem D.9 If $f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x})\cdot V^\dagger_r$, where $V_r = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_r$, then $df^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r+1}) = [\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})]\cdot V_{r+1}$, where $V_{r+1} = \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v}_{r+1}$.

Proof The proof is a direct verification.

$$[\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})]\cdot V^\dagger_{r+1} = [\underset{\sim}{v}_{r+1}\wedge\ldots\wedge\underset{\sim}{v}_1]\cdot[\nabla_{\underset{\sim}{x}}\wedge F^r(\underset{\sim}{x})]$$

identity 0.42
$$= \{[\underset{\sim}{v}_{r+1}\wedge\ldots\wedge\underset{\sim}{v}_1]\cdot\nabla_{\underset{\sim}{x}}\}\cdot F^r(\underset{\sim}{x})$$

identity 0.40
$$= \left\{\sum_{i=1}^{r+1}(-1)^{i+1}[\underset{\sim}{v}_{r+1}\wedge\ldots\wedge\check{\underset{\sim}{v}}_i\wedge\ldots\wedge\underset{\sim}{v}_1]\underset{\sim}{v}_i\cdot\nabla_{\underset{\sim}{x}}\right\}\cdot F^r(\underset{\sim}{x}$$

$$= \sum_{i=1}^{r+1}(-1)^{i+1}\underset{\sim}{v}_i\cdot\nabla_{\underset{\sim}{x}}[\underset{\sim}{v}_{r+1}(\underset{\sim}{x})\wedge\ldots\wedge\check{\underset{\sim}{v}}_i(\underset{\sim}{x})\wedge\ldots\wedge$$

$$\underset{\sim}{v}_1(\underset{\sim}{x})]\cdot F^r(\underset{\sim}{x}) + \left[\left[\sum_{i=1}(-1)^i\{\underset{\sim}{v}_i\cdot\nabla_{\underset{\sim}{x}}[\underset{\sim}{v}_{r+1}(\underset{\sim}{x})\wedge\ldots\wedge\check{\underset{\sim}{v}}_i\wedge\ldots\wedge\underset{\sim}{v}_1]\}\right]\right]\cdot F^r(\underset{\sim}{x}).$$

But,

$$[[\_\_\_\_]] = \sum_{i=1}^{r+1}\sum_{\substack{j=1\\ j\neq i}}^{r+1}(-1)^i[\underset{\sim}{v}_{r+1}\wedge\ldots\wedge(\underset{\sim}{v}_i\cdot\nabla_{\underset{\sim}{x}}\underset{\sim}{v}_j)\wedge\ldots\wedge\check{\underset{\sim}{v}}_i\wedge\ldots\wedge\underset{\sim}{v}_1]$$

$$= \sum_{i=1}^{r+1} \sum_{\substack{j=1 \\ j\neq i}}^{r+1} (-1)^{i+j} [v_{r+1} \wedge \cdots \wedge \check{v}_j \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1] \wedge [v_i \cdot \nabla_x v_j]$$

$$= \sum_{i<j} (-1)^{i+j} [v_{r+1} \wedge \cdots \wedge \check{v}_j \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1] \wedge [v_i \cdot \nabla_x v_j] + \sum_{i<j}$$

$$(-1)^{i+j} [v_{r+1} \wedge \cdots \wedge \check{v}_j \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1] \wedge [v_j \cdot \nabla_x v_i]$$

$$= \sum_{i<j} (-1)^{i+j} [v_{r+1} \wedge \cdots \wedge \check{v}_j \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1] \wedge [v_i , v_j] \ .$$

Thus,

$$[\nabla_x \wedge F^r(x)] \cdot V_{r+1}^{\dagger} = \sum_{i=1}^{r+1} (-1)^{i+1} v_i \cdot \nabla_x [v_{r+1}(x) \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1(x)] \cdot F^r(x) +$$

$$\sum_{i<j} (-1)^{i+j} \{[v_{r+1} \wedge \cdots \wedge \check{v}_j \wedge \cdots \wedge \check{v}_i \wedge \cdots \wedge v_1] \wedge [v_i , v_j]\} \cdot F^r(x)$$

$$\equiv df_x^r (v_1 , \ldots , v_{r+1}) \ .$$

XXXX

The following properties of the exterior derivative of forms follow easily by using the previous theorem, and corresponding properties of ∇_x .

<u>Theorem D.10</u> If f_x^r , and g_x^s are two forms on $\mathcal{X}_m$, then

(i) $d(f_x^r + g_x^r) = df_x^r + dg_x^r$, when $s = r$

(ii) $d(f_x^r \wedge g_x^s) = (df_x^r) \wedge g_x^s + (-1)^r f_x^r \wedge dg_x^s$

(iii) $d(df_{\underset{\sim}{x}}^{r}) = 0$.

<u>Proof</u> Let $f_{\underset{\sim}{x}}^{r}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x}) \cdot V_r^{\dagger}$, and $g_{\underset{\sim}{x}}^{s}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_s) = G^s(\underset{\sim}{x}) \cdot W_s^{\dagger}$, where $F^r(\underset{\sim}{x})$, and $G^s(\underset{\sim}{x})$ are the corresponding multivector fields for $f_{\underset{\sim}{x}}^{r}$ and $g_{\underset{\sim}{x}}^{s}$ given by theorem D.2(ii). The proof of the theorem follows from the properties of the gradient operator listed below:

(i) $\nabla_{\underset{\sim}{x}} \wedge [F^r(\underset{\sim}{x}) + G^r(\underset{\sim}{x})] = \nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}} \wedge G^r(\underset{\sim}{x})$.

(ii) $\nabla_{\underset{\sim}{x}} \wedge [F^r(\underset{\sim}{x}) \wedge G^s(\underset{\sim}{x})] = [\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] \wedge G^s(\underset{\sim}{x})$

$+ (-1)^r F^r(\underset{\sim}{x}) \wedge [\nabla_{\underset{\sim}{x}} \wedge G^s(\underset{\sim}{x})]$.

(iii) $\nabla_{\underset{\sim}{x}} \wedge [\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] = 0$. (property 2.13)

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c) The Contraction Operator

The contraction operator $C_{\underset{\sim}{v}}$, for $\underset{\sim}{v} \in \mathcal{D}_{\underset{\sim}{x}}^{1}$, is a mapping of r-forms into (r-1)-forms. It is defined below. (See [12, p.91] or [4, p.69] for an equivalent definition.)

<u>Definition D.11</u> For an r-form $f_{\underset{\sim}{x}}^{r}$ on $\mathcal{X}_m$,

$$C_{\underset{\sim}{v}} f_{\underset{\sim}{x}}^{r}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}) \equiv f_{\underset{\sim}{x}}^{r}(\underset{\sim}{v}, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}) .$$

Let $\underset{\sim}{f}^r_x$ be an r-form, and $F^r_{\underset{\sim}{x}}$ be the corresponding r-vector field given by theorem D.2.

<u>Theorem D.12</u> $\quad C_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}}\,(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}) = [\underset{\sim}{v}\cdot F^r_{\underset{\sim}{x}}]\cdot V^{\dagger}_{r-1}$,

where $V_{r-1} = \underset{\sim}{v}_1 \wedge\ldots\wedge \underset{\sim}{v}_{r-1}$.

<u>Proof</u> $\quad [\underset{\sim}{v}\cdot F^r_{\underset{\sim}{x}}]\cdot V^{\dagger}_{r-1} = F^r_{\underset{\sim}{x}}\cdot(V^{\dagger}_{r-1}\wedge\underset{\sim}{v})$ $\qquad$ identity 0.42

$$\equiv f^r_{\underset{\sim}{x}}\,(\underset{\sim}{v}, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \quad .$$

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The following theorem gives the basic properties of the contraction operator $C_{\underset{\sim}{v}}$. The proof, which is omitted, follows easily by using theorems D.2 and D.12, and algebraic properties of geometric algebra.

Let $f^r_{\underset{\sim}{x}}$ and $g^s_{\underset{\sim}{x}}$ be forms on $\mathcal{X}_m$. Then:

<u>Theorem D.13</u>

(i) $\quad (C_{\underset{\sim}{v}})^2\, f^r_{\underset{\sim}{x}} = 0$

(ii) $\quad C_{\underset{\sim}{v}}[f^r_{\underset{\sim}{x}} + g^s_{\underset{\sim}{x}}] = C_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}} + C_{\underset{\sim}{v}}\, g^s_{\underset{\sim}{x}}$

(iii) $\quad C_{\underset{\sim}{v}+\underset{\sim}{w}}\, f^r_{\underset{\sim}{x}} = C_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}} + C_{\underset{\sim}{w}}\, g^s_{\underset{\sim}{x}}$

(iv) $\quad C_{\underset{\sim}{v}}[f^r_{\underset{\sim}{x}}\wedge g^s_{\underset{\sim}{x}}] = [C_{\underset{\sim}{v}}\, f^r_{\underset{\sim}{x}}]\wedge g^s_{\underset{\sim}{x}} + (-1)^r\, f^r_{\underset{\sim}{x}}\wedge[C_{\underset{\sim}{v}}\, g^s_{\underset{\sim}{x}}]$

d) The Covariant Derivative

The covariant derivative operator $D_{\underset{\sim}{v}}$ for $\underset{\sim}{v} \in \mathcal{D}^1_{\underset{\sim}{x}}$, is a mapping of r-forms into r-forms. The definition for it given below is equivalent to that found in [12, p.94].

Let $f^r_{\underset{\sim}{x}}$ be an r-form on the surface $\mathcal{X}_m$, and let $\underset{\sim}{v} \in \mathcal{D}_{\underset{\sim}{x}}$. Then:

<u>Definition D.14</u> $D_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, \underset{\sim}{v}_r(\underset{\sim}{x})) - \sum_{i=1}^{r} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{i-1}, \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{v}_i(\underset{\sim}{x}), \underset{\sim}{v}_{i+1}, \ldots, \underset{\sim}{v}_r)$.

Let $f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r_{\underset{\sim}{x}} \cdot V^\dagger_r$, where $F^r_{\underset{\sim}{x}}$ is given by theorem D.2(ii), and $V_r = V_r(\underset{\sim}{x}) = \underset{\sim}{v}_1(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{v}_r(\underset{\sim}{x})$. The next theorem relates the covariant derivative of a form to the directional derivative of its corresponding multivector field. Its proof is an easy consequence of the identity

$$\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} V_r(\underset{\sim}{x}) = \sum_{i=1}^{r} \underset{\sim}{v}_1 \wedge \ldots \wedge \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} \underset{\sim}{v}_i(\underset{\sim}{x}) \wedge \ldots \wedge \underset{\sim}{v}_r .$$

<u>Theorem D.15</u> $D_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = [\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x})] \cdot V^\dagger_r$

Proof

$$[\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x})]\cdot V_r^\dagger = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}[F^r(\underset{\sim}{x})\cdot V_r^\dagger(\underset{\sim}{x})] - F^r\cdot[\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, V_r^\dagger(\underset{\sim}{x})]$$

$$\equiv D_{\underset{\sim}{v}}\, f_{\underset{\sim}{x}}^r \;,$$

using the identity given above, and definition D.14.

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e) The Lie Derivative

The Lie derivative $L_{\underset{\sim}{v}}$ is a mapping of r-forms into r-forms. Its definition is given below. See [12, p.93] or [1, p.64].

Definition D.16 $L_{\underset{\sim}{v}}\, f_{\underset{\sim}{x}}^r\,(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) \equiv \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, f_{\underset{\sim}{x}}^r\,(\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, \underset{\sim}{v}_r(\underset{\sim}{x})) - \sum_{i=1}^{r} f_{\underset{\sim}{x}}^r\,(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{r-1}, [\underset{\sim}{v},\underset{\sim}{v}_i], \underset{\sim}{v}_{i+1}, \ldots, \underset{\sim}{v}_r)$.

Theorem D.17 (i) $L_{\underset{\sim}{v}} f_{\underset{\sim}{x}}^r\,(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = \underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}[F^r(\underset{\sim}{x})\cdot V_r^\dagger(\underset{\sim}{x})] - F^r(\underset{\sim}{x})\cdot[\underset{\sim}{v},V_r]^\dagger$, where $f_{\underset{\sim}{x}}^r\,(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x})\cdot V_r^\dagger$, and $[\underset{\sim}{v},V_r]$ is the Lie bracket on multivector fields defined in section 10.

(ii) $L_{\underset{\sim}{v}} f_{\underset{\sim}{x}}^r = \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F_{\underset{\sim}{x}}^r + \nabla_{\underset{\sim}{x}_1}\wedge[\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F_{\underset{\sim}{x}}^r]\}\cdot V_r^\dagger$.

Proof (i) The proof is direct and uses theorem 10.10(i) for the decomposition of $[\underset{\sim}{v},V_r]$.

$$L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) \cdot V^\dagger_r(\underset{\sim}{x}) - \sum_{i=1}^{r} F^r(\underset{\sim}{x}) \cdot [\underset{\sim}{v}_1 \wedge$$

$$\cdots \wedge \underset{\sim}{v}_{i-1} \wedge [\underset{\sim}{v}, \underset{\sim}{v}_i] \wedge \underset{\sim}{v}_{i+1} \wedge \cdots \wedge \underset{\sim}{v}_r]^\dagger$$

theorem 10.10(i) $$= \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) \cdot V^\dagger_r(\underset{\sim}{x}) - F^r(\underset{\sim}{x}) \cdot [\underset{\sim}{v}, V_r]^\dagger .$$

(ii) follows from (i) by the short computation given below.

$$L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) \cdot V^\dagger_r(\underset{\sim}{x}) - F^r \cdot [\underset{\sim}{v}, V_r]^\dagger \qquad \text{using (i)}$$

$$= \underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) \cdot V^\dagger_r(\underset{\sim}{x}) - F^r \cdot [\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} V^\dagger_r(\underset{\sim}{x})] +$$

$$\{\nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}(\underset{\sim}{x}_1) \cdot F^r]\} \cdot V^\dagger_r$$

$$= \{\underset{\sim}{v} \cdot \nabla_{\underset{\sim}{x}} F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}(\underset{\sim}{x}_1) \cdot F^r]\} \cdot V_r^\dagger .$$

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Well-known properties of the Lie derivative are given in the next theorem and are proved using theorem D.17. Let $f^r_{\underset{\sim}{x}} (\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r) = F^r(\underset{\sim}{x}) \cdot V^\dagger_r$, and $g^s_{\underset{\sim}{x}} = G^s(\underset{\sim}{x}) \cdot W^\dagger_s$, then :

<u>Theorem D.18</u> (i) $L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = C_{\underset{\sim}{v}} \, df^r_{\underset{\sim}{x}} + d \, C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}$.

(ii) $L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} \wedge g^s_{\underset{\sim}{x}} = (L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}) \wedge g^s_{\underset{\sim}{x}} + f^r_{\underset{\sim}{x}} \wedge L_{\underset{\sim}{v}} g^s_{\underset{\sim}{x}}$

(iii) $d \, L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}} = L_{\underset{\sim}{v}} \, df^r_{\underset{\sim}{x}}$.

[illegible]

identity, [illegible] $\quad - [illegible]$

$$= (\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}} f^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1}\wedge[\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot f^r])\cdot V_r^{\dagger}$$

theorem D.17(ii) $\quad \equiv L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r)$.

The proofs of (ii) and (iii) follow from (i) by using the properties proved for $C_{\underset{\sim}{v}}$ and d in theorems D.10 and D.12.

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f) The Pull Back of Forms

Let $y\colon \mathcal{X}_m \to \mathcal{Y}_k$. The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ induces a linear mapping y^* called the "pull back," of r-forms $g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_k$ into r-forms $y^* g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{X}_m$. The following definition of y^* is equivalent to that given in [12, p.53].

<u>Definition D.19</u> $\quad y^* g^r_{\underset{\sim}{y}}(\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, \underset{\sim}{v}_r(\underset{\sim}{x})) \equiv g^r_{\underset{\sim}{y}}(y_\dagger \underset{\sim}{v}_1(\underset{\sim}{x}), \ldots, y_\dagger \underset{\sim}{v}_r(\underset{\sim}{x}))$, where $y_\dagger$ is the differential mapping of the tangent space $\mathcal{D}^1_{\underset{\sim}{x}}$ into the tangent space $\mathcal{D}^1_{\underset{\sim}{y}}$ defined in section 3.

Proof

(i) $C_{\underset{\sim}{v}}\, df^r_{\underset{\sim}{x}} + d\, C_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}$

$$= \{\underset{\sim}{v}\cdot[\nabla_{\underset{\sim}{x}} \wedge F^r(\underset{\sim}{x})] + \nabla_{\underset{\sim}{x}} \wedge [\underset{\sim}{v}(\underset{\sim}{x})\cdot F^r(\underset{\sim}{x})]\}\cdot V_r^{\dagger}$$

identity D.38 $$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x}) - \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}\cdot F^r(\underset{\sim}{x}_1)] + \nabla_{\underset{\sim}{x}} \wedge [\underset{\sim}{v}(\underset{\sim}{x})\cdot F^r(\underset{\sim}{x})]\}\cdot V_r^{\dagger}$$

$$= \{\underset{\sim}{v}\cdot\nabla_{\underset{\sim}{x}}\, F^r(\underset{\sim}{x}) + \nabla_{\underset{\sim}{x}_1} \wedge [\underset{\sim}{v}(\underset{\sim}{x}_1)\cdot F^r]\}\cdot V_r^{\dagger}$$

theorem D.17(ii) $$\equiv L_{\underset{\sim}{v}} f^r_{\underset{\sim}{x}}\, (\underset{\sim}{v}_1, \ldots , \underset{\sim}{v}_r) .$$

The proofs of (ii) and (iii) follow from (i) by using the properties proved for $C_{\underset{\sim}{v}}$ and d in theorems D.10 and D.12.

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f) The Pull Back of Forms

Let $y: \mathcal{X}_m \to \mathcal{Y}_k$. The mapping $\underset{\sim}{y} = y(\underset{\sim}{x})$ induces a linear mapping y^* called the "pull back," of r-forms $g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{Y}_k$ into r-forms $y^* g^r_{\underset{\sim}{y}}$ on the surface $\mathcal{X}_m$. The following definition of y^* is equivalent to that given in [12, p.53].

<u>Definition D.19</u> $y^* g^r_{\underset{\sim}{y}}\, (\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots , \underset{\sim}{v}_r(\underset{\sim}{x})) \equiv g^r_{\underset{\sim}{y}}\, (y_{\dagger}\underset{\sim}{v}_1(\underset{\sim}{x}), \ldots , y_{\dagger}\underset{\sim}{v}_r(\underset{\sim}{x}))$, where $y_{\dagger}$ is the differential mapping of the tangent space $\mathcal{D}^1_{\underset{\sim}{x}}$ into the tangent space $\mathcal{D}^1_{\underset{\sim}{y}}$ defined in section 3.

Let the r-form $g^r_{\underset{\sim}{y}}$ be given by $g^r_{\underset{\sim}{y}}(\underline{w}_1, \ldots, \underline{w}_r) = G^r(\underset{\sim}{y}) \cdot W^\dagger_r$, where $W_r = \underline{w}_1 \wedge \ldots \wedge \underline{w}_r$, and $G^r(\underset{\sim}{y})$ is the r-vector field for $g^r_{\underset{\sim}{y}}$ given by theorem D.2.

<u>Theorem D.20</u> $y^* g^r_{\underset{\sim}{y}}(\underline{v}_1, \ldots, \underline{v}_r) = [y^\dagger G^r(\underset{\sim}{y})] \cdot V^\dagger_r$, where $V_r = \underline{v}_1 \wedge \ldots \wedge \underline{v}_r \in \mathcal{D}^r_{\underset{\sim}{x}}$.

<u>Proof</u> $y^* g^r_{\underset{\sim}{y}}(\underline{v}_1, \ldots, \underline{v}_r) = G^r_{\underset{\sim}{y}} \cdot [y_\dagger \underline{v}_1 \wedge \ldots \wedge y_\dagger \underline{v}_r]^\dagger$

theorem 3.3(i) $\qquad = G^r_{\underset{\sim}{y}} \cdot [y_\dagger(\underline{v}_1 \wedge \ldots \wedge \underline{v}_r)]^\dagger$

$\qquad = G^r_{\underset{\sim}{y}} \cdot y_\dagger V^\dagger_r$

cor. 3.6 $\qquad = (y^\dagger G^r_{\underset{\sim}{y}}) \cdot V^\dagger_r$

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The following properties of y^* now follow easily from the properties of $y^\dagger$, and the preceding theorems of this appendix.

<u>Theorem D.21</u> Let $f^r_{\underset{\sim}{y}}$ and $g^s_{\underset{\sim}{y}}$ be forms on $\mathcal{Y}_k$. Then,

(i) $y^*(f^r_{\underset{\sim}{y}} + g^r_{\underset{\sim}{y}}) = y^* f^r_{\underset{\sim}{y}} + y^* g^r_{\underset{\sim}{y}}$, for $r = s$.

(ii) $y^* f^r_{\underset{\sim}{y}} \wedge g^s_{\underset{\sim}{y}} = y^* f^r_{\underset{\sim}{y}} \wedge y^* g^s_{\underset{\sim}{y}}$

(iii) $d\, y^* f^r_{\underset{\sim}{y}} = y^*\, df^r_{\underset{\sim}{y}}$

Appendix E. The Intrinsic Gradient and Curvature

Throughout this paper the tangential gradient has been used. In this appendix another gradient called the intrinsic gradient is introduced. The relationships between the gradient ∇ on $\mathcal{E}_n$, the tangential gradient $\nabla_{\underset{\sim}{x}}$ on $\mathcal{X}_m$, and the intrinsic gradient $\not\nabla_{\underset{\sim}{x}}$ on $\mathcal{X}_m$ are studied, and a new formulation of the Gauss curvature equation is given.

a) The Gradients ∇, $\nabla_{\underset{\sim}{x}}$ and $\not\nabla_{\underset{\sim}{x}}$

Let $\mathcal{X}_m$ be an m-surface in $\mathcal{E}_n$.

The gradient ∇_x on the surface $\mathcal{X}_m$ is related to the gradient ∇ on $\mathcal{E}_n$ by the following equation:

$$\text{(E.1)} \qquad \nabla = \nabla_{\parallel} + \nabla_{\perp}\ , \text{ where } \nabla_{\underset{\sim}{x}} \equiv \nabla_{\parallel} \quad .$$

Equation (E.1) shows that if the gradient ∇ of $\mathcal{E}_n$ is decomposed into a tangential component $\nabla_{\parallel}$ and a normal component $\nabla_{\perp}$ to the surface $\mathcal{X}_m$ at the point $\underset{\sim}{x}$, then $\nabla_{\underset{\sim}{x}}$ is the tangential component.

The identification of $\nabla_{\underset{\sim}{x}}$ as being the tangential component of ∇ to the surface $\mathcal{X}_m$ ensures that it will obey all the familiar operational properties of the gradient ∇ on $\mathcal{E}_n$. (In fact $\nabla_{\underset{\sim}{x}}$ can be thought of as the gradient of the tangent m-plane to the surface $\mathcal{X}_m$ at the point $\underline{x}$.) However, it differs from ∇ in one crucial respect, and that is it doesn't preserve tangent multivector fields on $\mathcal{X}_m$. I.e., if $F(\underline{x})$ is a tangent multivector field on $\mathcal{E}_n$, then $\nabla F(\underline{x})$ will be also, but if $F(\underline{x})$ is a tangent multivector field on $\mathcal{X}_m$, $\nabla_{\underline{x}} F(\underline{x})$ will in general have both tangential and normal components to the surface $\mathcal{X}_m$.

The following equation decomposes $\nabla_{\underline{x}} F(\underline{x})$ into tangential and normal components, and at the same time identifies the intrinsic gradient applied to $F(\underline{x})$.

$$\text{(E.2)} \qquad \nabla_{\underline{x}} F(\underline{x}) = [\nabla_{\underline{x}} F(\underline{x})]_{\parallel} + [\nabla_{\underline{x}} F(\underline{x})]_{\perp} ,$$

where $\not\nabla_{\underline{x}} F(\underline{x}) \equiv [\nabla_{\underline{x}} F(\underline{x})]$.

In words, (E.2) says that if $\nabla_{\underline{x}} F(\underline{x})$ is decomposed into tangential and normal components to the surfaces $\mathcal{X}_m$ at the point $\underline{x}$, then the intrinsic gradient of $F(\underline{x})$ is defined to be the tangential part.

Thus where $\nabla_{\underline{x}} F(\underline{x})$ suffers the "defect" of not preserving tangent fields on $\mathcal{X}_m$, $\not\nabla_{\underline{x}}$ removes this defect by "throwing away" the normal part to the surface.

A more formal definition of $\not\nabla_{\underline{x}}$ in terms of $\nabla_{\underline{x}}$ is now given.

<u>Definition E.3</u> $\quad \not\nabla_{\underline{x}} F(\underline{x}) \equiv \nabla_{\underline{x}_1} F(\underline{x}_1) \cdot p_{\underline{x}}\, p_{\underline{x}}^{\dagger} = [\nabla_{\underline{x}} F(\underline{x})] \cdot p_{\underline{x}}\, p_{\underline{x}}^{\dagger}$,

where $p_{\underline{x}}$ is a unit pseudoscalar field on $\mathcal{X}_m$, and $F(\underline{x}) \in \{F(\underline{x})\}_{\underline{x}}$

Important special cases of this definition are:

(E.3a) $\not\partial_{x} \cdot F(x) = \nabla_{x_1} \cdot [F(x_1) \cdot p_x \, p_x^{\dagger}]$

(E.3b) $\not\partial_{x} \wedge F(x) = \nabla_{x_1} \wedge [F(x_1) \cdot p_x \, p_x^{\dagger}]$

(E.3c) $A_r \cdot \not\partial_{x} F(x) = A_r \cdot \nabla_{x_1} [F(x_1) \cdot p_x] p_x^{\dagger}$

(E.3d) $A_r \wedge \not\partial_{x} F(x) = A_r \wedge \nabla_{x_1} [F(x_1) \cdot p_x] p_x^{\dagger}$.

The theorem below relates properties of the intrinsic gradient $\not\nabla_{x}$ to properties of the tangential gradient ∇_{x} .

Let $[A_r/B_s]$ denote the Lie bracket operation defined in section 10, but with respect to the intrinsic gradient.

Theorem E.4 (i) $\not\nabla_{x} \cdot F(x) = \nabla_{x} \cdot F(x)$

(ii) $[A_r/B_s] = [A_r, B_s]$, for A_r, $B_s \in \{F(x)\}_x$

(iii) $a \cdot \nabla_{x} b(x) - a \cdot \not\nabla_{x} b(x) = -[b \wedge S(a)] p_x^{\dagger}$,

for $a(x)$, $b(x) \in \{F(x)\}_x$, and where $S(a)$ is the shape operator defined in section 13.

Proof (i) The proof of (i) follows immediately from definition E.3 and theorem 10.12.

(ii) By the decomposition given in lemma 10.11(ii),

$$\nabla_{\underset{\sim}{x}}\cdot(A_r \wedge B_s) = (\nabla_{\underset{\sim}{x}}\cdot A_r)\wedge B_s + (-1)^r A_r \wedge(\nabla_{\underset{\sim}{x}}\cdot B_s) + (-1)^{r+1}[A_r, B_s] \; .$$

The same decomposition applied to $\not\nabla_{\underset{\sim}{x}}$ gives

$$\not\nabla_{\underset{\sim}{x}}\cdot(A_r \wedge B_s) = (\not\nabla_{\underset{\sim}{x}}\cdot A_r)\wedge B_s + (-1)^r A_r \wedge(\not\nabla_{\underset{\sim}{x}}\cdot B_s) + (-1)^{r+1}[A_r / B_s] \; .$$

But by (i), $\not\nabla_{\underset{\sim}{x}}\cdot(A_r \wedge B_s) = \nabla_{\underset{\sim}{x}}\cdot(A_r \wedge B_s)$, $\not\nabla_{\underset{\sim}{x}}\cdot A_r = \nabla_{\underset{\sim}{x}}\cdot A_r$, and $\not\nabla_{\underset{\sim}{x}}\cdot B_s = \nabla_{\underset{\sim}{x}}\cdot B_s$. Hence it follows that $[A_r / B_s] = [A_r, B_s]$.

(iii) $\underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}(\underset{\sim}{x}) - \underset{\sim}{a}\cdot\not\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}(\underset{\sim}{x})$

$$= \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}_1}\, \underset{\sim}{b}(\underset{\sim}{x}_1)\cdot p_{\underset{\sim}{x}}\, p_{\underset{\sim}{x}}^{\dagger} + \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}_1}\, \underset{\sim}{b}(\underset{\sim}{x}_1)\wedge p_{\underset{\sim}{x}}\, p_{\underset{\sim}{x}}^{\dagger} - \underset{\sim}{a}\cdot\not\nabla_{\underset{\sim}{x}}\, \underset{\sim}{b}(\underset{\sim}{x})$$

def. E.3c $\qquad = \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}_1}\, \underset{\sim}{b}(\underset{\sim}{x}_1)\wedge p_{\underset{\sim}{x}}\, p_{\underset{\sim}{x}}^{\dagger}$

theorem 13.2(i) $\qquad = -[\underset{\sim}{b}\wedge S(\underset{\sim}{a})]p_{\underset{\sim}{x}}^{\dagger}$

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Part (iii) of the last theorem shows that the difference between the tangential and intrinsic directional derivatives of a vector field is completely determined by the shape of the surface. (See [12, p.75] for a similar result.)

b) Gaussian Curvature

The proof of the following theorem makes use of the fact that $\not\nabla_{\underset{\sim}{x}}\, p(\underset{\sim}{x}) = 0$. This is an easy consequence of theorem 13.2(iii) and definition E.3.

<u>Theorem E.5</u> $\not\nabla_{\underset{\sim}{x}} \wedge \not\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x}) = \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}) \cdot p(\underset{\sim}{x}_1)] \cdot p^{\dagger}(\underset{\sim}{x}_2)$,

where $F(\underset{\sim}{x}) \in \{F(\underset{\sim}{x})\}_{\underset{\sim}{x}}$, and $p(\underset{\sim}{x})$ is a unit pseudoscalar field.

<u>Proof</u> $\not\nabla_{\underset{\sim}{x}} \wedge \not\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x}) = \not\nabla_{\underset{\sim}{x}} \wedge \not\nabla_{\underset{\sim}{x}} [F(\underset{\sim}{x}) \cdot p(\underset{\sim}{x})] \cdot p^{\dagger}(\underset{\sim}{x})$

def. E.3 $= \not\nabla_{\underset{\sim}{x}} \wedge \nabla_{\underset{\sim}{x}_1} [F(\underset{\sim}{x}_1) \cdot p(\underset{\sim}{x})] \cdot \dot{p}^{\dagger}(\underset{\sim}{x})$

def. E.3 $= \nabla_{\underset{\sim}{x}_2} \wedge \nabla_{\underset{\sim}{x}_1} [F(\underset{\sim}{x}_1) \cdot p(\underset{\sim}{x})]_2 \cdot p^{\dagger}(\underset{\sim}{x})$

property 2.13 $= \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}_1) \cdot p(\underset{\sim}{x}_2)] \cdot p^{\dagger}(\underset{\sim}{x})$

$$= \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}_1) \cdot p(\underset{\sim}{x}_2)] \cdot p^{\dagger}(\underset{\sim}{x}_1) - \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}) \cdot p(\underset{\sim}{x}_2)] \cdot p^{\dagger}(\underset{\sim}{x}_1)$$

$$= \nabla_{\bar{\underset{\sim}{x}}_2} F(\underset{\sim}{x}_1) [p(\underset{\sim}{x}_2) \cdot p^{\dagger}(\underset{\sim}{x}_1)] - \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}) \cdot p(\underset{\sim}{x}_2)] \cdot p^{\dagger}(\underset{\sim}{x}_1)$$

theorem 13.2(iii),(iv) $= \nabla_{\bar{\underset{\sim}{x}}_2} [F(\underset{\sim}{x}) \cdot p(\underset{\sim}{x}_1)] \cdot p^{\dagger}(\underset{\sim}{x}_2)$.

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Theorem E.5 shows that $\not\nabla_{\underset{\sim}{x}} \wedge \not\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x})$ is completely determined by the shape of the surface, and is independent of the field properties of $F(\underset{\sim}{x})$.

<u>Corollary E.6</u> $\quad \not\nabla_{x} \wedge \not\nabla_{x}\, v(x) = \frac{1}{2} \nabla_{\bar{x}_2}\, v \cdot [p(x_1)\, p^{\dagger}(x_2)]_{bi}$,

where $v(x) \in \{F(x)\}^{1}_{x}$, and "bi" stands for bivector part.

<u>Proof</u> The proof is an algebraic simplification of theorem E.5 with $F(x) = v(x)$.

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The following definition for curvature is analogous to that given in [12, p.59] .

<u>Definition E.7</u> Call $R(a,b) \equiv (b \wedge a)\cdot(\not\nabla_{x} \wedge \not\nabla_{x})$ the curvature operator of the vectors a , b $\in \mathcal{D}^{1}_{x}$.

Applying theorem 10.6(i) to the intrinsic gradient $\not\nabla_{x}$ gives the following identity for $R(a,b)$, when $a(x), b(x) \in \{F(x)\}_{x}$

(E.8) $\quad R(a,b) \equiv [a\cdot\not\nabla_{x},\, b\cdot\not\nabla_{x}] - [a,b]\cdot\not\nabla_{x}$.

Applying $R(a,b)$ to a vector field $v(x)$ and using corollary E.6, gives a form of what is known as the Gauss curvature equation for a surface in $\mathcal{E}_{n}$. (See [12, p.76].)

<u>Theorem E.9</u> $\quad R(a,b)\, v = [S(a)\, S^{\dagger}(b)]_{2}\cdot v$

<u>Proof</u> The proof is direct using corollary E.6.

$$R(a,b)\,\underset{\sim}{v} = (\underset{\sim}{b}\wedge\underset{\sim}{a})\cdot(\nabla_{\underset{\sim}{x}}\wedge\nabla_{\underset{\sim}{x}})\,\underset{\sim}{v}(\underset{\sim}{x})$$

cor. E.6
$$= \frac{1}{2}(\underset{\sim}{b}\wedge\underset{\sim}{a})\cdot\nabla_{\bar{x}_2}\,\underset{\sim}{v}\cdot[p(\underset{\sim}{x}_1)\,p^{\dagger}(\underset{\sim}{x}_2)]_2$$

$$= \frac{1}{2}\underset{\sim}{v}\cdot[S(\underset{\sim}{b})\,S^{\dagger}(\underset{\sim}{a}) - S(\underset{\sim}{a})\,S^{\dagger}(\underset{\sim}{b})]_2$$

$$= [S(\underset{\sim}{a})\,S^{\dagger}(\underset{\sim}{b})]_2\cdot\underset{\sim}{v} \quad ;$$

XXXX

Finally theorem E.9 will be applied to a hypersurface $\mathcal{X}_{n-1}$ of $\mathcal{E}_n$ to show more clearly the relationship of this theorem to more usual formulations. Let I be a unit pseudoscalar element of $\mathcal{E}_n$, then $\underset{\sim}{n}(\underset{\sim}{x}) \equiv p(\underset{\sim}{x})I$ is an orthonormal vector field to $\mathcal{X}_{n-1}$.

The following definition and theorem are given in [12, p.77].

<u>Definition E.10</u> Call $L(\underset{\sim}{a}) \equiv \underset{\sim}{a}\cdot\nabla_{\underset{\sim}{x}}\,\underset{\sim}{n} = S(\underset{\sim}{a})\,I$ the Weingarten mapping for $\underset{\sim}{a} \in \mathcal{D}^1_{\underset{\sim}{x}}$.

<u>Theorem E.11</u> For the hypersurface $\mathcal{X}_{n-1}$,

$$R(\underset{\sim}{a},\underset{\sim}{b})\,\underset{\sim}{v} = \underset{\sim}{v}\cdot L(\underset{\sim}{b})\,L(\underset{\sim}{a}) - \underset{\sim}{v}\cdot L(\underset{\sim}{a})\,L(\underset{\sim}{b}) \;.$$

<u>Proof</u>
$$R(\underset{\sim}{a},\underset{\sim}{b})\,\underset{\sim}{v} = [S(\underset{\sim}{a})\,S^{\dagger}(\underset{\sim}{b})]_2\cdot\underset{\sim}{v}$$

$$= [S(\underset{\sim}{a})\,I\,I^{\dagger}S^{\dagger}(\underset{\sim}{b})]_2\cdot\underset{\sim}{v}$$

$$= [L(\underset{\smile}{a}) \wedge L(\underset{\smile}{b})]\cdot\underset{\smile}{v}$$

identity 0.39 $$= \underset{\smile}{v}\cdot L(\underset{\smile}{b})\, L(\underset{\smile}{a}) - \underset{\smile}{v}\cdot L(\underset{\smile}{a})\, L(\underset{\smile}{b}) \,.$$

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BIBLIOGRAPHY

1. Abraham, Ralph, Foundations of Mechanics, W.A. Benjamin, Inc., New York, 1967.

2. Apostol, Tom M., Calculus, Vol. II, Blaisdell Publishing Co., New York, 1962.

3. ________, Mathematical Analysis, Addison Wesley Publishing Co., Massachusetts, 1957.

4. Bishop, Richard L., and Crittenden, Richard J., Geometry of Manifolds, Academic Press, New York, 1964.

5. Buck, R. Creighton, Advanced Calculus, 2nd Edition, McGraw-Hill Book Company, New York, 1965.

6. Gantmacher, F. R., Matrix Theory, Vol. I, Chelsea Publishing Company, New York, 1960.

7. Greub, W.H., Linear Algebra, 3rd Edition, Springer-Verlag New York Inc., New York, 1967.

8. ________, Multilinear Algebra, Springer-Verlag New York Inc., New York, 1967.

9. Hestenes, David, "Multivector Calculus," Journal of Mathematical Analysis and Applications, Vol. 24, No. 2, November 1968, Academic Press, New York.

10. ________, "Multivector Functions," Journal of Mathematical Analysis and Applications, Vol. 24, No. 3, December 1968, Academic Press, New York.

11. _______, Space-Time Algebra, Documents on Modern Physics, Gordon and Breach Science Publishers, Inc., New York, 1966.

12. Hicks, Noel J., Notes on Differential Geometry, D. Van Nostrand Company, Inc., Princeton, 1965.

13. Hu, S. T., Differentiable Manifolds, Holt, Rinehart and Winston, Inc., New York, 1969.

14. Kreyszig, Erwin, Introduction to Differential Geometry and Riemannian Geometry, University of Toronto Press, Toronto, 1968.

15. Lehti, Raimo, "On the Introduction of Dyadics into Elementary Vector Algebra," Commentationes Physico-Mathematicae, XXVI, 2, Helsinki-Helsingfors, 1962.

16. O'Neill, Barrett, Elementary Differential Geometry, Academic Press, New York, 1966.

17. Segercrantz, Jerry, "Note on the Proof of the Cayley-Hamilton Theorem," Commentationes Physico-Mathematicae, Vol 34, No. 3, Helsinki-Helsingfors, 1968.

18. Whitney, Hassler, Geometric Integration Theory, Part I, Princeton University Press, Princeton, New Jersey, 1957.

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Garret Eugene Sobczyk was born in Cambridge, Massachusetts on April 3, 1943. He received his elementary education in Los Alamos, New Mexico, and his secondary education at P. K. Young High School in Gainesville, Florida. In 1961 he attended Illinois Institute of Technology. In 1962 he attended the University of Florida. In May 1964 he graduated from Western State College (Gunnison, Colorado) with a Bachelor of Arts degree in mathematics. He entered graduate school at the University of Virginia in September 1965. Since September 1966 he has held a graduate assistantship in the Department of Mathematics at Arizona State University, while studying for the degree of Doctor of Philosophy. He is single.

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Geometric Algebra and Linear Transformations: In process of being written by D. Hestenes and myself and based in part on results from my dissertation.