Unitary Geometric Algebra

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Abstract. The geometric significance of the imaginary unit in a complex geometric algebra has troubled the author for 40 years. In the *unitary geometric algebra* presented here, the imaginary i is a unit (pseudo) vector with square minus one which anti commutes with all of the real vectors. The resulting natural hermitian inner product and hermitian outer product induce a grading of the algebra into complex k-vectors. Basic orthogonality relationships are studied.

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1. Orthogonality relationships in Euclidean space

We begin by expressing basic orthogonality relationships in the real Euclidean space $\mathbb{R}^n$ utilizing the tools of its geometric algebra $\mathbb{G}_n = \mathbb{G}_n(\mathbb{R}^n)$. Whereas much of this material has appeared elsewhere [4], we review it here to fix notation and because later we shall generalize it to hermitian unitary spaces.

Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are said to be *orthogonal* in $\mathbb{R}^n$ if $\mathbf{a} \cdot \mathbf{b} = 0$. By a *basis* $(\mathbf{b})_{(n)}$ of $\mathbb{R}^n$, we mean the *row* of linearly independent vectors

$$(\mathbf{b})_{(n)} := \begin{pmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{pmatrix}.$$

By $(\mathbf{b})^T_{(n)}$, we mean the *column* of vectors

$$(\mathbf{b})^T_{(n)} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$$

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which is the *transpose* of $(\mathbf{b})_{(n)}$. We also use the notation

$$\mathbf{b}_{(k)} := \wedge(\mathbf{b})_{(k)} = \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k.$$

One of the basic tasks is given a basis $(\mathbf{a})_{(n)}$ of $\mathbb{R}^n$, to find a closely related *orthogonal basis* $(\mathbf{b})_{(n)}$ of $\mathbb{R}^n$ which satisfies the following two conditions,

$$\mathbf{b}_i \cdot \mathbf{b}_j = 0 \qquad \text{and} \qquad \mathbf{b}_{(k)} = \mathbf{a}_{(k)}, \tag{1.1}$$

for each $1 \le i < j \le n$ and $1 \le k \le n$.

This task is immediately completed by the following recursive construction,

$$\mathbf{b}_1 = \mathbf{a}_1 \qquad \text{and} \qquad \mathbf{b}_k = \frac{\mathbf{b}^{\dagger}_{(k-1)} \cdot (\mathbf{b}_{(k-1)} \wedge \mathbf{a}_k)}{|\mathbf{b}_{(k-1)}|^2},$$

for all $1 < k \le n$, and where $\mathbf{b}^{\dagger}_{(k-1)} = \mathbf{b}_{k-1} \wedge \cdots \wedge \mathbf{b}_1$ is the *reverse* of $\mathbf{b}_{(k-1)}$. Note, since the $\mathbf{b}_j$'s are orthogonal,

$$|\mathbf{b}_{(k)}|^2 = |\mathbf{b}_1 \cdots \mathbf{b}_k|^2 = |\mathbf{b}_1|^2 \cdots |\mathbf{b}_k|^2.$$

The above construction is often called the *Gram-Schmidt orthogonalization process.*

Now that we have constructed a corresponding orthogonal basis $(\mathbf{b})_{(n)}$ for the basis $(\mathbf{a})_{(n)}$, satisfying the conditions (1.1), let us see what we can do with it. Let f be the linear operator which takes the basis $(\mathbf{a})_{(n)}$ into the basis $(\mathbf{b})_{(n)}$, i.e., $f(\mathbf{a})_{(n)} = (\mathbf{a})_{(n)}[f] = (\mathbf{b})_{(n)}$ where $[f]$ is the matrix of f. We can solve this relationship directly for the matrix $[f]$. Using the fact that $(\mathbf{b}^{-1})^T_{(n)} \cdot (\mathbf{b})_{(n)}$ is the identity $n \times n$ matrix, we get

$$[f] = \left[(\mathbf{b}^{-1})^T \cdot (\mathbf{a})_{(n)}\right]^{-1} = \begin{pmatrix} \mathbf{b}_1^{-1} \cdot \mathbf{a}_1 & \dots & \mathbf{b}_1^{-1} \cdot \mathbf{a}_n \\ \vdots & \ddots & \vdots \\ \mathbf{b}_n^{-1} \cdot \mathbf{a}_1 & \dots & \mathbf{b}_n^{-1} \cdot \mathbf{a}_n \end{pmatrix}^{-1}$$

where $\mathbf{b}_j^{-1} = \frac{\mathbf{b}_j}{|\mathbf{b}_j|^2}$ for $j = 1, \dots, n$.

By the *Gram matrix* of the basis $(\mathbf{a})_{(n)}$, we mean the matrix

$$A = (\mathbf{a})^T_{(n)} \cdot (\mathbf{a})_{(n)} = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \dots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \dots & \mathbf{a}_n \cdot \mathbf{a}_n \end{pmatrix}.$$

But the relationship

$$(\mathbf{a})_{(n)}[f] = (\mathbf{b})_{(n)} \qquad \Longleftrightarrow \qquad [f]^T(\mathbf{a})^T_{(n)} = (\mathbf{b})^T_{(n)}$$

implies

$$[f]^T A [f] = [f]^T (\mathbf{a})^T_{(n)} \cdot (\mathbf{a})_{(n)}[f] = (\mathbf{b})^T_{(n)} \cdot (\mathbf{b})_{(n)} = B$$

where B is the diagonal Gram matrix of the orthogonal basis $(\mathbf{b})_{(n)}$. Thus, we have diagonalized the quadratic form defined by the matrix A. Since the relationship (1.1) implies that $\det[f] = 1$, it also follows that $\det A = \det B$.

The Gram determinant of the basis $(\mathbf{a})_{(n)}$ is defined to be the determinant of Gram matrix A of $(\mathbf{a})_{(n)}$. We have

$$\det A = \det[(\mathbf{a})_{(n)}^T \cdot (\mathbf{a})_{(n)}] = \mathbf{a}_{(n)}^\dagger \cdot \mathbf{a}_{(n)}.$$

Again, because of the properties (1.1) of the related basis $(\mathbf{b})_{(n)}$, it follows that in the geometric algebra $\mathbb{G}_n(\mathbb{R}^n)$

$$\mathbf{a}_{(k)}^\dagger \cdot \mathbf{a}_{(k)} = \mathbf{b}_{(k)}^\dagger \cdot \mathbf{b}_{(k)} \geq 0$$

for $1 \leq k \leq n$. In the case when $n = 2$, we have the *Schwarz inequality* [7, p. 218]

$$\mathbf{a}_{(2)}^\dagger \cdot \mathbf{a}_{(2)} = (\mathbf{a}_2 \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \mathbf{a}_2) = |\mathbf{a}_1|^2 |\mathbf{a}_2|^2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2 \geq 0.$$

We see that the classical proofs of the above results reduce to basic algebraic identities in the geometric algebra $\mathbb{G}_n$ of $\mathbb{R}^n$.

We generalize these fundamental results to the unitary geometric algebra $\mathbb{U}_n$ of the unitary space $\mathbb{H}^n$ in the last section of this paper.

2. Unitary geometric algebra of pseudo Euclidean space

We now show how all of the previous identities in the geometric algebra $\mathbb{G}_n$ of $\mathbb{R}^n$ can be generalized to hold in a larger *complex geometric algebra*, which I call the *unitary geometric algebra* $\mathbb{U}_{p,q}$ of the geometric algebra $\mathbb{G}_{p,q}$ of the pseudo Euclidean space $\mathbb{R}^{p,q}$. Unlike in my previous approach [8], or the usual methods of the Clifford analysis groups [1,2], we extend the geometric algebra $\mathbb{G}_{p,q}$ by a new unit vector which has square minus one and anti commutes with all the vectors in $\mathbb{R}^{p,q}$. In this approach, the geometric interpretation of a complex scalar is unambiguous and the Hermitian inner product arises in a natural way. This larger complexified geometric algebra is necessary in order to have all of the tools of geometric algebra available for the study of hermitian spaces and their generalizations [6].

By a *(complex) vector* $\mathbf{x}$ in the hermitian space $\mathbb{H}^{p,q}$, we mean the quantity $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{p,q} \subset \mathbb{R}^{p,q+1}$ and

$$i = \mathbf{e}_{p+q+1} \in \mathbb{R}^{p,q+1} \subset \mathbb{G}_{p,q+1}$$

with $i^2 = -1$. The complex vector $\mathbf{x}$ is thus the sum of the *real* vector part $\mathbf{x}_1 \in \mathbb{R}^{p,q}$, and the *imaginary* vector part $i\mathbf{x}_2$, which is a bivector in $\mathbb{G}_{p,q+1}$. Since $i = \mathbf{e}_{p+q+1}$ is a *pseudovector*[1] in $\mathbb{R}^{p,q+1}$ with square -1, which is orthogonal to all of the basis vectors in $(\mathbf{e})_{(p+q)} \subset (\mathbf{e})_{(p+q+1)}$, it follows that

$$\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2 = \mathbf{x}_1 - \mathbf{x}_2 i.$$

We say that $\overline{\mathbf{x}} = \mathbf{x}_1 - i\mathbf{x}_2 = \mathbf{x}^\dagger$ is the *complex conjugate* of $\mathbf{x}$, where $\overline{A}$ for $A \in \mathbb{G}_{p,q+1}$ is operation of *complex conjugation.* The conjugate $\overline{A}$ of A

[1] The term "pseudovector" is used in mathematics in different ways. Since we use the term "pseudo Euclidean space", it seems appropriate that a pseudovector would be a vector which has negative square.

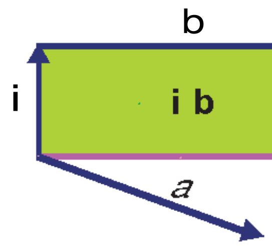

FIGURE 1. The complex vector $\mathbf{a} + i\mathbf{b}$.

agrees with the previously defined operation of reversal when the argument A is a complex vector, but it does not reverse the order of the terms in a geometric product. The conjugation $\overline{A}$ of any geometric number $A \in \mathbb{U}_{p,q}$ can most simply be obtained by replacing i by $-i$ wherever i appears in A, and is the natural generalization of the conjugation of a complex number[2]. By a *complex scalar* $\alpha \in \mathbb{U}_{p,q}$, we mean $\alpha = a_1 + ia_2$ where $a_1, a_2 \in \mathbb{R}$, and $\overline{\alpha} = a_1 - ia_2$. For $A, B \in \mathbb{G}_{p,q+1}$, we have

$$\overline{A+B} = \overline{A} + \overline{B} \qquad \text{and} \qquad \overline{AB} = \overline{A}\,\overline{B}.$$

Note that our complex conjugation is not the *Clifford-conjugation* defined in [5, p. 29].

The unitary geometric algebra

$$\mathbb{U}_{p,q} = \mathbb{U}_{p,q}(\mathbb{H}^{p+q}) = \mathbb{G}_{p,q+1} \tag{2.1}$$

of the hermitian space $\mathbb{H}^{p+q} = \mathbb{R}^{p+1} + i\mathbb{R}^{p+q}$ has exactly the same elements as the geometric algebra $\mathbb{G}_{p,q+1}(\mathbb{R}^{p+q+1})$. The geometric product of the elements of $\mathbb{U}_{p,q}$ are also exactly the same as for the same elements in $\mathbb{G}_{p,q+1}(\mathbb{R}^{p+q+1})$. What is different is what we mean by *complex vectors*, how we *grade* the algebra into *complex k-vectors*, and how we define the *hermitian inner product* and the *hermitian outer product* of the *complex multivectors* in $\mathbb{U}_{p+q}$. By a *complex k-vector* $\mathbf{C}_k \in \mathbb{U}_{p,q}$, we mean $\mathbf{C}_k = \mathbf{A}_k + i\mathbf{B}_k$ where $\mathbf{A}_k, \mathbf{B}_k$ are k-vectors in $\mathbb{G}^k_{p,q}$. Alternatively, $\mathbb{U}_{p,q} = \mathbb{G}_{p,q} + i\mathbb{G}_{p,q}$ can be thought of as being generated by taking all geometric sums of geometric products of the complex vectors in $\mathbb{H}^{p+q}$.

We are now ready to define the hermitian inner and outer products of complex vectors $\mathbf{x}, \mathbf{y} \in \mathbb{H}^{p+q}$. The *hermitian inner product* is defined by

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(\mathbf{x}\mathbf{y} + \overline{\mathbf{y}}\,\overline{\mathbf{x}}) = \frac{1}{2}(\mathbf{x}\mathbf{y} + (\mathbf{x}\mathbf{y})^{\dagger}),$$

[2] In terms of the inner product of the geometric algebra $\mathbb{G}_{p,q+1}$, $\overline{A} = A + 2i(i \cdot A)$.

and the *hermitian outer product* is defined by

$$\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{x}\mathbf{y} - \overline{\mathbf{y}}\,\overline{\mathbf{x}}) = \frac{1}{2}(\mathbf{x}\mathbf{y} - (\mathbf{x}\mathbf{y})^\dagger),$$

from which it follows that $\mathbf{x}\mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}$. Letting $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 + i\mathbf{y}_2$ for the complex vectors $\mathbf{x}, \mathbf{y} \in \mathbb{H}^{p+q}$, we calculate

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \mathbf{x}_1 \cdot \mathbf{y}_1 + \mathbf{x}_2 \cdot \mathbf{y}_2 + i(\mathbf{x}_2 \cdot \mathbf{y}_1 - \mathbf{x}_1 \cdot \mathbf{y}_2) = \langle \mathbf{x}\mathbf{y} \rangle_{\mathbb{C}}, \\
\mathbf{x} \cdot \overline{\mathbf{y}} &= \mathbf{x}_1 \cdot \mathbf{y}_1 - \mathbf{x}_2 \cdot \mathbf{y}_2 + i(\mathbf{x}_2 \cdot \mathbf{y}_1 + \mathbf{x}_1 \cdot \mathbf{y}_2) = \langle \mathbf{x}\overline{\mathbf{y}} \rangle_{\mathbb{C}}, \\
\mathbf{x} \wedge \mathbf{y} &= \mathbf{x}_1 \wedge \mathbf{y}_1 + \mathbf{x}_2 \wedge \mathbf{y}_2 + i(\mathbf{x}_2 \wedge \mathbf{y}_1 - \mathbf{x}_1 \wedge \mathbf{y}_2) = \langle \mathbf{x}\mathbf{y} \rangle_B, \\
\mathbf{x} \wedge \overline{\mathbf{y}} &= \mathbf{x}_1 \wedge \mathbf{y}_1 - \mathbf{x}_2 \wedge \mathbf{y}_2 + i(\mathbf{x}_2 \wedge \mathbf{y}_1 + \mathbf{x}_1 \wedge \mathbf{y}_2) = \langle \mathbf{x}\overline{\mathbf{y}} \rangle_B,
\end{aligned} \tag{2.2}$$

where $\langle \mathbf{x}\mathbf{y} \rangle_{\mathbb{C}}$ and $\langle \mathbf{x}\mathbf{y} \rangle_B$ denote the complex scalar and complex bivector parts of the geometric product $\mathbf{x}\mathbf{y}$, respectively. Note also that the hermitian inner and outer products are *hermitian symmetric* and *hermitian antisymmetric*, respectively, i.e.,

$$\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}} = \overline{\mathbf{y}} \cdot \overline{\mathbf{x}} \qquad \text{and} \qquad \mathbf{x} \wedge \mathbf{y} = -\overline{\mathbf{y} \wedge \mathbf{x}} = -\overline{\mathbf{y}} \wedge \overline{\mathbf{x}}.$$

The hermitian inner and outer products reduce to the ordinary inner and outer product in $\mathbb{G}_{p,q}$ when the complex vectors $\mathbf{x}, \mathbf{y} \in \mathbb{U}^{p+q}$ are real, i.e., $\mathbf{x} = \overline{\mathbf{x}}$ and $\mathbf{y} = \overline{\mathbf{y}}$, and, amazingly, the closely related identities satisfied by the hermitian inner and outer products of complex k-vectors reduce to their real counterparts in $\mathbb{G}_{p,q}$. Let $\mathbf{x} \in \mathbb{H}^{p+q}$ be a complex vector. In the standard

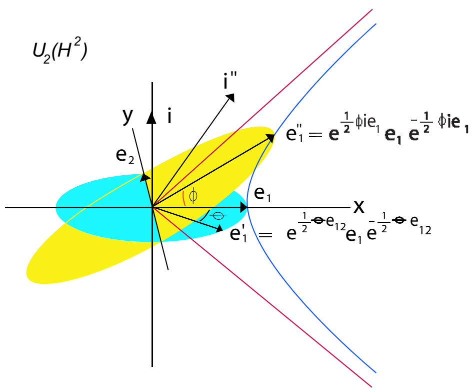

Figure 2. The vector $\mathbf{e}_1$ is rotated in the xy-plane of $\mathbf{e}_{12}$ into the vector $\mathbf{e}_1'$. The vector $\mathbf{e}_1$ is *boosted* into the *relative vector* $\mathbf{e}_1''$ of the *relative plane* of the *relative bivector* $\mathbf{e}_{12}'' = \mathbf{e}_1''\mathbf{e}_2$. The relative plane of $\mathbf{e}_{12}''$ has the relative velocity of $\frac{\mathbf{v}}{c} = i\mathbf{e}_1 \tanh\phi$ with respect to the plane of $\mathbf{e}_{12}$, where c is the speed of light.

row basis $(\mathbf{e})_{(n)}$ of $\mathbb{H}^{p+q}$, we write

$$\mathbf{x} = (\mathbf{e})_{(n)}(x)^{(n)} = \sum_{i=1}^{n} \mathbf{e}_i x^i = \sum_{i=1}^{n} \overline{x}_i \mathbf{e}^i = (\overline{x})_{(n)}(\mathbf{e})^{(n)} = \mathbf{x}^* \tag{2.3}$$

where $(x)^{(n)}$ is the column of complex scalar components of $\mathbf{x}$, and $(\overline{x})_{(n)} = [(x)^{(n)}]^*$ is the corresponding *hermitian transposed* row of the complex scalar components of $\mathbf{x}$. The Hermitian transpose reduces to ordinary transposition when $\overline{\mathbf{x}} = \mathbf{x}$.

Let us now explore the corresponding hermitian versions of the identities in the unitary geometric algebra $\mathbb{U}_{p,q}$ that were given for the real geometric algebra $\mathbb{G}_{p,q}$. The inner and outer products between a vector $\mathbf{a}$ and a k-vector $\mathbf{B}_k$ are now generalized to the hermitian inner and outer products of a complex vector and a complex k-vector. For *odd* $k \geq 1$,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{B}_k &:= \frac{1}{2}(\mathbf{a}\mathbf{B}_k + \overline{\mathbf{B}}_k\, \overline{\mathbf{a}}) = \langle \mathbf{a}\mathbf{B}_k \rangle_{k-1}, \\ \mathbf{a} \cdot \mathbf{B}_{k+1} &:= \frac{1}{2}(\mathbf{a}\mathbf{B}_{k+1} - \overline{\mathbf{B}}_{k+1}\, \mathbf{a}) = \langle \mathbf{a}\mathbf{B}_{k+1} \rangle_{k}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathbf{a} \wedge \mathbf{B}_k &:= \frac{1}{2}(\mathbf{a}\mathbf{B}_k - \overline{\mathbf{B}}_k\, \overline{\mathbf{a}}) = \langle \mathbf{a}\mathbf{B}_k \rangle_{k+1}, \\ \mathbf{a} \wedge \mathbf{B}_{k+1} &:= \frac{1}{2}(\mathbf{a}\mathbf{B}_{k+1} + \overline{\mathbf{B}}_{k+1}\, \mathbf{a}) = \langle \mathbf{a}\mathbf{B}_{k+1} \rangle_{k+2}, \end{aligned} \tag{2.5}$$

so that $\mathbf{a}\mathbf{B}_j = \mathbf{a} \cdot \mathbf{B}_j + \mathbf{a} \wedge \mathbf{B}_j$ for all $j \geq 1$. In the case that $\mathbf{a}$ and $\mathbf{B}_k$ are real, the identities (2.4) and (2.5) reduce to real counterparts.

We also have the identities

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{C}_k) = (\mathbf{a} \cdot \mathbf{b})\mathbf{C}_k - \overline{\mathbf{b}} \wedge (\overline{\mathbf{a}} \cdot \mathbf{C}_k).$$

When r and s are both odd, we have

$$\mathbf{a} \cdot (\mathbf{A}_r \wedge \mathbf{B}_s) = (\mathbf{a} \cdot \mathbf{A}_r) \wedge \mathbf{B}_s - \overline{\mathbf{A}}_r \wedge (\overline{\mathbf{a}} \cdot \mathbf{B}_s) = -(\overline{\mathbf{A}}_r \wedge \overline{\mathbf{B}}_s) \cdot \mathbf{a}.$$

In the last identity, when r and s are both even, the identity must be modified to read

$$\mathbf{a} \cdot (\mathbf{A}_r \wedge \mathbf{B}_s) = (\mathbf{a} \cdot \mathbf{A}_r) \wedge \mathbf{B}_s + \overline{\mathbf{A}}_r \wedge (\mathbf{a} \cdot \mathbf{B}_s) = -(\overline{\mathbf{A}}_r \wedge \overline{\mathbf{B}}_s) \cdot \mathbf{a}.$$

The rules for the other two case of this identity when $r + s$ is odd are left to the reader.

From these identities, many other useful identities for the hermitian geometric algebra $\mathbb{U}_{p,q}$ can be derived. For example, for complex vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{H}^{p+q}$, we have

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - \overline{\mathbf{b}}(\overline{\mathbf{a}} \cdot \mathbf{c}) = \mathbf{a} \cdot \mathbf{b}\ \mathbf{c} - \mathbf{a} \cdot \overline{\mathbf{c}}\ \overline{\mathbf{b}} = -(\overline{\mathbf{b}} \wedge \overline{\mathbf{c}}) \cdot \mathbf{a}.$$

With care, anyone familiar with the rules of real geometric algebra can quickly become adept to the closely related rules in the unitary geometric algebra $\mathbb{U}_{p,q}$.

However, there are peculiarities that must be given careful consideration. For example, in general $\mathbf{a}\wedge\mathbf{a}\neq 0$, but always $\mathbf{a}\wedge\overline{\mathbf{a}}=0$. Also, a complex vector $\mathbf{a}\in\mathbb{H}^{p+q}$ is not always invertible, as follows from

$$\mathbf{a}\,\overline{\mathbf{a}}=\mathbf{a}\cdot\overline{\mathbf{a}}+\mathbf{a}\wedge\overline{\mathbf{a}}=\mathbf{a}\cdot\overline{\mathbf{a}}$$

which implies that the inverse

$$\mathbf{a}^{-1}=\overline{\mathbf{a}}\left(\frac{1}{\mathbf{a}\cdot\overline{\mathbf{a}}}\right)=\left(\frac{1}{\overline{\mathbf{a}}\cdot\mathbf{a}}\right)\overline{\mathbf{a}}$$

of the complex vector $\mathbf{a}$, with respect to the unitary geometric product, exists only when $\mathbf{a}\cdot\overline{\mathbf{a}}\neq 0$. Examining the identities (2.2) shows that $\mathbf{a}\cdot\mathbf{a}=0$ for $\mathbf{a}\in\mathbb{H}^n$ if and only if $\mathbf{a}=0$. However, $\mathbf{a}\wedge\mathbf{a}=0$ for $\mathbf{a}=\mathbf{a}_1+i\mathbf{a}_2\in\mathbb{H}^{p+q}$ if and only if $\mathbf{a}_1=\alpha\mathbf{a}_2$ for some $\alpha\in\mathbb{R}$. On the other hand, $\mathbf{a}\cdot\overline{\mathbf{a}}=0$ for $\mathbf{a}\in\mathbb{H}^{p+q}$ if and only if $\mathbf{a}_1^2=\mathbf{a}_2^2$ and $\mathbf{a}_1\cdot\mathbf{a}_2=0$.

By the *magnitude* of a complex vector we mean $|\mathbf{a}|=\sqrt{\mathbf{a}\cdot\mathbf{a}}$. In the case of the unitary geometric algebra $\mathbb{U}_n$ of the Euclidean space $\mathbb{R}^n$ the magnitude is positive definite, but there are still non-invertible complex vectors with positive magnitude. For example, for $\mathbf{a}=\mathbf{e}_1+i\mathbf{e}_2\in\mathbb{H}^2$, $\mathbf{a}\cdot\mathbf{a}=2$, but $\mathbf{a}^{-1}$ does not exist since

$$\mathbf{a}\cdot\overline{\mathbf{a}}=(\mathbf{e}_1+i\mathbf{e}_2)\cdot(\mathbf{e}_1-i\mathbf{e}_2)=\mathbf{e}_1^2-\mathbf{e}_2^2+2i\mathbf{e}_1\cdot\mathbf{e}_2=0.$$

Sometimes, however, we will write $\mathbf{a}^{-1}=\frac{\mathbf{a}}{|\mathbf{a}|^2}$ for which $\mathbf{a}\cdot\mathbf{a}^{-1}=1$ with respect to the hermitian inner product.

Suppose now that a column $(\mathbf{a})^{(n)}$ of n complex vectors in $\mathbb{H}^{p,q}$ is given. In terms of the standard basis $(\mathbf{e})_{(n)}$ of $\mathbb{H}^{p,q}$, we can write

$$(\mathbf{a})^{(n)}=A(\mathbf{e})^{(n)}=\left[(\mathbf{e})_{(n)}A^*\right]^T, \tag{2.6}$$

where the k^{th} row of the $n\times n$ complex matrix A consists of the components of the corresponding complex vectors $\mathbf{a}^k=\mathbf{a}_k$, and $A^*=\overline{A}^T$ is the conjugate transpose of the matrix A. For example, for $n=2$ we have

$$\begin{aligned}(\mathbf{a})^{(2)}=A(\mathbf{e})^{(2)}&=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\\mathbf{e}_2\end{pmatrix}=\begin{pmatrix}a_{11}\mathbf{e}_1+a_{12}\mathbf{e}_2\\a_{21}\mathbf{e}_1+a_{22}\mathbf{e}_2\end{pmatrix}\\&=\begin{pmatrix}\mathbf{e}_1\overline{a}_{11}+\mathbf{e}_2\overline{a}_{12}\\\mathbf{e}_1\overline{a}_{21}+\mathbf{e}_2\overline{a}_{22}\end{pmatrix}=\left[\begin{pmatrix}\mathbf{e}_1&\mathbf{e}_2\end{pmatrix}\begin{pmatrix}\overline{a}_{11}&\overline{a}_{21}\\\overline{a}_{12}&\overline{a}_{22}\end{pmatrix}\right]^T=\left[(\mathbf{e})_{(2)}A^*\right]^T.\end{aligned} \tag{2.7}$$

We can now relate the determinant function of the matrix A to the outer product of the complex vectors $(\mathbf{a})_{(n)}$. We have the rather strange looking relationship

$$\mathbf{a}_{(\overline{n})}\equiv\mathbf{a}_1\wedge\overline{\mathbf{a}}_2\wedge\cdots\wedge\mathbf{a}_{2k+1}\wedge\overline{\mathbf{a}}_{2k}\wedge\cdots=(\det A)\mathbf{e}_{(n)}. \tag{2.8}$$

The hermitian character of the outer product requires that we take the conjugation of every even numbered complex vector in the product. For the

2-dimensional example (2.7), we have

$$\mathbf{a}_1 \wedge \overline{\mathbf{a}}_2 = (a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2) \wedge (\overline{a}_{21}\mathbf{e}_1 + \overline{a}_{22}\mathbf{e}_2)$$
$$= (a_{11}a_{22} - a_{12}a_{21})\mathbf{e}_{(2)} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{e}_{12}. \tag{2.9}$$

It follows that a column of complex vectors $(\mathbf{a})^{(n)}$, defined in the equation (2.6), is (complex) linearly independent iff $\det A \neq 0$.

3. Hermitian orthogonality

We now generalize the results obtained in Section 1 for the Euclidean space $\mathbb{R}^n$ to the unitary space $\mathbb{H}^n$. Whereas almost all of these results are more generally true in the pseudo Euclidean spaces $\mathbb{R}^{p+q}$, and the corresponding unitary geometric algebras $\mathbb{U}_{p,q}$, the methods are amply illustrated in $\mathbb{H}^n$ using the tools of the unitary geometric algebra $\mathbb{U}_n$. Two complex vectors

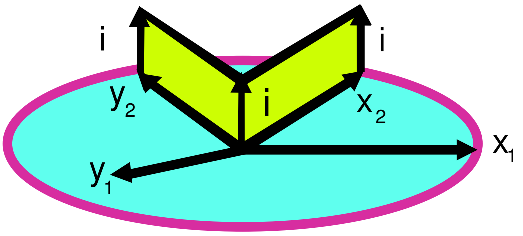

FIGURE 3. The complex vectors $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 + i\mathbf{y}_2$ are hermitian orthogonal.

$\mathbf{a}, \mathbf{b} \in \mathbb{H}^n$ are said to be *hermitian orthogonal* in $\mathbb{H}^n$ if $\mathbf{a} \cdot \mathbf{b} = 0$; they are said to be *conjugate orthogonal* if $\mathbf{a} \cdot \overline{\mathbf{b}} = 0$. By the *magnitude* of a complex vector $\mathbf{a} \in \mathbb{H}^n$ we mean $|a| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. If $\mathbf{a} = \mathbf{a}_1 + i\mathbf{a}_2$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are real vectors, then

$$|\mathbf{a}|^2 = (\mathbf{a}_1 + i\mathbf{a}_2) \cdot (\mathbf{a}_1 + i\mathbf{a}_2) = |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 \geq 0,$$

so the magnitude $|\mathbf{a}| = 0$ iff $\mathbf{a} = 0$.

A row $(\mathbf{a})_{(k)} = (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ of k complex vectors $\mathbf{a}_i \in \mathbb{H}^n$ is said to be *(complex) linearly independent* if for every column $(\alpha)^T_{(k)}$ of not all zero complex scalars $\alpha_i \in \mathbb{C}$,

$$(\mathbf{a})_{(k)}(\alpha)^T_{(k)} = \begin{pmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \sum_{j=1}^k \mathbf{a}_j \alpha_j = \sum_{j=1}^k \overline{\alpha}_j \mathbf{a}_j$$
$$= \begin{pmatrix} \overline{\alpha}_1 & \ldots & \overline{\alpha}_k \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \end{pmatrix} = (\overline{\alpha})_{(k)}(\mathbf{a})^T_{(k)} \neq 0. \tag{3.1}$$

We can easily characterize the linear dependence or independence of a row of complex vectors by using the hermitian outer product. But first we introduce some necessary notation. If $(\mathbf{a})_{(k)}$ is a row of complex vectors, where k is even, we define the row $(\mathbf{a})_{(\overline{k})}$ by

$$(\mathbf{a})_{(\overline{k})} = \begin{pmatrix}\mathbf{a}_1 & \overline{\mathbf{a}}_2 & \mathbf{a}_3 & \overline{\mathbf{a}}_4 & \dots & \overline{\mathbf{a}}_k\end{pmatrix}.$$

On the other hand, if k is odd, we have

$$(\mathbf{a})_{(\overline{k})} = \begin{pmatrix}\mathbf{a}_1 & \overline{\mathbf{a}}_2 & \mathbf{a}_3 & \overline{\mathbf{a}}_4 & \dots & \mathbf{a}_k\end{pmatrix}.$$

In both rows, only the even terms of the rows are conjugated. We define the corresponding alternatingly conjugated k-complex vectors, consistent with (2.8), by

$$\mathbf{a}_{(\overline{k})} = \wedge(\mathbf{a})_{(\overline{k})}.$$

With this notation, a row of complex vectors $(\mathbf{a})_{(k)}$ is linearly independent if and only if

$$\mathbf{a}_{(\overline{k})} = \wedge(\mathbf{a})_{(\overline{k})} \neq 0.$$

The *magnitude* $|\mathbf{a}_{(k)}|$ of the complex k-vector $\mathbf{a}_{(k)}$ is defined by

$$|\mathbf{a}_{(k)}| = \sqrt{\overline{\mathbf{a}}^{\dagger}_{(k)} \cdot \mathbf{a}_{(k)}}.$$

We will shortly see that the magnitude of a complex k-vector is also positive definite.

Given a basis $(\mathbf{a})_{(n)}$ of $\mathbb{H}^n$, we want to find the closely related *hermitian orthogonal basis* $(\mathbf{b})_{(n)}$ of $\mathbb{H}^n$ which satisfies the following two conditions,

$$\mathbf{b}_i \cdot \mathbf{b}_j = 0 \qquad \text{and} \qquad \mathbf{b}_{(\overline{k})} = \mathbf{a}_{(\overline{k})} \tag{3.2}$$

for all $1 \le i < j \le n$ and $1 \le k \le n$.

This task is immediately completed by the following recursive construction. For $k = 1$, we set $\mathbf{b}_1 = \mathbf{a}_1$. For all k, $2 \le k \le n$,

$$\mathbf{b}_k = \frac{\overline{\mathbf{b}}_{(\overline{k-1})} \cdot (\mathbf{b}^{\dagger}_{(\overline{k-1})} \wedge \mathbf{a}_k)}{|\mathbf{b}_{(\overline{k-1})}|^2}. \tag{3.3}$$

Note, since the $\mathbf{b}_j$'s are orthogonal,

$$|\overline{\mathbf{b}}_{(\overline{k-1})}|^2 = \overline{\mathbf{b}}_{(\overline{k-1})} \cdot \mathbf{b}^{\dagger}_{(\overline{k-1})} = |\mathbf{b}_1|^2 \cdots |\mathbf{b}_{k-1}|^2.$$

Our Gram-Schmidt orthogonalization (3.3) is closely related to [3, p. 258].

Now that we have constructed a corresponding hermitian orthogonal basis $(\mathbf{b})_{(n)}$ for the basis $(\mathbf{a})_{(n)}$, satisfying the conditions (3.2), let us proceed with the same calculations that we did earlier for an orthogonal basis of $\mathbb{R}^n$.

Let f be the linear operator which takes the basis $(\mathbf{a})_{(n)}$ into the basis $(\mathbf{b})_{(n)}$, i.e., $f(\mathbf{a})_{(n)} = (\mathbf{a})_{(n)}[f] = (\mathbf{b})_{(n)}$ where $[f]$ is the matrix of f. We can solve this relationship directly for the matrix $[f]$, getting

$$[f] = \left[(\mathbf{b}^{-1})^{(n)} \cdot (\mathbf{a})_{(n)}\right]^{-1} = \begin{pmatrix}\mathbf{b}_1^{-1} \cdot \mathbf{a}_1 & \dots & \mathbf{b}_1^{-1} \cdot \mathbf{a}_n \\ \vdots & \ddots & \vdots \\ \mathbf{b}_n^{-1} \cdot \mathbf{a}_1 & \dots & \mathbf{b}_n^{-1} \cdot \mathbf{a}_n\end{pmatrix}^{-1}$$

where each $\mathbf{b}_i^{-1} = \frac{\mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i} = \frac{\mathbf{b}_i}{|\mathbf{b}_i|^2}$. But the relationship

$$(\mathbf{a})_{(n)}[f] = (\mathbf{b})_{(n)} \quad \iff \quad [f]^*(\mathbf{a})_{(n)}^T = (\mathbf{b})_{(n)}^T,$$

implies

$$[f]^* A[f] = [f]^*(\mathbf{a})_{(n)}^T \cdot (\mathbf{a})_{(n)}[f] = (\mathbf{b})_{(n)}^T \cdot (\mathbf{b})_{(n)} = B$$

where B is the diagonal Gram matrix of the orthogonal basis $(\mathbf{b})_{(n)}$. Thus, we have diagonalized the Hermitian quadratic form defined by the matrix A. Since the relationship (3.2) implies that $\det[f] = 1$, it also follows that $\det A = \det B$.

The Gram determinant of the basis $(\mathbf{a})_{(n)}$ is defined to be the determinant of the Gram matrix A of $(\mathbf{a})_{(n)}$. We have

$$\det A = \det[(\mathbf{a})_{(n)}^T \cdot (\mathbf{a})_{(n)}] = \overline{\mathbf{a}}^\dagger_{(\overline{n})} \cdot \mathbf{a}_{(\overline{n})}.$$

Again, because of the properties (3.2) of the related basis $(\mathbf{b})_{(n)}$, it follows that in the unitary geometric algebra $\mathbb{U}_n(\mathbb{H}^n)$,

$$\overline{\mathbf{a}}^\dagger_{(\overline{k})} \cdot \mathbf{a}_{(\overline{k})} = \overline{\mathbf{b}}^\dagger_{(\overline{k})} \cdot \mathbf{b}_{(\overline{k})} \geq 0,$$

for $1 \leq k \leq n$. In the case when $n = 2$, we have the *Schwarz inequality* [7, p. 218],

$$\overline{\mathbf{a}}^\dagger_{(\overline{2})} \cdot \mathbf{a}_{(\overline{2})} = (\overline{\mathbf{a}}_2 \wedge \mathbf{a}_1) \cdot (\mathbf{a}_1 \wedge \overline{\mathbf{a}}_2) = |\mathbf{a}_1|^2 |\mathbf{a}_2|^2 - |\mathbf{a}_1 \cdot \mathbf{a}_2|^2 \geq 0.$$

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