Notes on Plücker's Relations in Geometric Algebra

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Abstract

Grassmannians are of fundamental importance in projective geometry, algebraic geometry, and representation theory. A vast literature has grown up utilizing using many different languages of higher mathematics, such as multilinear and tensor algebra, matroid theory, and Lie groups and Lie algebras. Here we explore the basic idea of the Plücker relations in Clifford's geometric algebra. We discover that the Plücker Relations can be fully characterized in terms of the geometric product, without the need for a confusing hodgepodge of many different formalisms and mathematical traditions found in the literature.

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1 Plücker's relations in geometric algebra

Clifford's *geometric algebra* is considered here to be the natural *geometrization* of the real and complex number sytems; an overview is provided in [8]. We use exclusively the language of geometric algebra, developed in the books [5, 10]. Plücker's coordinates are briefly touched upon on page 30 of [5], but Plücker's relations are not discussed. These relations characterize when an r-vector B in the geometric algebra $\mathbb{G}_n$ of an n-dimensional (real or complex) Euclidean vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$), can be expressed as the outer product of r-vectors. Several references to the relevant literature, and its more advanced applications, are [1, 2], [3, pp.227-231], [4], [6, pp.144-148], and [11].

Let $B = \sum_J B_J e_J$ be an r-vector expanded in the standard orthonormal basis of r-vector blades $e_{J_m} \in \mathbb{G}_n^r$, ordered lexicographically, and let $\#(J_p \cap J_m)$ be the number of integers that the sequences J_p and J_m have in common. Given an r-vector $B \in \mathbb{G}_n^r$, the *rank space* V_B of B is

$$V_B := \{x \in V^n |\; x \wedge B = 0\} \subset V^n,$$

where $V^n := V_{e_{1\cdots n}} \equiv \mathbb{R}^n$ is the underlying vector space of the geometric algebra $\mathbb{G}_n$. The *dimension* of the subspace V_B of V^n defined by B is denoted by $\#V_B$.

Lemma 1 *Given an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{G}_n$, an r-vector $B \in \mathbb{G}_n^r$, expanded in this basis, is given by*

$$B = \sum_m B_{J_m} e_{J_m}, \tag{1}$$

where $B_{J_m} = B \cdot e_{J_m}^\dagger$.

Proof: In the orthonormal r-vector basis $\{e_{J_m}\}$, the r-vector $B \in \mathbb{G}_n^r$ is given by

$$B = \sum_{p=1}^{\binom{n}{r}} B_{J_p} e_{J_p}$$

Dotting both sides of this equations with $e_{J_m}^\dagger$, the *reverse* of e_{J_m}, shows that $B_{J_m} = B \cdot e_{J_m}^\dagger$, since $e_{J_p} \cdot e_{J_m}^\dagger = \delta_{p,m}$.

□

Definition 1 *A non-zero r-vector $B \in \mathbb{G}_n^r$ is said to be divisible by a non-zero k-blade $K \in \mathbb{G}_n^k$, where $k \le r$, if $KB = \langle KB \rangle_{r-k}$. If $k = r$, B is said to be totally decomposable.*

If B is divisible by L, then

$$B = \frac{L(LB)}{L^2} = L\frac{L \cdot B}{L^2}.$$

If B is totally decomposable by L, then $B = \beta L$ for $\beta = \frac{L \cdot B}{L^2} \in \mathbb{R}$, in which case B is an r-blade.

A classical result is that an r-vector $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$ is an r-blade iff B satisfies the *Plücker relations*. The Plücker relations, [9, p.3], are

$$(A \cdot B) \wedge B = 0 \tag{2}$$

for all $(r-1)$-vectors $A \in \mathbb{G}_n^{r-1}$. In particular, for all coordinate $(r-1)$-vectors $e_{K_p} \in \mathbb{G}_n^{r-1}$,

$$(e_{K_p} \cdot B) \wedge B = 0.$$

Since an r-blade trivially satisfies the Plücker relations, it follows that if there is a coordinate $(r-1)$-blade e_K, such that $(e_K \cdot B) \wedge B \neq 0$, then B is not an r-blade.

For every coordinate $(r-1)$-blade e_{K_p}, there is a coordinate r-blade e_{J_p}, generally not unique, and a coordinate vector e_{j_p}, satisfying

$$e_{J_p} = \pm e_{j_p} e_{K_p} \quad \iff \quad e_{j_p} e_{J_p} = \pm e_{K_p}. \tag{3}$$

Indeed, when $e_{K_p} \cdot B \neq 0$, we always pick the coordinate vector e_{j_p}, and rearrange the order of $K_p = \pm K_{j_p}$, in such a way that $B_{J_{j_p}} = (e_{j_p} \wedge e_{K_{j_p}})^\dagger \cdot B \neq 0$. It follows that the Plücker relations (2) are equivalent to

$$e_{j_p} \cdot \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge B\Big] = B_{J_{j_p}} B - \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge \Big(e_{j_p} \cdot B\Big)\Big] = 0,$$

or equivalently,

$$B = B_{J_{j_p}}^{-1} \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge \Big(e_{j_p} \cdot B\Big)\Big], \tag{4}$$

for all coordinate $(r-1)$-blades $e_{K_p} \in \mathbb{G}_n^{r-1}$ for which $e_{K_p} \cdot B \neq 0$. It follows that when the Plücker relation (2) is satisfied, then the vector $e_{K_p} \cdot B$ divides B.

For a given non-zero r-vector $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$, let $\#_{max} S$ be the maximal number of linearly independent vectors in the set

$$S_B = \{e_{K_j} \cdot B | \quad e_{K_j} \in \mathbb{G}_n^{r-1}\}. \tag{5}$$

Lemma 2 *Given an r-vector $B \neq 0$, and the set S_B defined in (5). Then $r \leq \#_{max} S \leq n$, and $\#_{max} = r$ only if B is an r-blade*

Proof: Since $B \neq 0$, it follows that $B_{J_m} \neq 0$ for some

$$1 \leq m \leq \binom{n}{r}.$$

For each such e_{J_m}, there are r possible choices for e_{K_j} obtained by picking $K_j \subset J_m$, and the subset of S_B generated by them are linearly independent. It follows that $r \leq \#_{max} S_B \leq n$.

Suppose now that $r < \#_{max} S_B$. To complete the proof of the Lemma, we show that there is a Plücker relation that is not satisfied for B, so it cannot be an r-blade. Let

$$L = (e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_k} \cdot B) \neq 0,$$

be the outer product of the maximal number of linearly independent vectors from S_B. Then $k > r$ and $LB = L \cdot B \neq 0$.

Letting

$$w = \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \in \mathbb{G}_n^1,$$

it follows that $w \neq 0$, and

$$w^2 = \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot (B \wedge w) \neq 0.$$

Since $w^2 \neq 0$, it follows that $w \wedge B \neq 0$. Noting the identity

$$w = \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B$$

$$= (e_{K_1} \cdot B)\big[(e_{K_2} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B$$
$$-(e_{K_2} \cdot B)\big[(e_{K_1} \cdot B) \wedge \cdots \vee^2 \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B$$
$$+ \cdots + (-1)^r (e_{K_{r+1}} \cdot B)\big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_r} \cdot B)\big] \cdot B,$$

it follows that

$$w \wedge B = \sum_{i=1}^{r} (-1)^{i+1} \alpha_i (e_{K_i} \cdot B) \wedge B \neq 0,$$

where $\alpha_i := \big[(e_{K_1} \cdot B) \wedge \cdots \vee^i \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \in \mathbb{R}$. But this implies that $\alpha_j (e_{K_j} \cdot B) \wedge B \neq 0$ for some j, so the proof of the Lemma is complete.

□

It follows that if $\#_{max} S = r$, then

$$\alpha B = (e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_r} \cdot B).$$

Furthermore, using (3) and (4), we can find distinct coordinate vectors $e_{j_1}, \cdots e_{j_r}$, and coordinate $(r-1)$-vectors $e_{K_{j_i}} = \pm e_{K_i}$, for $i = 1, \cdots, r$, such that

$$\alpha = B e_{j_r \cdots j_1} = B \cdot e_{j_r \cdots j_1} = B_{J_{j_2}} \cdots B_{J_{j_r}}.$$

Theorem (Plücker's Relations) 1 *For $2 \leq r \leq n$, a non-zero r-vector $B \in \mathbb{G}_n^r$ is an r-blade iff the Plücker relations (2) are satisfied. When the Plücker relations are satified, then*

$$B = B_{J_{j_2}}^{-1} \cdots B_{J_{j_r}}^{-1} (e_{K_{j_1}} \cdot B) \wedge \cdots \wedge (e_{K_{j_r}} \cdot B).$$

Proof:
The proof follows directly from previous comments, and Lemma 2.

□

2 Examples

Everybodies' favorite example is when $r = 2$ and $n = 4$. A general non-zero 2-vector $B \in \mathbb{G}_4^2$ has the form

$$B = \sum_{m=1}^{6} B_{J_m} e_{J_m} \in \mathbb{G}_4^2.$$

Since $B \neq 0$, at least one component $B_{J_p} e_{ij} \neq 0$. Without loss of generality, we can assume $e_{ij} = e_{12}$. It follows that the Plücker relation (4) for this component is

$$B = B_{12}^{-1} \Big[\big(e_1 \cdot B\big) \wedge \big(e_2 \cdot B\big) \Big]$$
$$= B_{12}^{-1} \Big[(B_{12} e_2 + B_{13} e_3 + B_{14} e_4) \wedge (-B_{12} e_1 + B_{23} e_3 + B_{24} e_4) \Big]. \qquad (6)$$

Since

$$2(e_i \cdot B) \wedge B = e_i \cdot (B \wedge B) = (B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23})e_i e_{1234},$$

it follows that B is a 2-blade iff $B \wedge B = 0$, or equivalently,

$$B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = 0. \tag{7}$$

This occurs when B is totally decomposable, given in (6). More generally, a similar argument holds true for $r = 2$ and any $n > 4$, but with more Plücker relations (7) to be satisfied.

To get more insight into what is going on, let $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$ be a general r-vector. Following [7], we calculate

$$B^2 = \sum_{m,p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} = \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m \neq p} B_{J_m} B_{J_p} e_{J_m} e_{J_p}$$

$$= \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m<p} B_{J_m} B_{J_p} \Big(1 + (-1)^{r-k}\Big) e_{J_m} e_{J_p}, \tag{8}$$

where $k = \# J_m \cap J_p$, showing that $B^2 = \langle B^2 \rangle_0$ if r and k have different parity for all values of $m < p$.

Consider $B = e_{123} + e_{456} \in \mathbb{G}_6^3$. It is easy to show, using (2), that B is not a 3-blade, since

$$(e_{12} \cdot B) \wedge B = e_3 \wedge B = e_{3456} \neq 0. \tag{9}$$

Note, however, that $B^2 = -2$, as also follows from (8) since k and r have opposite parities. Clearly, the condition $B^2 - \langle B^2 \rangle_0 = 0$ is not sufficient to guarantee that B is an r-blade, but $B^2 \neq \langle B^2 \rangle_0$ guarantees that B is **not** an r-blade.

In [7], Nguyen expresses the Plücker Relations differently, and gives a different proof. The following Theorem shows that Nguyen's Plücker Relations are equivalent to our definition (2). Let e_K denote an arbitrary coordinate $(r-1)$-vector in $\mathbb{G}_n^{r-1}$.

Theorem 2 *A non-zero r-vector $B \in \mathbb{G}_n^r$ is an r-blade iff both the conditions*

$$B^2 = B \cdot B, \quad \text{and} \quad BvB \in \mathbb{G}_n^1 \quad \text{for all} \quad v = e_K \cdot B \in \mathbb{G}_n^1,$$

are satisfied.

Proof: It is easy to show that if $B \in \mathbb{G}_n^r$ is an r-blade, then both of the conditions in the Theorem are true. To complete the proof we must only show that if both conditions are not satisfied, then B is not an r-blade. If $B^2 \neq B \cdot B$, then B is not an r-blade, so it is only left to show that if $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^n$, then B is not an r-blade.

The following identity is needed:

$$BvB = B(v \cdot B + v \wedge B) = B(v \cdot B) + B(v \wedge B)$$

$$= B \cdot (v \cdot B) + B \cdot (v \wedge B) + \langle B(vB) \rangle_{3,5,\cdots} \tag{10}$$

Suppose now that $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^1$ for some $v = e_K \cdot B$. The identity (10) implies that $\langle BvB \rangle_{3,5,\cdots} \neq 0$. We also know that

$$2B \wedge v = Bv + (-1)^r vB \quad \iff \quad Bv = 2B \wedge v - (-1)^r vB,$$

so that

$$\langle BvB \rangle_{3,5,\cdots} = \langle \big(2B \wedge v - (-1)^r vB\big) B \rangle_{3,5,\cdots} = 2 \langle \big(B \wedge v\big) B \rangle_{3,5,\cdots} \neq 0,$$

since $B^2 = B \cdot B$. But this implies that $B \wedge (e_K \cdot B) \neq 0$, so by (2), B is not an r-blade. □

Note that (2) is fully equivalent to the conditions in the Theorem. If $B^2 \neq B \cdot B$ then there is at least one $e_K \in \mathbb{G}_1^{r-1}$ such that $(e_K \cdot B) \wedge B \neq 0$. This follows from the coordinate expansion of B given in (8), because the parity condition allows us to find at least one non-zero $term(m,p) := B_{J_m} B_{J_p} e_{J_m} e_{J_p}$, with overlap $k = \#(S_{J_m} \cap S_{J_p})$, such that $(e_K \cdot B) \wedge B \neq 0$. We simply pick e_K in such a way that $K \subset S_{J_m}$, but $\#(K \cap J_p) \leq r - 2$, see Figure 1.

It is also interesting to note that just as $B^2 = B \cdot B$ whenever B is an r-vector satisfying the Plücker Relations (2), $BvB = (-1)^{r+1}(B \cdot B)v$ for all $v = e_K \cdot B$ whenever B is an r-vector satisfying the Plücker Relations (2). This follows from the identity

$$BvB = B[2v \wedge B + (-1)^{r+1} Bv] = (-1)^{r+1} B^2 v \quad \iff \quad v \wedge B = 0$$

for all $v = e_K \cdot B$, whenever B satisfies the Plücker Relations (2).

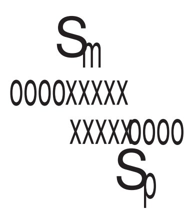

Figure 1: Shows S_m and S_p in the case for odd $r = 9$, with an odd overlap $k = 5$. In this case we choose $e_K = e_{j_2 \cdots j_9}$, the $j_2, \cdots j_9$ are the last 8 digits of S_m. A similar construction applies when r and k are even.

An example of a decomposable 3-vector $B \in \mathbb{G}_5^3$ is

$$B = e_{125} + e_{234} + 2e_{124} + e_{235} + e_{123} + e_{245},$$

satisfying the condition

$$B_{125}B_{234} - B_{124}B_{235} + B_{123}B_{245} = 0.$$

One factorization is $B =$

$$(-e_1 + e_2 + e_3 + e_4) \wedge (-2e_1 + e_2 + e_3 - e_5) \wedge (-3e_1 + e_2 + 2e_3 + e_4 - e_5).$$

To gain further insight, consider $B \in \mathbb{G}_3^6$ and where $B^2 = \langle B^2 \rangle_0$. For example, let

$$B = e_{123} + e_{456} + e_{124} + e_{356} + e_{125} + e_{346} + e_{126} + e_{345}.$$

Then,

$$\begin{aligned}(e_{21} \cdot B) \wedge B = (B_{123}B_{456} - B_{124}B_{356} + B_{125}B_{346} - B_{126}B_{345})e_{3456} \\ +(B_{123}B_{145} - B_{124}B_{135} + B_{125}B_{134})e_{1345} + \cdots \\ +(B_{124}B_{156} - B_{125}B_{146} + B_{126}B_{145})e_{1456} = 0,\end{aligned}$$

but

$$(e_{13} \cdot B) \wedge B = e_2 \wedge B = e_{2456} + e_{2356} + e_{2346} + e_{2345} \neq 0.$$

Since B satisfies the first condition, it is divisible by $e_{12} \cdot B$. But since it does not satisfy the second condition, it is not totally decomposable.

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