Notes on Plücker's Relations in Geometric Algebra

Garret Sobczyk
Universidad de las Américas-Puebla
Departamento de Físico-Matemáticas
72820 Puebla, Pue., México
Email: garretudla@gmail.com

January 21, 2019

Abstract

Grassmannians are of fundamental importance in projective geometry, algebraic geometry, and representation theory. A vast literature has grown up utilizing using many different languages of higher mathematics, such as multilinear and tensor algebra, matroid theory, and Lie groups and Lie algebras. Here we explore the basic idea of the Plücker relations in Clifford's geometric algebra. We discover that the Plücker relations can be fully characterized in terms of the geometric product, without the need for a confusing hodgepodge of many different formalisms and mathematical traditions found in the literature.

AMS Subject Classification: 14M15, 14A25, 15A66, 15A75

Keywords: Clifford geometric algebras, Grassmannians, matroids, Pauli matrices, Plücker relations

0 Introduction

The standard treatments of the Plücker relations rely heavily on the concept of a vector space, its dual vector space, and generalizations. More elementary treatments express the basic ideas in terms of familiar matrices and determinants. Here we exploit a powerful formalism that was invented 140 years ago by William Kingdon Clifford. Clifford himself dubbed this formalism, a generalization of the works of H. Grassmann and W. Hamilton, "geometric algebra", but it is often referred to today as "Clifford algebra".

It is well known that Clifford's geometric algebras are algebraically isomorphic to matrix algebras over the real or complex numbers. However, what is lacking is the recognition that whereas geometric algebras give a comprehensive geometrical interpretation to matrices, they also provide a way of unifying vectors and their duals into a rich algebraic structure that is missed in the standard treatment of vector spaces and their dual spaces as separate objects. Our study of the famous Plücker relations utilizes the powerful higher dimensional vector analysis of geometric algebra. When the Plücker relations are satisfied, an explicit factorization of an r-vector into the outer product of r vectors is found. Also, a new characterization of the Plücker relations is given.

Let a and b be two non-zero *null vector generators* of a real or complex associative algebra, $\mathbb{R}(a,b)$ or $\mathbb{C}(a,b)$, respectively, with the identity 1, and satisfying the rules

$$1)\ a^2 = 0 = b^2, \quad \text{and} \quad 2)\ ab + ba = 1. \tag{1}$$

From these rules, a multiplication table for these elements is easily constructed, given in Table 1.

Table 1: Multiplication Table.

	a	b	ab	ba
a	0	ab	0	a
b	ba	0	b	0
ab	a	0	ab	0
ba	0	b	0	ba

The null vectors a and b are *dual* to one another in the sense that the symmetric *inner product* of a and b, defined by

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}.$$

In addition, the *outer product* of a and b, gives the bivector

$$a \wedge b := \frac{1}{2}(ab - ba),$$

with the property

$$(a \wedge b)^2 = \frac{1}{4}(ab - ba)^2 = \frac{1}{4}(ab + ba) = \frac{1}{4} \quad \text{so} \quad |a \wedge b| := \frac{1}{2}.$$

Taking the sum of the above inner and outer products shows that the geometric (Clifford) product of the reciprocal null vectors a and b is the primitive idempotent

$$ab = a \cdot b + a \wedge b,$$

already identified in Table 1. More details for the interested reader of the construction of basic geometric algebras, obtained from $\mathbb{R}(a,b)$ and $\mathbb{C}(a,b)$, and by taking the Kronecker products of such algebras, is deferred to the Appendix.

Geometric algebras are considered here to be the natural *geometrization* of the real and complex number sytems; an overview is provided in [9]. We use exclusively the language of geometric algebra, developed in the books [6, 12].

Plücker's coordinates are briefly touched upon on page 30 of [6], but Plücker's relations are not discussed. These relations characterize when an r-vector B in the geometric algebra $\mathbb{G}_n := \mathbb{G}(\mathbb{R}^n)$, or $\mathbb{G}_n(\mathbb{C}) := \mathbb{G}(\mathbb{C}^n)$, of an n-dimensional (real or complex) Euclidean vector space $\mathbb{R}^n$, or $\mathbb{C}^n$, respectively, can be expressed as an outer product of r-vectors. Several references to the relevant literature, and its more advanced applications, are [1, 2], [3, pp.227-231], [5], [7, pp.144-148], and [13].

1 Plücker's relations in geometric algebra

A general geometric number $g \in \mathbb{G}_n$ is the sum of its k-vector parts,

$$g = \sum_{k=0}^{n} \langle g \rangle_k,$$

where the k-vector $\langle g \rangle_k \in \mathbb{G}_n^k$ is in the linear subspace $\mathbb{G}_n^k$ of k-vectors in $\mathbb{G}_n$, and the 0-vector part $\langle g \rangle_0 \in \mathbb{G}_n^0 \equiv \mathbb{R}$. A nonzero r-vector $B \in \mathbb{G}_n^r$ is said to be an *r-blade* if $B = v_1 \wedge \cdots \wedge v_r$ for vectors $v_1, \ldots, v_r \in \mathbb{G}_n^1$,

Let $B = \sum_{J_m} B_{J_m} e_{J_m}$ be an r-vector expanded in the standard orthonormal basis of r-vector blades

$$e_{J_m} := e_{j_1} e_{j_2} \cdots e_{j_r} \in \mathbb{G}_n^r,$$

ordered lexicographically, with $J_m = \{j_1, \cdots, j_r\}$ for $1 \le j_1 < \cdots < j_r \le n$, and let $\#(J_p \cap J_m)$ be the number of integers that the sequences J_p and J_m have in common. For example, if $J_p = \{12345\}$ and $J_m = \{45678\}$, then $\#(J_p \cap J_m) = 2$.

The *reverse* $g^\dagger$ of a geometric number $g \in \mathbb{G}_n$ is obtained by reversing the order of all of the geometric products of the vectors which define g.

Lemma 1 *In the standard orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{G}_n$, an r-vector $B \in \mathbb{G}_n^r$, expanded in this basis, is given by*

$$B = \sum_m B_{J_m} e_{J_m}, \tag{2}$$

where $B_{J_k} = B \cdot e_{J_k}^\dagger$.

Proof: In the orthonormal r-vector basis $\{e_{J_m}\}$, the r-vector $B \in \mathbb{G}_n^r$ is given by

$$B = \sum_{k=1}^{\binom{n}{r}} B_{J_k} e_{J_k}$$

Dotting both sides of this equations with $e_{J_m}^\dagger$, the reverse of e_{J_m}, shows that $B_{J_m} = B \cdot e_{J_m}^\dagger$, since $e_{J_k} \cdot e_{J_m}^\dagger = \delta_{k,m}$.

□

Definition 1 *A non-zero r-vector $B \in \mathbb{G}_n^r$ is said to be divisible by a non-zero k-blade $K \in \mathbb{G}_n^k$, where $k \le r$, if $KB = \langle KB \rangle_{r-k}$. If $k = r$, B is said to be totally decomposable by K.*

If B is divisible by the k-blade L, then

$$B = \frac{L(LB)}{L^2} = L\frac{L \cdot B}{L^2}. \tag{3}$$

If B is totally decomposable by L, then $B = \beta L$ for $\beta = \frac{L \cdot B}{L^2} \in \mathbb{R}$. Clearly, a totally decomposable r-vector is an r-blade.

A classical result is that an r-vector $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$ is an r-blade iff B satisfies the *Plücker relations*. The Plücker relations, [4, Thm1(1)], are

$$(A \cdot B) \wedge B = 0 \tag{4}$$

for all $(r-1)$-vectors $A \in \mathbb{G}_n^{r-1}$. In particular, for all coordinate $(r-1)$-vectors $e_{K_p} \in \mathbb{G}_n^{r-1}$,

$$(e_{K_p} \cdot B) \wedge B = 0.$$

Since an r-blade trivially satisfies the Plücker relations, it follows that if there is a coordinate $(r-1)$-blade e_K, such that $(e_K \cdot B) \wedge B \neq 0$, then B is not an r-blade.

For every coordinate $(r-1)$-blade e_{K_p}, there is a coordinate r-blade e_{J_p}, generally not unique, and a coordinate vector e_{j_p}, satisfying

$$e_{J_p} = \pm e_{j_p} e_{K_p} \quad \iff \quad e_{j_p} e_{J_p} = \pm e_{K_p}. \tag{5}$$

Indeed, when $e_{K_p} \cdot B \neq 0$, we always pick the coordinate vector e_{j_p}, and rearrange the order of $K_p = \pm K_{j_p}$, in such a way that $B_{J_{j_p}} = (e_{j_p} \wedge e_{K_{j_p}})^\dagger \cdot B \neq 0$. It follows that the Plücker relations (4) are equivalent to

$$e_{j_p} \cdot \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge B\Big] = B_{J_{j_p}} B - \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge \Big(e_{j_p} \cdot B\Big)\Big] = 0,$$

or equivalently,

$$B = B_{J_{j_p}}^{-1} \Big[\Big(e_{K_{j_p}} \cdot B\Big) \wedge \Big(e_{j_p} \cdot B\Big)\Big], \tag{6}$$

for all coordinate $(r-1)$-blades $e_{K_p} \in \mathbb{G}_n^{r-1}$ for which $e_{K_p} \cdot B \neq 0$. It follows that when the Plücker relation (4) is satisfied, then the vector $e_{K_p} \cdot B$ divides B.

For a given non-zero r-vector $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$, let $\#_{max} S$ be the maximal number of linearly independent vectors in the set

$$S_B = \{e_{K_j} \cdot B | \quad e_{K_j} \in \mathbb{G}_n^{r-1}\}. \tag{7}$$

Lemma 2 *Given an r-vector $B \neq 0$, and the set S_B defined in (7). Then $r \le \#_{max} S \le n$, and $\#_{max} = r$ only if B is an r-blade.*

Proof: Since $B \neq 0$, it follows that $B_{J_m} \neq 0$ for some

$$1 \leq m \leq \binom{n}{r}.$$

For each such e_{J_m}, there are r possible choices for e_{K_j} obtained by picking $K_j \subset J_m$, and the subset of S_B generated by them are linearly independent. It follows that $r \leq \#_{max} S_B \leq n$.

Suppose now that $r < \#_{max} S_B$. To complete the proof of the Lemma, we show that there is a Plücker relation that is not satisfied for B, so it cannot be an r-blade. Let

$$L = (e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_k} \cdot B) \neq 0,$$

be the outer product of the maximal number of linearly independent vectors from S_B. Then $k > r$ and $LB = L \cdot B \neq 0$.

Letting

$$w = \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \in \mathbb{G}_n^1,$$

it follows that $w \neq 0$, and

$$w^2 = \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot (B \wedge w) \neq 0.$$

Since $w^2 \neq 0$, it follows that $w \wedge B \neq 0$. Using the identity

$$\begin{aligned} w &= \big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \\ &= (e_{K_1} \cdot B)\big[(e_{K_2} \cdot B) \wedge \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \\ &-(e_{K_2} \cdot B)\big[(e_{K_1} \cdot B) \wedge \cdots \vee^2 \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \\ &+ \cdots + (-1)^r (e_{K_{r+1}} \cdot B)\big[(e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_r} \cdot B)\big] \cdot B, \end{aligned}$$

where $\vee^i$ means we are omitting the term $e_{K_i} \cdot B$ in the exterior product. It follows that

$$w \wedge B = \sum_{i=1}^{r} (-1)^{i+1} \alpha_i (e_{K_i} \cdot B) \wedge B \neq 0,$$

where $\alpha_i := \big[(e_{K_1} \cdot B) \wedge \cdots \vee^i \cdots \wedge (e_{K_{r+1}} \cdot B)\big] \cdot B \in \mathbb{R}$. But this implies that $\alpha_j (e_{K_j} \cdot B) \wedge B \neq 0$ for some j, so B is not an r-blade.

On the other hand, if $\#_{max} S = r$, then for $L = (e_{K_1} \cdot B) \wedge \cdots \wedge (e_{K_r} \cdot B)$, $LB = L \cdot B \in \mathbb{R}$, and using (3),

$$B = \frac{L \cdot B}{L^2} L$$

is an r-blade. □

As a direct consequence of Lemma 2, we have

Theorem 1 *For $2 \leq r \leq n$, a non-zero r-vector $B \in \mathbb{G}_n^r$ is an r-blade iff the Plücker relations (4) are satisfied. When the Plücker relations are satified, then $B = \frac{L \cdot B}{L^2} L$ for the r-blade L defined in the last paragraph of the proof of Lemma 2.*

2 Examples

Everybodies' favorite example is when $r = 2$ and $n = 4$. A general non-zero 2-vector $B \in \mathbb{G}_4^2$ has the form

$$B = \sum_{m=1}^{6} B_{J_m} e_{J_m} \in \mathbb{G}_4^2.$$

Since $B \neq 0$, at least one component $B_{J_p} e_{ij} \neq 0$. Without loss of generality, we can assume $e_{ij} = e_{12}$. It follows that the Plücker relation (6) for this component is

$$B = B_{12}^{-1}\Big[\big(e_1 \cdot B\big) \wedge \big(e_2 \cdot B\big)\Big]$$

$$= B_{12}^{-1}\Big[(B_{12}e_2 + B_{13}e_3 + B_{14}e_4) \wedge (-B_{12}e_1 + B_{23}e_3 + B_{24}e_4)\Big]. \quad (8)$$

Since

$$2(e_i \cdot B) \wedge B = e_i \cdot (B \wedge B) = (B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23}) e_i e_{1234},$$

it follows that B is a 2-blade iff $B \wedge B = 0$, or equivalently,

$$B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = 0. \quad (9)$$

This occurs when B is totally decomposable, given in (8). More generally, a similar argument holds true for $r = 2$ and any $n > 4$, but with more Plücker relations (9) to be satisfied.

To get more insight into what is going on, let $B = \sum_m B_{J_m} e_{J_m} \in \mathbb{G}_n^r$ be a general r-vector. Following [8], we calculate

$$B^2 = \sum_{m,p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} = \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m \neq p} B_{J_m} B_{J_p} e_{J_m} e_{J_p}$$

$$= \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m<p} B_{J_m} B_{J_p} \Big(1 + (-1)^{r-k}\Big) e_{J_m} e_{J_p}, \quad (10)$$

where $k = \#(J_m \cap J_p)$, showing that $B^2 = \langle B^2 \rangle_0$ if r and k have different parity for all values of $m < p$.

Consider $B = e_{123} + e_{456} \in \mathbb{G}_6^3$. It is easy to show, using (4), that B is not a 3-blade, since

$$(e_{12} \cdot B) \wedge B = e_3 \wedge B = e_{3456} \neq 0. \quad (11)$$

Note, however, that $B^2 = -2$, as also follows from (10) since k and r have opposite parities. Clearly, the condition $B^2 - \langle B^2 \rangle_0 = 0$ is not sufficient to guarantee that B is an r-blade, but $B^2 \neq \langle B^2 \rangle_0$ guarantees that B is **not** an r-blade.

In [8], Nguyen expresses the Plücker Relations differently, and gives a different proof. The following Theorem shows that Nguyen's Plücker Relations are equivalent to definition (4). Let e_K denote an arbitrary coordinate $(r-1)$-vector in $\mathbb{G}_n^{r-1}$.

Theorem 2 *A non-zero r-vector $B \in \mathbb{G}_n^r$ is an r-blade iff both the conditions*

$$B^2 = B \cdot B, \quad \text{and} \quad BvB \in \mathbb{G}_n^1 \quad \text{for all} \quad v = e_K \cdot B \in \mathbb{G}_n^1,$$

are satisfied.

Proof: It is easy to show that if $B \in \mathbb{G}_n^r$ is an r-blade, then both of the conditions in the Theorem are true. To complete the proof we must only show that if both conditions are not satisfied, then B is not an r-blade. If $B^2 \neq B \cdot B$, then B is not an r-blade, so it is only left to show that if $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^n$, then B is not an r-blade.

The following identity is needed:

$$BvB = B(v \cdot B + v \wedge B) = B(v \cdot B) + B(v \wedge B)$$
$$= B \cdot (v \cdot B) + B \cdot (v \wedge B) + \langle B(vB) \rangle_{3,5,\cdots} \qquad (12)$$

Suppose now that $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^1$ for some $v = e_K \cdot B$. The identity (12) implies that $\langle BvB \rangle_{3,5,\cdots} \neq 0$. We also know that

$$2B \wedge v = Bv + (-1)^r vB \quad \Longleftrightarrow \quad Bv = 2B \wedge v - (-1)^r vB,$$

so that

$$\langle BvB \rangle_{3,5,\cdots} = \langle \big(2B \wedge v - (-1)^r vB\big) B \rangle_{3,5,\cdots} = 2\langle \big(B \wedge v\big) B \rangle_{3,5,\cdots} \neq 0,$$

since $B^2 = B \cdot B$. But this implies that $B \wedge (e_K \cdot B) \neq 0$, so by (4), B is not an r-blade. $\square$

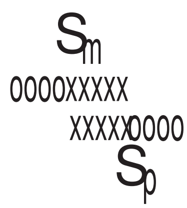

Figure 1: Shows S_m and S_p in the case for odd $r = 9$, with an odd overlap $k = 5$. In this case we choose $e_K = e_{j_2 \cdots j_9}$, the $j_2, \cdots j_9$ are the last 8 digits of S_m. A similar construction applies when r and k are even.

Note that (4) is fully equivalent to the conditions in the Theorem 2. If $B^2 \neq B \cdot B$ then there is at least one $e_K \in \mathbb{G}_1^{r-1}$ such that $(e_K \cdot B) \wedge B \neq 0$. This follows from the coordinate expansion of B given in (10), because the parity condition allows us to find at least one non-zero $B_{J_m} B_{J_p} e_{J_m} e_{J_p}$, with overlap $k = \#(S_{J_m} \cap S_{J_p})$, such that $(e_K \cdot B) \wedge B \neq 0$. We simply pick e_K in such a way that $K \subset S_{J_m}$, but $\#(K \cap J_p) \leq r - 2$, see Figure 1. In the example (11), B is not a 3-blade because even though $B^2 = B \cdot B$, for

$$e_{12} \cdot B = -e_3, \quad Be_3B = 2e_{12456} \in \mathbb{G}_6^5.$$

It is also interesting to note that just as $B^2 = B\cdot B$ whenever B is an r-vector satisfying the Plücker relations (4), $BvB = (-1)^{r+1}(B\cdot B)v$ for all $v = e_K \cdot B$ whenever B is an r-vector satisfying the Plücker Relations (4). This follows from the identity

$$BvB = B[2v\wedge B + (-1)^{r+1}Bv] = (-1)^{r+1}B^2 v \qquad \Longleftrightarrow \qquad v\wedge B = 0$$

for all $v = e_K \cdot B$, when B satisfies the Plücker relations (4).

Let us consider now some other examples. For the 3-vector $B \in \mathbb{G}_5^3$,

$$B := -2e_{134} + e_{145} - 2e_{234} + e_{245} + e_{315} + e_{325} + e_{345},$$

it is easily verified that the Plücker relations

$$(e_{41}\cdot B)\wedge B = (2e_3 + e_5)\wedge B = 0, \quad e_{54}\cdot B = (e_1+e_2+e_3)\wedge B = 0,$$

$$e_{13}\cdot B = (2e_4 + e_5)\wedge B = 0,$$

and that the vectors $(e_{41}\cdot B) = 2e_3 + e5$, $(e_{54}\cdot B) = (e_1+e_2+e_3)$, $e_{13}\cdot B = 2e_4 + e5$ are linearly independent. Applying Theorem 1, with

$$L = (e_1+e_2+e_3)\wedge(2e_3+e_5)\wedge(2e_4+e5)$$

$$B = \frac{L\cdot B}{L^2}L = -\frac{1}{2}L.$$

For

$$B = e_{123} + e_{124} + e_{125} + e_{126} + e_{345} + e_{346} + e_{356} + e_{456} \in \mathbb{G}_6^3,$$

we find that

$$(e_{21}\cdot B)\wedge B = (B_{123}B_{456} - B_{124}B_{356} + B_{125}B_{346} - B_{126}B_{345})e_{3456}$$
$$+(B_{123}B_{145} - B_{124}B_{135} + B_{125}B_{134})e_{1345} + \cdots$$
$$+(B_{124}B_{156} - B_{125}B_{146} + B_{126}B_{145})e_{1456} = 0,$$

but

$$(e_{13}\cdot B)\wedge B = e_2\wedge B = e_{2456} + e_{2356} + e_{2346} + e_{2345} \neq 0.$$

Since $(e_{21}\cdot B)\wedge B = 0$, it is divisible by $e_{12}\cdot B$. Using (6), we find that

$$B = (e_{21}\cdot B)\wedge(e_3\cdot B) = (e_3+e_4+e_5+e_6)\wedge(e_{12}+e_{56}+e_{46}+e_{45}).$$

But since it does not satisfy the second condition, it is not totally decomposable.

As a final example, consider $B = (e_1+e_2+e_3+e_4)\wedge(e_{23}+e_{34}+e_{46}) \in \mathbb{G}_6^3$. There are no non-zero vectors of the form $e_{ij}\cdot B$ such that

$$(e_{ij}\cdot B)\wedge B = 0.$$

However, the 2-blade $e_{36} - e_{46}$, satisfying

$$(e_{36} - e_{46})\cdot B = e_1 + e_2 + e_3 + e_4$$

does divide B, since $(S\cdot B)\wedge B = 0$.

Appendix: Geometric matrices

Let $[g] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ be a real matrix in $Mat_2(\mathbb{R})$. Since $\mathbb{R}(a,b)$ is an associative algebra, the module $Mat_2\big(\mathbb{R}(a,b)\big)$ is well-defined. We use the matrix $[g]$ to define the unique element $g \in \mathbb{R}(a,b)$, specified by

$$g := \begin{pmatrix} ba & a \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} ba \\ b \end{pmatrix} = g_{11}ba + g_{12}b + g_{21}a + g_{22}ab.$$

We say that $[g]$ is the *matrix* of $g \in \mathbb{R}(a,b)$ with respect to the *Witt basis* of null vectors, $\{a,b\}$, over the real numbers $\mathbb{R}$. It is easy to show that this establishes an algebraic isomorphism between $\mathbb{R}(a,b)$ and $Mat_2(\mathbb{R})$, [10].

The *standard orthonormal basis* $\{e_1, f_1\}$ of the geometric algebra

$$\mathbb{G}_{1,1} := \mathbb{R}(e_1, f_1) \equiv \mathbb{R}(a,b)$$

is

$$e_1 := a + b, \quad \text{and} \quad f_1 := a - b,$$

and satisfies the rules

$$1)\ e_1^2 = 1 = -f_1^2, \quad \text{and} \quad 2)\ e_1 f_1 = -f_1 e_1. \tag{13}$$

The geometric algebra $\mathbb{G}_{1,1} := \mathbb{R}(e_1, f_1)$, defined by the rules (13), is nothing more than a change of basis, from the Witt basis of null vectors (1) to standard orthonormal basis (13). As a linear space over $\mathbb{R}$,

$$\mathbb{G}_{1,1} = span_{\mathbb{R}}\{1, e_1, f_1, e_1 f_1\}.$$

In terms of the null basis, the bivector $e_1 f_1 = (a+b)(a-b) = ba - ab = 2b \wedge a$. Letting $u_1 = ba = \frac{1}{2}(1 + e_1 f_1)$, the matrices of the standard basis vectors, are

$$e_1 = \begin{pmatrix} ba & a \end{pmatrix} [e_1] \begin{pmatrix} ba \\ a \end{pmatrix} = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix},$$

and

$$f_1 = \begin{pmatrix} ba & a \end{pmatrix} [e_1] \begin{pmatrix} ba \\ a \end{pmatrix} = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.$$

The matrix of the bivector $e_1 f_1$, is then easily found to be

$$[e_1 f_1] = [e_1][f_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By *complexifying* the basis vector $f_1 \in \mathbb{G}_{1,1}$, we obtain the geometric algebra $\mathbb{G}_3$, a geometric interpretation of the famous *Pauli matrices*, and a geometric interpretation of the imaginary $i = \sqrt{-1}$. We define

$$\mathbb{G}_3 := \mathbb{R}(e_1, e_2, e_3),$$

where $e_2 := if_1$ and $e_3 := e_1 f_1$. It then follows that $e_{123} = e_1 i f_1 e_1 f_1 = i$, giving imaginary unit i the geometric interpretation of the unit trivector $e_{123} \in \mathbb{G}_3^3$. Note also the unit vector e_3, along the z-axis of $\mathbb{R}^3$ is identified with the bivector $e_1 f_1 \in \mathbb{G}_{1,1}^2$. The *Pauli matrices* $[e_k]$, satisfying

$$e_k = \begin{pmatrix} ba & a \end{pmatrix} [e_k] \begin{pmatrix} ba \\ b \end{pmatrix},$$

are given by

$$[e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [e_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Geometric matrices of real or complex higher dimensional geometric algebras $\mathbb{G}_{n,n} := \mathbb{G}(\mathbb{R}^{n,n})$, or $\mathbb{G}_{n,n}(\mathbb{C}) := \mathbb{G}(\mathbb{C}^n)$, are obtained by taking *directed Kronecker products* of blocks of *pairs* of null vectors $\{a_k, b_k\}$, each block satisfying the rules (1), in addition to being *anti-commutative*. That is

$$a_i a_j = -a_j a_i, \quad b_i b_j = -b_j b_i, \quad a_i b_j = -b_j a_i,$$

for all $1 \le i, j \le n$ and $i \ne j$. For example, the geometric matrix $[g] \in Mat_4(\mathbb{R})$ defines the geometric number $g \in \mathbb{G}_{2,2}$, given by $g :=$

$$\begin{pmatrix} 1 & a_1 \end{pmatrix} \overrightarrow{\otimes} \begin{pmatrix} 1 & a_2 \end{pmatrix} u_{12}[g] \begin{pmatrix} 1 \\ b_2 \end{pmatrix} \overleftarrow{\otimes} \begin{pmatrix} 1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_2 & a_{12} \end{pmatrix} u_{12}[g] \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_{21} \end{pmatrix},$$

where $u_{12} = u_1 u_2 = b_1 a_1 b_2 a_2$. Details of the general construction can be found in [11].

Acknowledgement

I thank Dr. Dung B. Nguyen for calling my attention to the problem of expressing and proving the Plücker relations in geometric algebra. This work is the result of many email exchanges with him, false starts, and trying to understand the roots of the vast literature on the subject and its generalizations. The reviewer's suggestions for improvements were extremely valuable to the author in making the final revision of the manuscript.

References

[1] F. Ardila, *The Geometry of Matroids*, Notices of the AMS, Vol. 65, No. 8, Sept. 2018.

[2] A. Dress and W. Wenzel, *Grassmann-Plücker Relations and Matroids with Coefficients*, Advances in Mathematics 86, 68-110 (1991).

[3] W. Fulton, J. Harris, *Representation Theory: A First Course*, Springer-Verlag 1991.

[4] M.G. Eastwood and P.W. Michor, *Some Remarks on the Plücker Relations*, Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. 63 (2000), 85-88. https://www.mat.univie.ac.at/˜ michor/plue-lon.pdf

[5] J. Harris, *Algebraic Geometry: A First Course*, Spring, p.63-71, 1992. http://math.rice.edu/ẽvanmb/math465spring11/ Harris-Grassmannians.pdf

[6] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, 2nd edition, Kluwer 1992.

[7] J. Milne, *Algebraic Geometry*, Version 6.06, March 2017, pp. 144-148. http://www.jmilne.org/math/CourseNotes/AG.pdf

[8] D. B. Nguyen, *Plücker's Relations, Geometric Algebra, and the Electromagnetic Field*, unpublished version: Sept. 3, 2018.

[9] G. Sobczyk, *Geometrization of the Real Number System*, July, 2017. http://www.garretstar.com/geonum2017.pdf

[10] G. Sobczyk, *Hyperbolic Numbers Revisited*, preprint 2017. http://www.garretstar.com/hyprevisited12-17-2017.pdf

[11] G. Sobczyk, *Geometric Matrices and the Symmetric Group*, 2018. https://arxiv.org/pdf/1808.02363.pdf

[12] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, New York 2013.

[13] C.T.C. Wall, *Plücker formulae for curves in high dimensions*, University of Liverpool, 2008. https://www.liverpool.ac.uk/˜ ctcw/pluckD.pdf