# Space-time algebra approach to curvature

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Introducing the notions of vector and bivector differentiation into the Dirac algebra, considered as a Clifford algebra, makes possible an extremely concise and geometrically transparent treatment of the curvature tensor and its properties, and of related topics such as Lorentz invariants, characteristic equations, Petrov types, and principal null directions by explicit construction.

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## INTRODUCTION

The Riemann curvature tensor is the crucial geometric ingredient in the study of general relativity. It is for this reason that the curvature tensor and its properties and invariants have attracted wide attention. The original classification of the curvature tensor for empty space was carried out by Petrov[1] using matrix methods. Subsequently, a number of different methods and refinements have been introduced. Noteworthy of mention is the spinor approach used by Witten,[2] and later refined by Penrose[3] in his systematic study of the coincidence patterns of the four principal null directions. But, as anyone who is familiar with calculations with spinors knows, these methods are only adapted to certain kinds of problems. Classical tensor methods have also been used with some success, for example,[4] but the computational aspects of this approach are formidable. Thorpe[5] notes that computations are considerably simplified by using the Hodge star operator to make the space of bivectors into a complex Euclidean space, but he ignores the possibility of utilizing the underlyng Lie algebra of bivectors. Stehney[6] modifies Thorpe's approach to the requirements of matrix methods and produces a classification scheme based on the minimal polynomial of a complex $3\times 3$ matrix, but her methods lack conceptual clarity, and her algorithm works only for repeated principal null directions.

The purpose of the present work is to cover much the same ground as the above authors, but in a coordinate-free formalism whose power, simplicity, and geometric transparency have yet to be recognized; a formalism which has all the advantages of each of the above mentioned approaches, and the defects of none.

In Sec. 1, following Hestenes,[7,8] we introduce the 16-dimensional Clifford algebra called the *Dirac algebra* of space-time in agreement with the name given its matrix representation. (Clifford algebra of $2^n$-dimensions has been extensively developed in the book, *Clifford Algebras and Geometric Calculus: A Unified Language for Mathematics and Physics*,[9] using an abstract approach,[10] rather than a matrix representation such as is used by Cartan,[11] and others). The even subalgebra, consisting of scalars, bivectors, and pseudoscalars of the Dirac algebra, make up the *Pauli algebra* of space. The Pauli algebra can be fruitfully compared to the popular Gibbs–Heaviside vector algebra, because many identities of the former are the "complexified" versions of the latter. A discussion of bivectors and null bivectors is given, and a multiplication table of basis elements is included.

In Sec. 2 we complement the algebraic machinery introduced in Sec. 1 by introducing the operations of vector and bivector differentiation. These operations simplify and generalize the operation of contraction in tensor algerbra. They were originally developed as a coordinate-free tool for the study of linear transformations in Ref. 12.

In Sec. 3 we study general properties of linear operators on the space of bivectors by decomposing it into the sum of dual and antidual operator parts. A dual operator is equivalent to a general linear operator on a complex three-dimensional Euclidean space. Using the new method of bivector derivatives, the determinant, characteristic polynomial and Cayley–Hamilton theorem are derived for dual operators. In our approach it is unnecessary to introduce the Hodge-star operator, because in the Dirac algebra duality is simply expressed by multiplication by the unit pseudoscalar element. Finally, we show that an antidual bivector operator can be expressed entirely in terms of two symmetric trace-free vector operators. In another paper[13] we show how the problem of the classification of these symmetric vector operators is directly correlated to the Petrov classification.

In Sec. 4 we give a complete classification of dual operators based on explicit construction of their principal null bivectors. The classification of a dual skew-symmetric operator is equivalent to the classification of an electromagnetic field by its principal null directions. A dual symmetric operator with vanishing trace is equivalent to the conformal curvature tensor. The Petrov–Penrose classification of dual symmetric operators is carried out by construction of its four principal null bivectors, based on a new canonical form involving a complex scalar, a bivector, and a null eigenbivector. This new canonical form provides a simple geometric criterion for the various coincidence patterns of the four principal null directions. In addition, it makes it trivial to give simple examples of conformal curvature tensors of any desired type.

In Sec. 5, curvature invariants, which are complex scalars, are defined in terms of the bivector derivative, and it is shown that a curvature operator has nine complex scalars, three of which are real. When the extra Bianchi identity is imposed, these 15 real scalars reduce down to the well

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known 14 real invariants of the Riemann curvature tensor. Various well known properties and identities of Riemann curvature are then derived in the spacetime algebra (STA) formalism. In each case the simplicity and geometric transparency of our methods are apparent. A table is included comparing the appearance of well known formulas in the tensor, STA, and spinor formalism. We believe a close examination of this table and the methods of this paper will show the judgment of Misner, Thorne, Wheeler[14] (p. 1165) that "the spinor formalism is a more powerful method than any other for deriving the Petrov–Pirani algebraic classification of the conformal curvature tensor, and for proving theorems about algebraic properties of curvature tensors," needs reexamination. See also Ref. 13.

## 1. SPACE-TIME ALGEBRA

Let $x$ be a generic point in spacetime. Following Hestenes,[7] we select a set of orthonormal vectors $e_0$, $e_1$, $e_2$, $e_3$ tangent to the point $x$, and subject them to the rules:

$$e_0^2 = 1, e_1^2 = e_2^2 = e_3^2 = -1, \tag{1.1}$$

$$e_u e_v = -e_v e_u \quad \text{for} \quad u,v = 0,1,2,3 \quad \text{and} \quad u \neq v. \tag{1.2}$$

The orthonormal vectors $\{e_u\}$, under the rules for geometric multiplication (1.1) and (1.2) generate a real Clifford Algebra of $2^4 = 16$ dimensions called the *Dirac Algebra* $\mathscr{D}$ in agreement with the name given its matrix representation. Symbollically we write $\mathscr{D} = \mathscr{D}_0 + \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3 + \mathscr{D}_4$, to express the Dirac algebra $\mathscr{D}$ as the sum of linear subspaces of scalars, vectors, bivectors, trivectors, and pseudoscalars, respectively.

For purposes of orientation and fixing the notation that will be used here, let us review some of the basic operations and identities in $\mathscr{D}$. Let $a$, $b$ be vectors in $\mathscr{D}_1$,

$$a = \alpha^u e_u \equiv \sum_{u=0}^{3} \alpha^u e_u, \quad b = \beta^u e_u,$$

then

$$a\cdot b \equiv \tfrac{1}{2}(ab + ba) = \alpha^0\beta^0 - \alpha^1\beta^1 - \alpha^2\beta^2 - \alpha^3\beta^3 \equiv g(a,b) \tag{1.3}$$

and

$$\begin{aligned} a\wedge b &\equiv \tfrac{1}{2}(ab - ba) \\ &= \begin{vmatrix}\alpha^1 \alpha^0\\ \beta^1\beta^0\end{vmatrix} e_1\wedge e_0 + \begin{vmatrix}\alpha^2 \alpha^0\\ \beta^2\beta^0\end{vmatrix} e_2\wedge e_0 \\ &+ \begin{vmatrix}\alpha^3 \alpha^0\\ \beta^3\beta^0\end{vmatrix} e_3\wedge e_0 + \begin{vmatrix}\alpha^3 \alpha^2\\ \beta^3\beta^2\end{vmatrix} e_3\wedge e_2 \\ &+ \begin{vmatrix}\alpha^1 & \alpha^3\\ \beta^1 & \beta^3\end{vmatrix} e_1\wedge e_3 + \begin{vmatrix}\alpha^2 \alpha^1\\ \beta^2\beta^1\end{vmatrix} e_2\wedge e_1. \end{aligned} \tag{1.4}$$

From the definitions (1.3) and (1.4), it is clear that

$$ab = \tfrac{1}{2}(ab + ba) + \tfrac{1}{2}(ab - ba) = a\cdot b + a\wedge b, \tag{1.5}$$

i.e., the geometric product of two vectors can be decomposed into the sum of an inner product or (real) scalar part, and an outer product or bivector part. The metric tensor $g(a, b)$ of spacetime is determined by the inner product and is, of course, invariant under local Lorentz transformations.

Define the bivectors

$$E_1 = e_1\wedge e_0 = e_1 e_0, \quad E_2 = e_2\wedge e_0, \quad E_3 = e_3\wedge e_0, \tag{1.6}$$

$$E_4 = e_3\wedge e_2 = e_3 e_2, \quad E_5 = e_1\wedge e_3, \quad E_6 = e_2\wedge e_1.$$

The unit pseudoscalar $I$, defined by

$$I = e_0\wedge e_1\wedge e_2\wedge e_3 = e_0 e_1 e_2 e_3 = E_1 E_2 E_3, \tag{1.7}$$

has the property $I^2 = -1$, and assigns a unique orientation to the Dirac algebra $\mathscr{D}$. The duality of the bivectors $E_1$, $E_2$, $E_3$ and $E_4$, $E_5$, $E_6$ has the simple algebraic expression

$$E_4 = IE_1 = E_1 I, \quad E_5 = IE_2, \quad E_6 = IE_3. \tag{1.8}$$

Note that the bivectors $E_1$, $E_2$, $E_3$ satisfy the following rules of multiplication:

$$E_1^2 = E_2^2 = E_3^2 = 1 \quad (E_4^2 = E_5^2 = E_6^2 = -1), \tag{1.9}$$

$$E_i E_j = -E_j E_i, \quad \text{for } i = j = 1,2,3 \text{ and } i \neq j, \tag{1.10}$$

and generate a $2^3 = 8$ dimensional Clifford algebra called the *Pauli albegra* $\mathscr{P}$, which is the even subalgebra of $\mathscr{D}$ consisting of the scalars, bivectors, and pseudoscalars.

Operations similar to (1.3) and (1.4) can be defined in the Pauli algebra $\mathscr{P}$. Thus, let $A$, $B$ be bivectors in $\mathscr{D}_2$, then

$$A = \alpha^i E_i \equiv \sum_{i=1}^{3} \alpha^i E_i, \quad B = \beta^i E_i,$$

where $\alpha^i$ and $\beta^i$ are "complex" scalars of the form

$$\alpha^i = \alpha'^i + \alpha''^i I \quad \text{and} \quad \beta^i = \beta'^i + \beta''^i I$$

and $I$ is the unit pseudoscalar defined in (1.7). Now define:

$$\begin{aligned} A\circ B &\equiv \tfrac{1}{2}(AB + BA) = \alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3 \\ &\equiv G(A,B) \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} A\times B \equiv \tfrac{1}{2}(AB - BA) &= \begin{vmatrix}\alpha^2 \alpha^3\\ \beta^2\beta^3\end{vmatrix} IE_1 + \begin{vmatrix}\alpha^3 \alpha^1\\ \beta^3\beta^1\end{vmatrix} IE_2 \\ &+ \begin{vmatrix}\alpha^1 \alpha^2\\ \beta^1\beta^2\end{vmatrix} IE_3. \end{aligned} \tag{1.12}$$

From (1.11) and (1.12) it follows that

$$AB \equiv \tfrac{1}{2}(AB + BA) + \tfrac{1}{2}(AB - BA) = A\circ B + A\times B, \tag{1.13}$$

i.e., the geometric product of bivectors can be decomposed into the sum of a symmetric product, or complex scalar part, and a Lie product, or bivector part. The metric tensor $G(A,B)$ defined by the symmetric product (1.11) turns the space of bivectors $\mathscr{D}_2$ into a complex Euclidean space, as is noted by Thorpe,[5] and like $g(a, b)$ is Lorentz invariant.

The operations $A\circ B$ and $A\times B$ in the Pauli algebra can be expressed entirely in terms of the operations (1.3) and (1.4) in the Dirac algebra. Thus, let $A = a\wedge b$ and $B = c\wedge d$, then

$$A\circ B = A\cdot B + A\wedge B = \text{scalar} + \text{pseudoscalar} \tag{1.14}$$

where

$$A\cdot B = (a\wedge b)\cdot(c\wedge d) = (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d),$$

$$A\wedge B = a\wedge b\wedge c\wedge d = \begin{vmatrix}\alpha^0 & \alpha^1 & \alpha^2 & \alpha^3\\ \beta^0 & \beta^1 & \beta^2 & \beta^3\\ \gamma^0 & \gamma^1 & \gamma^2 & \gamma^3\\ \delta^0 & \delta^1 & \delta^2 & \delta^3\end{vmatrix} I,$$

and

$$\begin{aligned} A\times B &= (a\wedge b)\times(c\wedge d) \\ &= a\wedge(b\cdot(c\wedge d)) + (a\cdot(c\wedge d))\wedge b \qquad (1.15)\\ &= (b\cdot c)a\wedge d - (b\cdot d)a\wedge c \\ &+ (a\cdot c)d\wedge b - (a\cdot d)c\wedge b. \end{aligned}$$

Note also the duality relations

$$I(A\cdot B) = (IA)\wedge B \quad \text{and} \quad I(A\wedge B) = (IA)\cdot B \quad (1.16)$$

between $A\cdot B$ and $A\wedge B$.

There are two triple products in the Pauli algebra $\mathscr{P}$ built up from the symmetric and Lie products. They are given by

$$(A\times B)\circ C = \begin{vmatrix} \alpha^1 & \alpha^2 & \alpha^3 \\ \beta^1 & \beta^2 & \beta^3 \\ \gamma^1 & \gamma^2 & \gamma^3 \end{vmatrix} I = A\circ(B\times C) \qquad (1.17)$$

and

$$A\times(B\times C) = (A\circ B)C - (A\circ C)B. \qquad (1.18)$$

From (1.17), it follows that three bivectors $A, B, C$ are linearly independent over the complex scalars iff their triple product (1.17) is nonvanishing. The identities (1.11), (1.12), (1.17), and (1.18) of the Pauli algebra obviously parallel their Gibbs–Heaviside vector algebra counterparts, and this suggests that the former are in some sense the "complexified" version of the latter.

We conclude this section with a discussion and classification of bivectors.[11] A bivector $B$ is said to be *simple* if

$$B^2 = B\cdot B + B\wedge B = B\cdot B, \qquad (1.19)$$

i.e., $B^2$ is a (real) scalar. The bivector $B$ is said to be *null* if

$$B^2 = 0 \text{ and } B\neq 0. \qquad (1.20)$$

A simple bivector can always be factored into the product of two anticommuting (orthogonal) Dirac vectors, i.e., $B = ab = -ba$. A non-null bivector $C\neq 0$ can always be uniquely expressed in the form

$$C = \rho e^{I\theta}A, \text{ for } \rho\geqslant 0,\ 0\leqslant\theta<2\pi, \text{ and } A^2 = 1, \qquad (1.21)$$

and a null bivector $N$ can always be uniquely expressed in the form

$$\begin{aligned} &N = \rho(1+A_1)A_2, \quad \rho>0, \text{ and} \\ &A_1^2 = A_2^2 = 1, \text{ and } A_1A_2 = -A_2A_1. \qquad (1.22) \end{aligned}$$

To prove (1.21), note that we can define $\rho^2e^{2I\theta}\equiv C^2\neq 0$, and $A = \rho^{-1}e^{-I\theta}C$, from which the required properties easily follow. For the case of the null bivector $N$, there exists an orthonormal basis $a_u$ related to the orthonormal basis $e_u$ of (1.1) by a proper Lorentz transformation, which satisfies:

$$\begin{aligned} N &= \rho a_2 n = \rho a_2(a_0+a_1) = \rho a_2 a_0 a_0(a_0+a_1) \qquad (1.23)\\ &= \rho A_2(1-A_1) = \rho(1+A_1)A_2, \end{aligned}$$

where $n = a_0 + a_1$ is a null vector, $A_1 = a_1\wedge a_0 = a_1a_0$, and $A_2 = a_2\wedge a_0 = a_2a_0$.

The following is a multiplication table for $A_1, A_2, A_{12} \equiv A_1A_2$, and a null bivector $N = (1+A_1)A_2$:

| | $A_1$ | $A_2$ | $A_{12}$ | $N$ |
|---|---|---|---|---|
| $A_1$ | $1$ | $A_{12}$ | $A_2$ | $N$ |
| $A_2$ | $-A_{12}$ | $1$ | $-A_1$ | $1-A_1$ |
| $A_{12}$ | $-A_2$ | $A_1$ | $-1$ | $-1+A_1$ |
| $N$ | $-N$ | $1+A_1$ | $-1-A_1$ | $0$ |

(1.24)

## 2. VECTOR AND BIVECTOR DIFFERENTIATION

Two notions of differentiation are fundamental to the methods of this work, the vector derivative $\partial_v$, defined for differentiable $\mathscr{D}$-valued functions of a vector variable $f:\mathscr{D}_1\to\mathscr{D}$, and the bivector derivative $\partial_B$, defined for differentiable functions of a bivector variable $F:\mathscr{D}_2\to\mathscr{D}$. The vector derivative is characterized by two properties:

$$\partial_B \text{ behaves algebraically like a vector in } \mathscr{D}_1. \qquad (2.3)$$

$$a\cdot\partial f \equiv a\cdot\partial_v f(v) \equiv (d/dt)f(v+ta)|_{t=0}. \qquad (2.2)$$

The bivector derivative is characterized by two similar properties:

$$\partial_B \text{ behaves algebraically like a vector in } \mathscr{D}_2. \qquad (2.3)$$

$$A\cdot\partial F \equiv A\cdot\partial_B F(B) \equiv (d/dt)F(B+tA)|_{t=0}. \qquad (2.4)$$

We shall not be precise in specifying conditions for vector and bivector differentiability, because we shall be concerned here only with derivatives of linear functions, which always exist.

Because of property (2.1), $\partial_v$ can be expressed in terms of the orthonormal basis $\{e_u\}$ by

$$\partial \equiv \partial_v = e_0e_0\cdot\partial_v - e_1e_1\cdot\partial_v - e_2e_2\cdot\partial_v - e_3e_3\cdot\partial_v. \qquad (2.5)$$

Simple but important formulas for the vector derivative are

$$a\cdot\partial v = a = \partial v\cdot a, \qquad (2.6)$$

$$\partial v = 4 \Leftrightarrow \partial\cdot v = 4 \quad \text{and} \quad \partial\wedge v = 0, \qquad (2.7)$$

$$a\wedge\partial v = 3a = \partial v\wedge a, \qquad (2.8)$$

$$\partial_v\wedge\partial_u u\wedge v = 12 = \partial_v\partial_u u\wedge v, \qquad (2.9)$$

which can be easily derived from (2.1), (2.2), (2.5) and algebraic identities from Sec. 1. For example, to prove (2.6), note that

$$a\cdot\partial_v v \equiv (d/dt)(v+ta)|_{t=0} = a.$$

Identity (2.7) follows by using (2.5) and (2.6) to get

$$\begin{aligned} \partial_v v &= e_0e_0\cdot\partial_v v - e_1e_1\cdot\partial_v v - e_2e_2\cdot\partial_v v - e_3e_3\cdot\partial_v v \\ &= e_0^2 - e_1^2 - e_2^2 - e_3^2 = 4. \end{aligned}$$

Identity (2.8) then follows by using (1.5), (2.1), (2.7), and (2.6) to get

$$a\wedge\partial v = a\partial v - a\cdot\partial v = 4a - a = 3a.$$

Because of its property (2.3), the bivector derivative $\partial_B$ can be expressed in terms of the orthonormal timelike bivector basis $\{E_i\}$ by

$$\partial \equiv \partial_B = E_1E_1\circ\partial_B + E_2E_2\circ\partial_B + E_3E_3\circ\partial_B. \qquad (2.10)$$

Simple, but important, formulas for the bivector derivative are

$$A\cdot\partial B = A = \partial B\cdot A, \tag{2.11}$$

$$A\wedge\partial B = -I(IA)\cdot\partial B = A = \partial B\wedge A, \tag{2.12}$$

$$A\circ\partial B = A\cdot\partial B + A\wedge\partial B = 2A = \partial B\circ A \tag{2.13}$$

$$\partial B = 6 \Leftrightarrow \partial\circ B = 6 \text{ and } \partial\times B = 0, \tag{2.14}$$

$$A\times\partial B = 4A = \partial B\times A, \tag{2.15}$$

$$\partial_B\times\partial_A A\times B = 24 = \partial_B\partial_A A\times B, \tag{2.16}$$

$$\partial_C\circ\partial_B\times\partial_A A\times B\circ C = 48 = \partial_C\partial_B\partial_A A\times B\circ C, \tag{2.17}$$

and these formulas can be derived from (2.3), (2.4), (2.10) and the algebraic identities in Sec. 1. For example, to prove the left-hand side of (2.11), use definition (2.4) to get

$$A\cdot\partial B = (d/dt)(B+tA)|_{t=0} = A|_{t=0} = A.$$

The left-hand side of (2.12) is a consequence of (2.11) and (1.16). To prove the right-hand side of (2.11), we use (2.10), (1.14), and what we have just proved, to get

$$\begin{aligned}\partial B\cdot A &= E_1E_1\circ\partial B\cdot A + E_2E_2\circ\partial B\cdot A + E_3E_3\circ\partial B\cdot A\\ &= E_1(E_1\cdot\partial - I(IE_1)\cdot\partial)B\cdot A + \cdots\\ &= E_1(E_1\cdot A - I(IE_1)\cdot A) + \cdots\\ &= E_1E_1\circ A + E_2E_2\circ A + E_3E_3\circ A = A.\end{aligned}$$

The right-hand side of (2.12) now easily follows from the right-hand sides of (2.11) and (1.16). Finally, to see that (2.16) is a consequence of (2.14) and (2.13), first use (2.3) and (1.14) and write

$$A\times\partial B = A\partial B - A\circ\partial B = 6A - 2A = 4A.$$

There is a close relationship between the vector and bivector derivatives of a linear function $F(B)$. It is given by

$$\partial_B F(B) = \partial_v\wedge\partial_u F(B) = \tfrac{1}{2}\partial_v\wedge\partial_u F(u\wedge v), \tag{2.18}$$

where $B=\frac{1}{2}u\wedge v$. This relationship is checked for the identity $F(B)=B$ by comparing (2.9) and (2.14). The vector and bivector derivatives, and their natural generalization to $2^n$-dimensional Clifford algebra were originally developed as coordinate-free tools for use in linear algebra and differential geometry in Ref. 12, and since have been extensively used in Ref. 9.

## 3. BIVECTOR OPERATORS

By a bivector operator $F(B)$ we mean a linear bivector-valued function of the bivector variable $B$. If in addition $F$ satisfies

$$F(IB) = IF(B), \tag{3.1}$$

we say that $F$ is *dual*. If instead $F$ satisfies

$$F(IB) = -IF(B), \tag{3.2}$$

we say that $F$ is *antidual*. A bivector operator can always be split into the sum of dual and antidual parts, as is evident in

$$F(B) = S(B) + T(B), \tag{3.3}$$

where

$$S(B) = \tfrac{1}{2}[F(B) - IF(IB)],$$

and

$$T(B) = \tfrac{1}{2}[F(B) + IF(IB)].$$

Using formulas (2.11), (2.12), and (2.13), we calculate derivatives of $F(B)$, finding

$$A\circ\partial F = A\circ\partial S + A\circ\partial T = 2S(A), \tag{3.4}$$

since

$$A\circ\partial S = S(A\circ\partial B) = 2S(A) \tag{3.5}$$

and

$$A\circ\partial T = T(A\cdot\partial B) - T(A\wedge\partial B) = T(A) - T(A) = 0. \tag{3.6}$$

From (3.4) and (2.13) it follows that

$$\partial F = \tfrac{1}{2}\partial_A A\circ\partial F = \partial S = \partial\circ S + \partial\times S, \tag{3.7}$$

which shows that the bivector derivative of $F$ is completely determined by the bivector derivative of its dual part. As a consequence of this, it follows that

$$\partial T = 0, \tag{3.8}$$

i.e., the derivative of an antidual operator vanishes.

An operator $F(B)$ is said to be *symmetric* (with respect to the metric $g$) if

$$F(A)\cdot B = A\cdot F(B) \Leftrightarrow F(A) = F^\dagger(A) \equiv \partial_B F(B)\cdot A \tag{3.9}$$

and *skew-symmetric* (w.r.t.g) if

$$F(A)\cdot B = -A\cdot F(B) \Leftrightarrow F(A) = -F^\dagger(A). \tag{3.10}$$

Differentiating the first expressions in (3.9) and (3.10) by $\partial_A\partial_B$ gives, with the help of (2.11),

$$\partial\times F = \tfrac{1}{2}(\partial F - F\overleftarrow{\partial}) = 0 \tag{3.11}$$

and

$$\partial\circ F = \tfrac{1}{2}(\partial F + F\overleftarrow{\partial}) = 0, \tag{3.12}$$

respectively, where $\overleftarrow{\partial}$ differentiates to the left. Thus, symmetric operators have vanishing *curl*, whereas skew-symmetric operators have vanishing *trace*. Symmetric bivector operators are known in the literature as curvature operators, and will be studied in Sec. 5.

An operator is said to be *dual symmetric* if it is both dual and symmetric, and *dual skew-symmetric* if it is both dual and skew-symmetric. An operator is dual symmetric iff

$$F(A)\circ B = A\circ F(B), \tag{3.13}$$

i.e., $F$ is symmetric w.r.t. the metric $G$, or equivalently, iff

$$A\circ\partial F = 2F(A) = \partial F\circ A. \tag{3.14}$$

To establish (3.13), note by using (1.16) that

$$\begin{aligned}F(A)\wedge B &= -I(IF(A))\cdot B = -IF(IA)\cdot B\\ &= -I(IA)\cdot F(B) = A\wedge F(B)\end{aligned}$$

and combine this result with (3.9). Property (3.14) follows directly from (3.13) and (2.13). An operator is dual skew-symmetric iff

$$F(A)\circ B = -A\circ F(B), \tag{3.15}$$

i.e., $F$ is skew-symmetric w.r.t. the metric $G$, or, equivalently, iff

$$F(B) = \tfrac{1}{4}B\times(\partial\times F). \tag{3.16}$$

The proof of (3.15) is similar to that of (3.13). The proof of (3.16) follows by using (1.18), (3.15), and (2.13) to get

$$B\times(\partial\times F) = B\circ\partial F - \partial F\circ B = 4F(B).$$

There is an important identity satisfied by dual operators $F(B)$. It is given by

$$F(A\times C)+F(A)\times C+A\times F(C) = \tfrac{1}{2}\partial\circ FA\times C+\tfrac{1}{2}(A\times C)\times(\partial\times F). \quad (3.17)$$

In the special case that $F$ is also symmetric, (3.17) reduces to

$$F(A\times C)+F(A)\times C+A\times F(C)=\tfrac{1}{2}\partial\circ FA\times C. \quad (3.18)$$

In the special case that $F$ is dual skew-symmetric, (3.17) reduces to (3.16). Identity (3.18) follows by equating the right sides of the identitites

$$A\times(C\times\partial)F-C\times(A\times\partial)F = \partial\circ FA\times C-2F(A\times C)$$

and

$$\begin{aligned}A\times(C\times\partial)F-C\times(A\times\partial)F &= \partial\circ FA\circ C-2CF(A)-\partial\circ FCA+2AF(C)\\ &= -2C\times F(A)-2F(C)\times A,\end{aligned}$$

and the general identity (3.17) follows by combining (3.18) and (3.16).

We will now find the determinant, the characteristic equation, and the Cayley–Hamilton theorem for a dual operator $F(B)$. Define

$$\det(F)=(1/48)\partial_C\circ\partial_B\times\partial_A\, F(A)\times F(B)\circ F(C). \quad (3.19)$$

In terms of the orthonormal basis $\{E_i\}$, with the help of (1.17) and (2.10), it is not difficult to check that

$$\det(F)=-IF(E_1)\times F(E_2)\circ F(E_3)=|F(E_i)\circ E_j|. \quad (3.20)$$

For $F(B)=B$, from (2.17) or (1.7) it can be seen that $\det(F)=1$, as would be expected. Carrying out the indicated differentiation in (3.19) gives

$$\det(F)=1/48[8\partial\circ F^3-6\partial\circ F\ \partial\circ F^2+(\partial\circ F)^3], \quad (3.21)$$

which expresses the $\det(F)$ in terms of the complex scalars $\partial\circ F$, $\partial\circ F^2$, and $\partial\circ F^3$. Note the these three complex scalars correspond to six real scalars, and are Lorentz invariant; more about them later. In the case that $F$ is dual skew-symmetric, $\det(F)=0$, since $\partial\circ F=0=\partial\circ F^3$.

To obtain the characteristic polynomial for $F$, define

$$F'\equiv F-\lambda\equiv F(B)-\lambda B. \quad (3.22)$$

Then $\psi(\lambda)$ is given by

$$\psi(\lambda)=\det(F')=\det(F-\lambda). \quad (3.23)$$

Using (3.23), and (3.19) or (3.21), we compute

$$\begin{aligned}\psi(\lambda)&=\lambda^3-\tfrac{1}{2}\partial\circ F\lambda^2-\tfrac{1}{4}[\partial\circ F^2-\tfrac{1}{2}(\partial\circ F)^2]\lambda\\ &\quad-1/48[8\partial\circ F^3-6\partial\circ F\partial\circ F^2+(\partial\circ F)^3].\end{aligned} \quad (3.24)$$

In the case that $F$ is dual skew-symmetric, $\psi(\lambda)$ simplifies to $\psi(\lambda)=\lambda[\lambda+\tfrac{1}{2}(\partial\circ F^2)^{1/2}][\lambda-\tfrac{1}{2}(\partial\circ F^2)^{1/2}]$.

The Cayley–Hamilton for $F$ says simply that

$$\psi(F)\equiv 0, \quad (3.26)$$

i.e., $F$ satisfies its characteristic equation. The method of proof of (3.26) is to decompose $\det(F)A$, which is the last term in $\psi(F)$, into the sum of the other terms. This is accomplished in the following steps:

$$\begin{aligned}48A\det(F)&=A\partial_3\circ\partial_2\times\partial_1\, F_1\times F_2\circ F_3\\ &=6\partial_2\times\partial_1\, F_1\times F_2\circ F(A)\\ &=6\partial_2\times\partial_1\, F_1\times F_2F(A)-6\partial_2\times\partial_1(F_1\times F_2)\times F(A)\\ &=[6(\partial\circ F)^2-12\partial\circ F^2]F(A)-24\partial\circ FF^2(A)\\ &\quad+48F^3(A).\end{aligned}$$

This formulation and proof of the Cayley–Hamilton theorem was first found for linear transformations in Ref. 12.

We will now show that an antidual operator $T(B)$ can be expressed entirely in terms of two symmetric trace-free vector operators. First consider the identity

$$\begin{aligned}T(B)&=\tfrac{1}{2}(B\cdot\partial_v)\cdot\partial_u T(u\wedge v)\\ &=\tfrac{1}{2}B\cdot\partial_v\partial_u T(u\wedge v)-B\times\partial_A T(A)\\ &=\tfrac{1}{2}B\cdot\partial_v\partial_u T(u\wedge v).\end{aligned} \quad (3.27)$$

The last equality is a consequence of (3.6) and (3.8), since

$$B\times\partial_A T(A)=B\partial_A T(A)-B\circ\partial_A T(A)=0.$$

Now define the vector operators

$$t(v)\equiv\partial_u\cdot T(u\wedge v) \quad \text{and} \quad \bar{t}(v)\equiv\partial_u\cdot T(u\wedge vI). \quad (3.28)$$

An easy consequence of (3.8) is that $t(v)$ and $\bar{t}(v)$ satisfy

$$\partial_v t(v)=0=\partial_v\bar{t}(v),$$

which means $t(v)$ and $\bar{t}(v)$ are symmetric trace-free operators. We can now express (3.27) in the form

$$\begin{aligned}T(B)&=\tfrac{1}{2}B\cdot\partial_v[\partial_u\cdot T(u\wedge v)+\partial_u\cdot T(u\wedge vI)I]\\ &=E(B)+D(B),\end{aligned} \quad (3.29)$$

where

$$E(B)\equiv\tfrac{1}{2}B\cdot\partial_v t(v)=E^\dagger(B) \quad (3.30)$$

is an antidual symmetric bivector operator, and

$$D(B)\equiv\tfrac{1}{2}B\cdot\partial_v\bar{t}(v)I=-D^\dagger(B) \quad (3.31)$$

is an antidual skew-symmetric bivector operator. The symmetry of $E(B)$ follows from the steps

$$\begin{aligned}E^\dagger(B)&\equiv\partial_A E(A)\cdot B=\tfrac{1}{2}\partial_A[A\cdot\partial_v t(v)]\cdot B=\tfrac{1}{2}\partial_A A\cdot[\partial_v t(v)\cdot B]\\ &=\tfrac{1}{2}[SB{:}\partial{:}v]t(v)\cdot B=E(B),\end{aligned}$$

and the skew-symmetry of $D(B)$ can be similarly established.

We have the following important properties of $E(B)$ and $D(B)$:

$$\partial_u\wedge E^k(u\wedge v)=0, \quad \text{for} \quad k=1,2,\cdots \quad (3.32)$$

and

$$\partial_u\wedge D^{2k}(u\wedge v)=0=\partial_u\cdot D^{2k-1}(u\wedge v), \quad \text{for} \quad k=1,2,\cdots, \quad (3.33)$$

which can be proved by using induction on $k$ and the symmetry of $t(v)$ and $\bar{t}(v)$.

Combining the results of (3.3), (3.13), (3.16), and (3.29), we find that a general bivector operator can always be decomposed into

$$F(B)=[H(B)+E(B)]+[J(B)+D(B)], \quad (3.34)$$

where $H(B)$ is dual symmetric, $E(B)$ is antidual symmetric, $J(B)$ is dual skew-symmetric, and $D(B)$ is antidual skew-symmetric. The classification of trace-free symmetric vector operator is carried out in Ref. 13 by reducing the problem to the Petrov classification of a correlated Weyl tensor.

## 4. CLASSIFICATION OF DUAL OPERATORS

Let $F$ be a dual operator, i.e., one satisfying (3.1). The $F$ has the characteristic polynomial $\psi(\lambda)$ given by (3.24), and setting

$$\psi(\lambda) = 0 \tag{4.1}$$

gives the characteristic equation for $F$. The solutions $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $F$. Writing

$$\psi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3), \tag{4.2}$$

we find, on expanding the right-hand side of (4.2) and equating the coefficients of $\lambda$ with those in (3.24), that

$$\tfrac{1}{2}\partial\circ F^k = \lambda_1^k + \lambda_2^k + \lambda_3^k \quad \text{for} \quad k = 1,2,3. \tag{4.3}$$

The characteristic roots of (4.1) have multiplicity 1, 2, or 3 according to whether

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \quad \text{for multiplicity 1}, \tag{4.4}$$

$$\lambda_1 \neq \lambda_2 = \lambda_3 \quad \text{for multiplicity 2}, \tag{4.5}$$

$$\lambda_1 = \lambda_2 = \lambda_3 \quad \text{for multiplicity 3}, \tag{4.6}$$

Conditions for (4.4), (4.5), (4.6) can be given in terms of $\partial\circ F$, $\partial\circ F^2$, $\partial\circ F^3$.[3,13,15]

We see from (3.23) and (3.20) that, for each eigenvalue $\lambda_k$,

$$[F(E_1) - \lambda_k E_1] \times [F(E_2) - \lambda_k E_2] \circ [F(E_3) - \lambda_k E_3] = 0,$$

which implies, because of (1.17), that there exist eigenbivectors satisfying

$$F(C_k) = \lambda_k C_k \quad \text{for} \quad k = 1,2,3. \tag{4.7}$$

We will consider the classification of dual symmetric and dual skew-symmetric operators separately. This is justified by the fact that we can always decompose $F$ into

$$F(B) = H(B) + J(B), \tag{4.8}$$

where

$$H(B) \equiv \tfrac{1}{2}[F(B) + F^\dagger(B)] = \tfrac{1}{2}\partial_B F(B)\cdot B$$

is dual symmetric, and

$$J(B) = \tfrac{1}{2}[F(B) - F^\dagger(B)] = \tfrac{1}{4}B \times (\partial \times F)$$

is dual skew-symmetric.

Let $J(B)$ be a dual skew-symmetric operator. Then by (3.16), $J(B)$ can be writtern in the form

$$J(B) = B \times Q, \tag{4.9}$$

where $Q = \frac{1}{4}\partial \times J$. From (4.9) we calculate

$$J^2(B) = (B \times Q) \times Q = B\, Q\circ Q - B\circ Q\, Q \tag{4.10}$$

and

$$J^3(B) = B \times Q\, Q^2, \tag{4.11}$$

from which is follows that $\partial J^k = 4Q^k$, which implies

$$\partial \circ J = 0, \quad \partial \circ J^2 = 4Q^2, \quad \partial \circ J^3 = 0. \tag{4.12}$$

The characteristic polynomial (3.25) of $J(B)$ can be written in terms of $Q^2$, getting

$$\psi(\lambda) = \lambda\,[\lambda + (Q^2)^{1/2}][\lambda - (Q^2)^{1/2}]. \tag{4.13}$$

From (4.13) it is clear that the key to the classification of $F(B)$ is the bivector $Q$. The canonical forms (1.21) and (1.22) for a bivector tell us that

$$Q = 0 \quad \text{or} \quad Q = \rho e^{I\theta} A_1 \quad \text{or} \quad Q = \rho(1 + A_1)A_2.$$

The case $Q = 0$ is trivial.

For the case $Q = \rho e^{I\theta} A_1$, we construct the null bivectors $N = (1 + A_1)A_2$, and $M = (1 - A_1)A_2$, and note, with the help of table (1.23), that

$$A_1 = \tfrac{1}{2}N \times M, \quad N\circ M = 2, \quad A_1\circ N = 0 = A_1\circ M. \tag{4.14}$$

It then follows, using (1.18), that

$$\begin{aligned} J(B) = B \times Q &= \tfrac{1}{2}\rho e^{I\theta} B \times (N \times M) \\ &= \tfrac{1}{2}\rho e^{I\theta}(B\circ N\, M - B\circ M\, N). \end{aligned} \tag{4.15}$$

From the canonical form (4.15) of $J(B)$, with the help of (4.14), we can read off the eigenbivectors and eigenvalues of $J$. Thus,

$$J(A_1) = 0A_1, \quad J(N) = -\rho e^{I\theta} N, \quad J(M) = \rho e^{I\theta} M. \tag{4.16}$$

For the case $Q = \rho N$, where $N = (1 + A_1)A_2$, $Q^2 = 0$,

$$J(B) = \rho B \times N = \rho(B\circ A_1 N - B\circ N A_1) \tag{4.17}$$

is the desired canonical form. We calculate

$$\begin{aligned} &J(N) = 0, \quad J(A_{12}) = \rho A_1, \quad J(A_1) = \rho N, \\ &J(A_2) = -\rho A_1, \end{aligned} \tag{4.18}$$

from which it follows that $N$ is the only eigenbivector of $J(B)$. The above cases can be summarized in the following table enumerating the number of null eigenbivectors of $J(B)$:

$$\begin{array}{ccc} & 11 & Q^2 \neq 0 \\ 1(Q \neq 0) & \text{—}(Q = 0) & Q^2 = 0. \end{array} \tag{4.19}$$

Of course it closely parallels that given by Penrose,[3] in his spinor classification of an electromagnetic field. The bivector $Q$ represents an electromagnetic field at a point in spacetime.

We will now carry out the classification of a dual symmetric operator $H(B)$ into the so-called Petrov types. Because of (4.7), $H$ has eigenbivectors and values satisfying

$$H(C_k) = \lambda_k C_k, \quad \text{for} \quad k = 1,2,3. \tag{4.20}$$

That orthogonal bivectors correspond to distinct eigenvalues follows from the standard argument:

$$(\lambda_i - \lambda_j) C_i \circ C_j = H(C_i)\circ C_j - C_i \circ H(C_j) = 0. \tag{4.21}$$

Furthermore, because of the bivector classifications (1.21) and (1.22), and the fact that $H$ is dual, each eigenbivector $C$ of $H$ can be replaced by a time-like eigenbivector $A$, with $A^2 = 1$, or by a null bivector $N = (1 + A_1)A_2$, having the same eigenvalue as $C$. We will always assume that the eigenbivectors $C_k$ of $H$ have been so normalized. The operator $H(B)$ is said to be of Petrov

Type I: if $\{C_k\}$ spans a three-dim. space,
Type II: if $\{C_k\}$ spans a two-dim. space,
Type III: if $\{C_k\}$ spans one-dim. space.

Suppose $H$ is Type I. If the eigenvalues $\lambda_k$ are distinct, then by (4.21) the $C_k$'s are orthogonal. This excludes the possibility that one or more of the eigenbivectors are null, because inspection of table (1.23) shows that if an eigenbivector $C$ is orthogonal to a null eigenbivector $N = (1 + A_1)A_2$, then $C$

must be of the form $C = A_1 + \alpha N$, so that the eigenbivectors $C_k$ could span at most a two-dim. space. If the eigenvalues are not distinct, simple orthonormal space-like bivectors can still be choosen with the same multiplicity as the repeated roots.

Suppose $H$ is type II. Then the eigenvalues of $H$ cannot all be distinct, for othewise, because of (4.21), $H$ would be Type I. Also, $H$ cannot have two orthonormal time-like eigenbivectors $A_1, A_2$, for in this case, letting $A_3 = IA_1 \times A_2$, we find by using (3.18) that

$$H(A_1\times A_2) + H(A_1)\times A_2 + A_1\times H(A_2) = \tfrac{1}{2}\partial\circ H A_1\times A_2,$$

or

$$H(A_3) = (\tfrac{1}{2}\partial\circ H - \lambda_1 - \lambda_2)A_3,$$

so that $A_3$ would be an eigenbivector also, contradicting the assumption that $H$ is Type II. Thus, $H$ must have a null eigenbivector $N = (1 + A_1)A_2$ satisfying

$$H(N) = \lambda_N N, \tag{4.22}$$

and a time-like bivector of the form $C_1 = A_1 + \alpha N$, satisfying

$$H(C_1) = \lambda_1 C_1. \tag{4.23}$$

Equations (4.22), (4.23) imply

$$H(A_1) = \lambda_1 A_1 + \beta_1 N, \tag{4.24}$$

where

$$\beta_1 \equiv A_1\circ H(A_2) = A_2\circ H(A_1) = \alpha(\lambda_1 - \lambda_N).$$

In the degenerate case when $\lambda_1 = \lambda_N$, (4.24) reduces to

$$H(A_1) = \lambda_1 A_1, \quad \text{and} \quad \beta_1 = 0. \tag{4.25}$$

Finally, note that $A_1\times N = N$, and using this in identity (3.18), together with (4.24) and (4.22), shows that

$$\tfrac{1}{2}\partial\circ H = \lambda_1 + 2\lambda_N, \tag{4.26}$$

for Type II.

Suppose $H$ is Type III. Then $H$ has one eigenbivector, a null bivector $N$, satisfying

$$H(N) = \lambda_N N, \tag{4.27}$$

and

$$\tfrac{1}{2}\partial\circ H = 3\lambda_N. \tag{4.28}$$

The above classification scheme can be refined by introducing the notion of principal null directions of $H$. These are null bivectors $M$ which satisfy

$$H(M)\circ M = 0 \quad \text{and} \quad M^2 = 0, \tag{4.29}$$

and were used by Penrose[3] in his refinement of the Petrov classification of the conformal curvature tensor using spinors. The condition (4.29) was first noted in a remark by Thorpe.[5] The principal null bevectors of $H$ are explicitly calculated below, and their coincidence patterns are specified by new and simple conditions.

For the case that the $\lambda_k$'s are distinct, $H(B)$ has a basis of orthonormal time-like eigenbivectors $A_1, A_2, A_3$. In terms of this basis we can write

$$H(B) = B\circ A_1\lambda_1 A_1 + B\circ A_2\lambda_2 A_2 + B\circ A_3\lambda_3 A_3. \tag{4.30}$$

Imposing the condition (4.29) leads to the equations

$$M\circ M = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0, \quad \text{or} \quad \alpha_3^2 = -(\alpha_1^2 + \alpha_2^2),$$

and

$$H(M)\circ M = \lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 - \lambda_3(\alpha_1^2 + \alpha_2^2) = 0,$$

where $M = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, which has solutions

$$\alpha_1 = \pm\sqrt{\lambda_2 - \lambda_3}, \quad \alpha_2 = \pm\sqrt{\lambda_3 - \lambda_1}, \quad \alpha_3 = \sqrt{\lambda_1 - \lambda_2} \tag{4.31}$$

which correspond to four distinct principal null directions.

For all other cases there will be a null eigenbivector $N = (1 + A_1)A_2$ for which $H(N) = \lambda_N N$. In these cases, we expand $H(B)$ in the basis $A_1, A_2, A_{12} = A_1A_2$, finding, with the help of (1.23) and (3.18),

$$H(B) = (B\circ A_1\lambda_1 + B\,N\beta_1)A_1 + B\circ H_2A_2 + (B\circ H_2 - \lambda_N B\circ N)A_{12}, \tag{4.32}$$

where $\beta_1 \equiv A_1\circ H(A_2)$, $H_2 \equiv H(A_2)$, and $\lambda_1 \equiv \tfrac{1}{2}\partial\circ H - 2\lambda_N$, $\lambda_N \equiv N\circ H_2$.

Note that $H(B)$ is defined entirely in terms of the independent quantities

$$H_2, \quad N, \quad \tfrac{1}{2}\partial\circ H, \tag{4.33}$$

where $H_2$ is an arbitrary bivector (six parameters), $N$ is an arbitrary null bivector (four parameters), and $\tfrac{1}{2}\partial\circ H$ is an arbitrary complex scalar (two parameters), making up 12 independent parameters in all.

We are now ready to solve for principal null bivectors by imposing (4.29) on the expansion (4.32). This is done in the steps below:

$$M = \alpha_1A_1 + \alpha_2A_2 + \alpha_3A_{12}, \quad M\circ M = \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = 0,$$

$$\begin{aligned} H(M) &= [\lambda_1\alpha_1 + (\alpha_2 - \alpha_3)\beta_1]A_1 + M\circ H_2A_2 \\ &\quad + [M\circ H_2 - \lambda_N(\alpha_2 - \alpha_3)]A_{12} \\ &= [\lambda_1\alpha_1 - y\beta_1]A_1 + [\beta_1\alpha_1 + \lambda_N\alpha_2 - \beta_2 y]A_2 \\ &\quad + [\beta_1\alpha_1 + (\lambda_N - \beta_2)y + \lambda_N\alpha_2]A_{12} \end{aligned}$$

and

$$H(M)\circ M = \beta_2 y^2 + (\delta x - 2\alpha_1\beta_1)y \tag{4.34}$$

in terms of the new variables:

$$x = \alpha_3 + \alpha_2, \quad y = \alpha_3 - \alpha_2, \quad \delta = \lambda_1 - \lambda_N,$$
$$\beta_1 = A_1\circ H_2, \quad \beta_2 = A_{21}\circ H_2.$$

Thus the equation

$$H(M)\circ M = 0 \Leftrightarrow \delta x\,y + \beta_2 y^2 = 2\beta_1\alpha_1 y, \tag{4.35}$$

and for $y \neq 0 \neq \beta_1$, we find

$$\alpha_1 = [1/2\beta_1](\delta x + \beta_2 y).$$

Squaring the equation in (4.35) leads to the equation

$$y^2[(\delta x + \beta_2 y)^2 - 4\beta_1^2 x\,y] = 0. \tag{4.36}$$

Analysis of equation (4.36) together with (4.35) leads to the following classification scheme of the principal null directions of $H(B)$:

| | | | |
|---|---|---|---|
| | | $1111(\neq\lambda_k$'s$)$ | |
| | $211(\beta_1^2 \neq \delta\beta_2)$ | $22\ (\beta_1^2 = \delta\beta_2)$ | $\delta \neq 0$ |
| $31(\beta_1 \neq 0)$ | $4(\beta_1 = 0 \neq \beta_2)$ | $—(\beta_1 = 0 = \beta_2)$ | $\delta = 0$ |
| III | II | I | (4.37) |

As an example of the kind of analysis involved in the above classification, we will carry it to completion in the more involved case $\beta_1 \neq 0 \neq \beta_2$ and $\delta \neq 0$. In this case, set $x = 1$ in (4.36), and factor the resulting quadratic equation in $y$, getting

$$y^2\left\{\left[y+\left(\frac{\delta}{\beta_2}-\frac{2\beta_1^2}{\beta_2^2}\right)+\epsilon\right]\left[y+\left(\frac{\delta}{\beta_2}-\frac{2\beta_1^2}{\beta_2^2}\right)-\epsilon\right]\right\} = 0, \qquad (4.38)$$

where

$$\epsilon = 2\frac{\beta_1}{\beta_2}\sqrt{\frac{\delta\beta_2-\beta_1^2}{\beta_2^2}}\,.$$

This equation reduces to

$$y^2\left(y-\frac{\beta_1^2}{\beta_2^2}\right)^2 = 0, \quad \text{when} \quad \delta\beta_2 = \beta_1^2. \qquad (4.39)$$

We see that for $\epsilon \neq 0$ (or $\delta\beta_2 \neq \beta_1^2$), Eq. (4.38) has double solution for $y = 0$, and two single solutions corresponding to the zeros of the other factors. The corresponding principal null bivectors can be exhibited explicitly by going back to the original variables. Similarly, (4.39) gives two double principal null bivectors for each of the roots of its repeated factors, when $\delta\beta_2 = \beta_1^2$).

In the case that $\frac{1}{2}\partial\circ H = \lambda_1 + \lambda_2 + \lambda_3 = 0$, the three Petrov types can be efficiently characterized by the canonical forms

$$H(B) = (2\lambda_1+\lambda_2)B\circ A_1A_1 + (2\lambda_2+\lambda_1)B\circ A_2A_2 - (\lambda_1+\lambda_1)B, \qquad (4.40)$$

for type I,

$$H(B) = \lambda_N B - 3\lambda_N B\circ C_1C_1 + \mu BoNN,$$

where

$$\begin{Bmatrix} C_1 = A_1 + \alpha N \\ \mu = \beta_2 + 3\lambda_N\alpha^2 \end{Bmatrix}, \qquad (4.41)$$

for type II, and

$$H(B) = B\circ NC^1 = B\circ C_1 N,$$

where

$$C_1 = \beta_1 A_1 + \tfrac{1}{2}\beta_2 N, \qquad (4.42)$$

for type III. The canonical form (4.40) can be derived immediately from (4.30) and the fact that $B \equiv \sum_{k=1}^{3} B\circ A_k A_k$. To derive (4.41), we use the properties (4.22)–(4.26), together with (1.18) and (1.23) and the fact that $C_1 \times N = N$, to calculate

$$H(B)\times N = H(B)\times(C_1\times N) = H(B)\circ C_1 N - H(B)\circ NC_1$$
$$= \lambda_N(B\times N - 3B\circ C_1 N),$$

which implies that

$$[H(B) - \lambda_N B + 3\lambda_N B\circ C_1C_1]\times N \equiv 0 \qquad \text{for all } B.$$

Applying $A_2 x$ to this last identity, and again utilizing (1.18) and (1.23), yields

$$H(B) - \lambda_N B + 3\lambda_N B\circ C_1C_1$$
$$= A_2\circ[H(B) - \lambda_N B + 3\lambda_N B\circ C_1C_1]N$$
$$= B\circ[H(A_2) - \lambda_N A_2 + 3\lambda_N\alpha C_1]N. \qquad (4.43)$$

Applying (4.43) with $B = A_2$ gives

$$H(A_2) - \lambda_N A_2 + 3\lambda_N + C_1 = [A_2\circ H_2 - \lambda_N + 3\lambda_N\alpha^2]N = \mu N, \qquad (4.44)$$

where $\mu \equiv \beta_2 + 3\lambda_N\alpha^2$. Then (4.41) now follows trivially from (4.43) and (4.44). Finally, to derive (4.42), we note from (4.32) that for type III, $H(B)$ reduces to the form

$$H(B) = B\circ N\beta_1 A_1 + B\circ H(A_2)N, \qquad (4.45)$$

from which it follows that

$$H(A_2) = \beta_1 A_1 + \beta_2 N. \qquad (4.46)$$

Together, (4.45) and (4.46) imply (4.42), where $C_1 \equiv \beta_1 A_1 + \frac{1}{2}\beta_2 N$.

## 5. RIEMANN CURVATURE: INVARIANTS AND PROPERTIES

Recall that a curvature operator $R(B)$ is a bivector operator satisfying (3.9). From (3.34) it follows tht $R(B)$ can be written in the form

$$R(B) = H(B) + E(B) = R^\dagger(B), \qquad (5.1)$$

where

$$H(B) \equiv \tfrac{1}{2}[R - I(RI)](B) = H^\dagger(B)$$

is a dual symmetric bivector operator, and

$$E(B) \equiv \tfrac{1}{2}[R + I(RI)](B) = E^\dagger(B)$$

is an antidual symmetric bivector operator. We shall now study the Lorentz invariants of $R$ in terms of complex scalars of $R$. By *complex scalars* of $R$ we mean all possible rational linear combinations of complex scalar derivatives of $R^k$ and its dual $(RI)^k$, for $k = 1, 2, \cdots$. Thus,

$$\partial\circ R + I\partial\circ R^2 + 3\partial\circ(RI)^2 - 2\partial\circ(RI)^4 \qquad (5.2)$$

is a complex scalar of $R$. Note that (5.2) is also a Lorentz invariant of $R$; we will show that all Lorentz invariants of $R$ can be so expressed.

Squaring both sides of (5.1), considered as an operator equation, leads to

$$R^2(B) = [H^2 + E^2](B) + [HE + EH](B), \qquad (5.3)$$

where

$$H^2(B) = \tfrac{1}{4}[R^2 - IR^2I](B) - \tfrac{1}{4}[(RI)^2 - I(RI)^2I](B)$$

and

$$E^2(B) = \tfrac{1}{4}[R^2 - IR^2I](B) + \tfrac{1}{4}[(RI)^2 - I(RI)^2](B)$$

are dual symmetric operators, and

$$[HE + EH](B) = \tfrac{1}{2}[R^2 + IR^2I](B)$$

is antidual. Since $H^2$ and $E^2$ are symmetric, it follows by (3.11) that

$$\partial\times H^2 = 0 = \partial\times E^2. \qquad (5.4)$$

Because of (3.8), derivatives of $R^2(B)$ can be entirely expressed in terms of $H^2$ and $E^2$, getting

$$\partial R^2 = \partial H^2 + \partial E^2 = \partial\circ H^2 + \partial\circ E^2. \qquad (5.5)$$

Now defining $K \equiv E^2$, note that

$$\partial \circ HK = KH(\partial_B)\circ B = \partial \circ KH, \tag{5.6}$$

$$\partial \circ EHE = \partial_B \circ [E(B)\circ \partial_A EH(A)] = E(\partial_A)\circ EH(A)$$
$$= \partial \cdot E^2 H - \partial \wedge E^2 H = \partial \cdot KH - \partial \wedge KH,$$

and more generally,

$$\partial \circ E^i H E^j$$
$$\text{fs1.05m} = \begin{cases} 0 \quad \text{if } i+j \text{ is odd} \\ (-1)^i \partial \wedge K^{\frac{1}{2}(i+j)} H + \partial \cdot K^{\frac{1}{2}(i+j)} H \\ \quad \text{if } i+j \text{ is even.} \end{cases} \tag{5.7}$$

From the above remarks it follows that the complex scalars of $R$ can be expressed entirely in terms of complex scalars of the form $\partial \circ H^i K^j$. But the characteristic equations of $H$, $K$ and $HK$ are all of the third order; with the help of the Cayley–Hamilton theorem (3.26), and (3.32), it follows that all complex scalars of $R$ can be expressed in terms of rational polynomials in

$$\partial \circ H,\ \partial \circ H^2,\ \partial \circ H^3;\quad \partial \cdot K,\ \partial \cdot K^2,\ \partial \cdot K^3;$$
$$\partial \circ HK,\ \partial \circ (HK)^2,\ \partial \circ (HK)^3 \tag{5.8}$$

and their complex conjugates. Thus, $R$ has a total of $3\times 6 - 3 = 15$ independent invariants, and, as we shall shortly see, the added symmetry of the Riemann curvature tensor reduces this number to the well known14. (If the same analysis of invariants is carried out for a general bivector operator given by (3.34), in addition to the 15 invariants found in (5.8), there are 15 more given by $\partial \circ J^2$; $\partial \cdot L$, $\partial \cdot L^2$, $\partial \cdot L^3$; $\partial \circ (HL)$, $\partial \circ (HL)^2$, $\partial \circ (HL)^3$; $\partial \circ HJ^2$, $\partial \circ H^2 J$,where$L \equiv D^2$,makingupa total of 30 independent scalar invariants.)

For the remainder of this section, let $R(B)$ be a bivector operator with the property

$$\partial_a \wedge R(a\wedge b) = 0. \tag{5.9}$$

An operator with the property (5.9) is called Riemann curvature, because it is equivalent to the usual Riemann curvature tensor $R_{ijkl}$ by way of the identification

$$R_{ijkl} \equiv R(e_i \wedge e_j)\cdot(e_k \wedge e_l), \tag{5.10}$$

the same as is made by Thorpe in Ref. 5. The identities

$$(a\wedge b)\cdot[\partial_d \wedge \partial_c \wedge R(c\wedge d)]$$
$$\equiv [(a\wedge b)\cdot \partial_d]\cdot[\partial_c \wedge R(c\wedge d)]$$
$$- R(a\wedge b) + R^\dagger(a\wedge b)$$

and

$$(a\wedge b\wedge c)\cdot[\partial_v \wedge R(v\wedge d)] \equiv R(c\wedge d)\cdot(a\wedge b) + R(a\wedge d)\cdot(b\wedge c) + R(b\wedge d)\cdot(c\wedge a),$$

together with (5.9), show that

$$R(a\wedge b)\cdot(c\wedge d) = (a\wedge b)\cdot R(c\wedge d) \tag{5.11}$$

and

$$R(a\wedge b)\cdot(c\wedge d) + R(b\wedge c)\cdot(a\wedge d) + R(c\wedge a)\cdot(b\wedge d) = 0. \tag{5.12}$$

Identity (5.11) say that $R(B)$ is a symmetric operator, and (5.12) is the famous Bianchi identity. (The other Bianchi identity in this formalism has the form $\nabla \wedge R = 0$, and can be found in (9); this paper is exclusively concerned with local properties of operators at a point $x$ in curved spacetime.) Thus, (5.9) is equivalent to the two well known conditions (5.11) and (5.12).

Now let $S(B) \equiv R(B)\cdot B$ be the *sectional curvature* defined by $R(B)$. The sectional curvature satisfies the important identity

$$\partial_v \wedge \partial_u S(u\wedge v)|_{u=a, v=b} = 2\partial_v \wedge [v\cdot R(a\wedge v)]|_{v=b}$$
$$= 6R(a\wedge b). \tag{5.13}$$

A well-known consequence of this identity is that $S(u\wedge v) \equiv 0$ iff $R(u\wedge v) \equiv 0$, for all $u, v \in \mathscr{D}_1$.

From the curvature operator $R(B)$ we construct the Ricci operator by contraction:

$$R(b) \equiv \partial_a \cdot R(a\wedge b). \tag{5.14}$$

The Ricci tensor is identified by

$$R_{ij} \equiv R(e_i)\cdot e_j, \tag{5.15}$$

and the property that the Ricci tensor is symmetric is equivalent to

$$\partial_B \times R(B) = \tfrac{1}{2}\partial_a \wedge [\partial_a \cdot R(a\wedge b)] = \tfrac{1}{2}\partial_b \wedge R(b) = 0. \tag{5.16}$$

Scalar curvature is constructed by contracting (5.14), getting

$$R \equiv \partial_b \cdot R(b) = R^i_{\ i} = 2\partial_B \cdot R(B). \tag{5.17}$$

Notice that we use only the domain to distinguish between Riemann, Ricci, and scalar curvature.

We now decompose $R(B)$, as is done in Refs. 15 and 4, by writing

$$R(B) = C(B) + E(B) + G(B), \tag{5.18}$$

where

$$C(B) = R(B) - \tfrac{1}{2}B\cdot\partial_v[R(v) - (1/6)vR],$$
$$E(B) = \tfrac{1}{2}B\cdot\partial_v[R(v) - (1/4)vR],$$

and

$$G(B) = (1/12)BR.$$

The conformal curvature operator $C(B)$ has the properties

$$\partial_a C(a\wedge b) = 0 = \partial_B C(B) \quad \text{and} \quad C(IB) = IC(B). \tag{5.19}$$

The Einstein operator $E(B)$ has the properties

$$E(B) = \tfrac{1}{2}B\cdot\partial_v E(v) = \tfrac{1}{2}[E(a)\wedge b + a\wedge E(b)] \text{ and}$$
$$E(IB) = -IE(B), \tag{5.20}$$

where$E(v) \equiv \partial_a \cdot E(a\wedge b) = R(v) - \tfrac{1}{4}vR$and$\partial_b E(b) = 0$. An important consequence of the fact that $E(B)$ is completely determined by the symmetric vector operator $E(v)$, as given in (5.20), is that

$$\partial_B E^k(B) = \partial_B \cdot E^k(B), \text{ for } k = 1,2,3,\cdots. \tag{5.21}$$

[Recall (3.32)]. The operator $G(B)$ satisfies

$$\partial_a G(a\wedge b) = \tfrac{1}{4}bR,\ \partial_B G(B) = \tfrac{1}{2}R,\ G(IB) = IG(B). \tag{5.22}$$

A comparison of the decompositions (5.1) and (5.18), together with the properties of $C$, $E$, and $G$ given above, shows that

$$H(B) = C(B) + G(B) \tag{5.23}$$

and that $E(B)$ has been correctly identified. Because of properties (5.19), (5.22), and (5.23), $\partial_B H(B) = \partial_B \cdot H(B)$ and therefore the 15 real invariants determined by (5.8) reduce to the well known $15 - 1 = 14$ for Riemann curvature.

If spacetime is not empty but is filled with sourceless electromagnetic fields, the Ricci operator (5.14) satisfies

$$R(v) = -QvQ, \tag{5.24}$$

where $Q$ is the electromagnetic bivector defining $J(B)$ in (4.9). It is easy to check that in this case the scalar curvature $R \equiv \partial_v \cdot R(v) = 0$, as is well known. From this it follows that $E(v) \equiv R(v)$, and from (5.20) we calculate

$$E(B) = B \cdot Q\, Q - B \wedge Q\, Q = (B \cdot Q - B \wedge Q)Q. \tag{5.25}$$

Equation (5.25) shows that the Einstein operator determines $Q$ uniquely up to a phase $e^{I\theta}$. Further discussion of these problems in the language of spinors can be found in (2), (3), and (15). There is a discussion of Maxwell's equation and properties of electromagnetic fields in the STA formalism in Ref. 7.

To demonstrate the geometric transparency of the spacetime algebra (STA) formalism, we give a new geometric argument for the well known numbers of independent parameters (IP) of the Riemann, Conformal Weyl, and Einstein tensors. Let $F(B)$ be a general bivector operator. Then $F(B)$ has $6 \times 6 = 36$ IP, since both the domain and range of $F$ are the six-dim. bivector space $\mathscr{D}_2$. Taking the contraction and curl of $F(B)$ defines the operators

$$f(b) \equiv \partial_a \cdot F(a \wedge b) \tag{5.26}$$

and

$$T(b) \equiv \partial_a \wedge F(a \wedge b). \tag{5.27}$$

The operator $T(b)$ determines $4 \times 4 = 16$ IP of $F(B)$, since the domain and range of $T$ are the four-dim. spaces $\mathscr{D}_1$ and $\mathscr{D}_3$. A similar argument shows that $f(b)$ also determines $4 \times 4 = 16$ IP of $F(B)$; but these degrees of freedom are not completely independent of those determined by $T(b)$, since it is easy to show that

$$\partial_b \cdot T(b) = \partial_b \wedge f(b). \tag{5.28}$$

The relation (5.28) shows that $f(b)$ and $T(b)$ have six parameters in common, i.e., they determine a common bivector. The proof that (5.28) is an integrability condition which guarantees the existence of an operator $F(B)$ satisfying (5.26) and (5.27) will be given elsewhere.

From the above considerations we can read off the numbers of IP for the various operators and their corresponding tensors. Thus, for Riemann curvature, (5.27) vanishes leaving $36 - 16 = 20$ IP. For conformal curvature, both (5.27) and (5.26) vanish, and taking into consideration (5.28), this leaves $36 - 16 - 16 + 6 = 10$ IP. For Einstein curvature, since it is completely determined by (5.26), and (5.28) vanishes, taking into account that $\partial_b \cdot E(b) = 0$, gives $16 - 6 - 1 = 9$ IP.

To bring out the advantages of the STA formalism over the tensor and spinor formalisms, we present the following table of how basic quantities and relationships find expression in each.

| TENSOR | STA | SPINOR | |
|---|---|---|---|
| $F_{uv} = -F_{vu} = (e_u \wedge e_v) \cdot Q$ | $Q$ | $\phi^{\alpha\beta}$ | |
| $F^{*}_{uv} = \frac{1}{2}\epsilon_{uvrs} F^{rs}$ | $IQ$ | $-i\phi^{\alpha\beta}$ | |
| $R_{uvrs} = (e_u \wedge e_v) \cdot R(e_r \wedge e_s)$ | $R(B) = C(B) + E(B) + G(B)$ | $\psi_{\alpha\beta\gamma\delta}, \phi_{\alpha\beta\dot{\mu}\dot{\nu}}, \Lambda$ | (5.29) |
| $\begin{Bmatrix} R_{uvrs} + R_{ruvs} + R_{vrus} = 0 \\ R_{uvrs} = R_{rsuv} \end{Bmatrix}$ | $\partial_a \wedge R(a \wedge b) = 0$ | $\begin{Bmatrix} \psi^{\dot{\alpha}}{}_{\beta}{}^{\dot{\beta}}{}_{\dot{\alpha}} = \psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} \\ \psi^{\alpha\beta\mu\nu} = \psi^{\mu\nu\alpha\beta}, \phi^{\dot{\alpha}\dot{\beta}\mu\nu} = \phi^{\mu\dot{\nu}\alpha\beta} \end{Bmatrix}$ | |
| $\begin{Bmatrix} E_{uvrs} E^{rs}{}_{pq} C^{pqtw} E_{tw}{}^{mn} E_{mn}{}^{uv} \\ + E^{*}_{uvrs} E^{rs}{}_{pq} C^{pqtw} E_{tw}{}^{mn} E_{mn}{}^{uv} \end{Bmatrix}$ | $\partial_B \circ E^4 C(B)$ | $\phi^{rs\dot{\alpha}\dot{\beta}} \phi_{\rho\sigma\dot{\alpha}\dot{\beta}} \psi^{\rho\sigma\kappa\lambda} \phi_{\kappa\lambda\dot{\mu}\dot{\nu}} \phi_{\gamma\delta}{}^{\dot{\mu}\dot{\nu}}$ | |

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