The Symmetric Group and Twisted Symmetric Product in Geometric Algebra

Garret Sobczyk
Universidad de Las Américas - Puebla,
72820 Cholula, Mexico,
email: garret_sobczyk@yahoo.com
URL: http://www.garretstar.com/

October 13, 2009

Abstract

The idea of a twisted symmetric product in geometric algebra arose in trying to understand the structure of the finite symmetric group directly in terms of the representing geometric algebra of multivectors, apart from the corresponding matrix representation. The concept has proved invaluable in constructing regular representations of the symmetric groups in terms of geometric numbers in the graded Clifford algebra of multivectors.
Subject Classification AMS 2010: 15A66, 20B30, 20B35, 20C05, 20C15, 20C30.
Keywords: Clifford algebra, geometric algebra, symmetric group, group representation, group ring.

Introduction

Over the last 112 years since Frobenius and Burnside initiated the study of representation theory, matrix algebra has played a central role with the indispensable idea of a group character [2]. But the concept of a matrix as an array of numbers, with an imposed ring structure of addition and multiplication, hardly provides a geometric way of looking at things. The present approach to representation theory, using geometric algebra, rectifies this defect by providing a new geometric perspective and new algebraic tools to an undeniably rich and powerful area of mathematics, with its many important applications to the physical sciences.

Section 1, introduces the *twisted symmetric product* on any associative algebra over the real or complex numbers. For many years the author has believed that there must be a direct way of representing a group in geometric algebra without having to use their matrix representation. D. Hestenes in [4], and with J. Holt in [5], have beautifully shown how to represent the crystallographic space groups directly as double covering spin groups in geometric algebra. In this work we tackle the left regular representation of the general symmetric group by exploiting the properties of the twisted symmetric product. The section is devoted to deriving basic identities that will be needed in the remainder of the paper. All calculations are done within the enveloping geometric algebra.

Section 2, gives a brief introduction to geometric algebra, emphasizing the point of view that geometric algebra and matrix algebra can be seamlessly knit together into one unified structure [11].

Section 3, develops basic identities of the twisted symmetric product when applied to the particular associative geometric algebra. It also lays the groundwork for the representation and studying of the basic properties of the entangled (adjacent) 2-cycle generators of the symmetric group.

Section 4, develops the recursive construction of regular representations of symmetric groups up to S_{16} in the geometric algebra $cl_{4,4}$. A general entangled 2-cycle in S_{16} is represented by a geometric number $g = a \circ b \circ c \circ d$ where a, b, c, d are commuting, square-one k-blades from the geometric algebra $cl_{4,4}$. All of the properties of such entangled 2-cycles are completely determined by the commutativity and anti-commutativity of their component blades, and the basic properties of the twisted symmetric product.

Section 5, is devoted to proving the validity of the recursive construction developed in Section 4 for the regular representation $\mathcal{R}_{2^n}$ of S_{2^n} in the geometric algebra $cl_{n,n}$.

1 The twisted symmetric product

Let $\mathcal{A}$ be any associative algebra over the real or complex numbers and with the unity 1. For elements $a, b \in \mathcal{A}$, let $ab \in \mathcal{A}$ denotes the product of a and b in

$\mathcal{A}$. For elements $a, b \in \mathcal{A}$, we define a new product

$$a \circ b = \frac{1}{2}(1 + a + b - ab) = 1 - \frac{1}{2}(1-a)(1-b), \tag{1}$$

which we call the *twisted symmetric product* of a and b in $\mathcal{A}$. In the definition of the twisted symmetric product, the order of the elements in the product is always respected. This makes very general identities hold, even if its arguments are not commutative.

For any element $a \in \mathcal{A}$, we define the special symbols

$$a^+ = a \circ 0 = \frac{1}{2}(1+a), \quad a^- = (-a) \circ 0 = \frac{1}{2}(1-a), \tag{2}$$

and note that $a^+ + a^- = 1$ and $a^+ - a^- = a$. Using these symbols, we can re-express (1) in the useful alternative forms

$$a \circ b = a^+ + a^- b = b^+ + ab^-. \tag{3}$$

Note also the curious special cases

$$1 \circ a = 1, \; a \circ (-1) = a, \quad \text{and} \quad 0 \circ 0 = \frac{1}{2}, \tag{4}$$

and if $a^2 = \pm 1$,

$$a \circ a = a, \quad a \circ a = 1 + a, \tag{5}$$

respectively.

It can be readily verified that for any $a_1, a_2, a_3 \in \mathcal{A}$ that

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3) = 1 - 2a_1^- a_2^- a_3^-, \tag{6}$$

so that the twisted symmetric product inherits the important associative property from the associative product in $\mathcal{A}$. More generally, for n elements $a_1, \ldots, a_n \in \mathcal{A}$, we have

$$a_1 \circ \cdots \circ a_n = 1 - 2a_1^- \cdots a_n^- \iff 2a_1^- \cdots a_n^- = 1 - a_1 \circ \cdots \circ a_n. \tag{7}$$

It should also be noticed that whenever any argument is replaced by its negative on the left side of the identity, then the corresponding sign of the term must be changed on the right side. For example,

$$(-a_1) \circ a_2 \circ \cdots \circ a_n = 1 - 2a_1^+ a_2^- \cdots a_n^-.$$

For three elements $a, b, c \in \mathcal{A}$, using the definition (1), we derive The distributive-like property

$$a \circ (b + c) = 1 - \frac{1}{2}(1-a)[(1-b) + (1-c) - 1] = a \circ b + a \circ c - a^+. \tag{8}$$

Letting $c = -b$ in the above identity, and using (2), we find the important identity that

$$2a^+ = a \circ b + a \circ (-b). \tag{9}$$

The *commutator* of the twisted symmetric product is also very useful. We find, using (7) that

$$a \circ b - b \circ a = -2(a^- b^- - b^- a^-) = -\frac{1}{2}(ab - ba), \tag{10}$$

as can be easily verified. Thus, the commutator of $a \circ b$ can be expressed entirely in terms of The product of b^- and a^-. There is one other important general property that needs to be mentioned. If $a, b, g, g^{-1} \in \mathcal{A}$, where $gg^{-1} = 1$, then it immediately follows from the definition (1) that

$$g(a \circ b)g^{-1} = (gag^{-1}) \circ (gbg^{-1}). \tag{11}$$

1.1 Special properties

Under certain conditions, the twisted symmetric product is also distributive over ordinary multiplication. Suppose for $k > 1$ that $a, b_1, \ldots, b_k \in \mathcal{A}$, $a^2 = 1$ and that $ab_i = b_i a$ for $i = 1, \ldots, k-1$. Then

$$a \circ (b_1 \cdots b_k) = \prod_{i=1}^{k} (a \circ b_i) = a^+ + a^- b_1 \cdots b_k. \tag{12}$$

The identity (12) is easily established. Using (3), we have

$$a \circ (b_1 \cdots b_k) = a^+ + a^- b_1 \cdots b_k$$

for the the left side of (12). For the right side, we use the fact that a^+ and a^- are mutually annihiliating idempotents, so that

$$\prod_{i=1}^{k} (a \circ b_i) = \prod_{i=1}^{k} (a^+ + a^- b_i) = a^+ + a^- b_1 \cdots b_k.$$

In the special case that $\alpha \in I\!R$ and $a^2 = 1$, we have

$$a \circ (\alpha b) = (a \circ \alpha)(a \circ b) = (a^+ + \alpha a^-) a \circ b. \tag{13}$$

For $\alpha = -1$, (13) reduces to $a \circ (-b) = a(a \circ b)$, which further reduces to

$$a \circ (-a) = a^2 = 1 \tag{14}$$

when $b = a$. Curiously when $a^2 = -1$, $a \circ (-a) = 0$, a divisor of zero.

Let us suppose now that for $k \in I\!N$ we have k mutually commuting elements $a_1, \ldots, a_k$ and that $a_1^2 = \cdots = a_k^2 = 1$. Then it is easy to show that

$$(a_1 \circ \cdots \circ a_k)^2 = 1, \tag{15}$$

as follows by induction on k. For $k = 1$, there is nothing to show. Suppose now that the hypothesis is true for some $k \in I\!N$ so that $(a_1 \circ \cdots \circ a_k)^2 = 1$. Then for $k + 1$, using (3), we have

$$[(a_1 \circ \cdots \circ a_k) \circ a_{k+1}]^2 = [a_{k+1}^+ + (a_1 \circ \cdots \circ a_k) a_{k+1}^-]^2$$

$$= (a_{k+1}^+)^2 + (a_1 \circ \cdots \circ a_k)^2 (a_{k-1}^-)^2 = a_{k-1}^+ + a_{k-1}^- = 1.$$

One other striking relationship that the twisted symmetric product satisfies when $a^2 = 1$ is that

$$a^+((ab) \circ c) = a^+(b \circ c), \quad \text{and} \quad a^-((ab) \circ c) = a^-((-b) \circ c), \tag{16}$$

as follows directly from the steps

$$a^+((ab) \circ c) = a^+ \frac{1}{2}(1 + ab + c - abc) = \frac{1}{2}(a^+ + a^+ b + a^+ c - a^+ bc)$$

$$= a^+ \frac{1}{2}(1 + b + c - bc) = a^+(b \circ c).$$

The other half of (16) follows similarly. Thus, the idempotents a^+ and a^- *absorb* the a in $(ab) \circ c$, leaving behind either 1 or -1 to give the stated results. In the special case that $b = 1$, (16) reduces to the even simpler relationship

$$a^+(a \circ c) = a^+, \quad \text{and} \quad a^-(a \circ c) = a^- c. \tag{17}$$

1.2 Basic relationships

We list here basic relationships, which follow from (13) and (14), that hold between two commutative elements a_1, a_2 with the property that $a_1^2 = a_2^2 = 1$.

1. $(a_1 \circ a_2)^2 = 1 \iff [((-a_1) \circ a_2]^2 = [a_1 \circ (-a_2)]^2 = [((-a_1) \circ (-a_2)]^2 = 1,$
2. $a_1(a_1 \circ a_2) = a_1 \circ (-a_2) \iff (a_1 \circ a_2)[a_1 \circ (-a_2)] = a_1,$
3. $a_2(a_1 \circ a_2) = (-a_1) \circ a_2 \iff (a_1 \circ a_2)[(-a_1) \circ a_2] = a_2,$
4. $a_1 a_2 (a_1 \circ a_2) = -(-a_1) \circ (-a_2) \iff (a_1 \circ a_2)[(-a_1) \circ (-a_2)] = -a_1 a_2.$

Because of the associativity of the twisted symmetric product, these relationships can be easily extended to the case of more commutative arguments which have square one. For example, the relationship 2 above for three arguments becomes

$$a_1(a_1 \circ a_2 \circ a_3) = a_1 \circ [-(a_2 \circ a_3)] \iff a_1 = (a_1 \circ a_2 \circ a_3)[a_1 \circ (-(a_2 \circ a_3))].$$

Using the absorbtion property (17), we also have

$$a_1(a_1 \circ a_2 \circ a_3) = (a_1^+ - a_1^-)(a_1 \circ a_2 \circ a_3) = a_1^+ - a_1^-(a_2 \circ a_3).$$

2 Geometric numbers in $\mathbb{G} = cl_{n,n}$

Let $\mathbb{G} = cl_{n,n}$ be a geometric algebra of the *neutral signature* (n, n). Geometric algebras have been discussed in the literature, eg., [3], [8] and [10]. We will use the notation and methods developed in [3]. We have carried out extensive calculations using the invaluable software package [9], and we therefore use the conventions in that program for the definition of basis elements. The geometric algebras $cl_{n,n}$ are particularly useful in relationship to square matrix algebras of order $2^n \times 2^n$ since they are algebraically isomorphic, [12]. We briefly review the main features below.

Let $n \geq 1$. The associative geometric algebra $\mathbb{G} = cl_{n,n}$ is a *graded algebra*

$$\mathbb{G} = \mathbb{G}^0 \oplus \mathbb{G}^1 \ldots \oplus \mathbb{G}^{2n},$$

of *scalars*, *vectors*, *bivectors*, etc., taken together with a *definite quadratic form* $\varphi(x, y)$. The k-vector part of a multivector $g \in \mathbb{G}$ is denoted by $< g >_k \in \mathbb{G}^k$ for $0 \leq k \leq 2n$.

The *standard orthonormal basis* $\mathcal{E}_{n,n}$ of $\mathbb{G} = cl_{n,n}$ is generate by taking the geometric products of the $2n$ basis vectors,

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}, \tag{18}$$

subject to the rules that $e_i^2 = \varphi(e_i, e_i) = 1$ for $1 \leq i \leq n$ and $e_i^2 = \varphi(e_i, e_i) = -1$ for $n < i \leq 2n$. The $2^{2n} - 1$ products of *distinct* basis vectors are *anti-commutative* and are denoted by

$$e_{\lambda_1, \ldots, \lambda_k} = e_{\lambda_1} \cdots e_{\lambda_k},$$

where $1 \leq \lambda_1 < \cdots < \lambda_k \leq 2n$. The element e_α for $\alpha = \lambda_1, \ldots, \lambda_k$ is called a *simple k-vector*. The special symbol $e_0 = 1$ is used to denote the identity element. Using this notation and the defining the index set $\mathcal{I}_{n,n}$ for the geometric algebra $cl_{n,n}$, the standard orthonormal basis of $cl_{n,n}$ can be compactly represented by

$$\mathcal{E}_{n,n} = (\, e_\alpha \,)_{\alpha \in \mathcal{I}_{n,n}} = \{e_0, e_1, \ldots, e_n, \ldots, e_{1,\ldots,2n}\}. \tag{19}$$

Given two vectors $a, b \in \mathbb{G}^1$, the *geometric product* ab can be decomposed into symmetric and antisymmetric parts,

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a \cdot b + a \wedge b.$$

The symmetric part $a \cdot b = \varphi(a, b)$ is called the *inner product* and the antisymmetric part $a \wedge b$ is called the *outer* or *exterior product* of the vectors a and b. This same kind of decompostion exists for the geometic product of a vector a with a k-vector B_k for $1 \leq k \leq 2n$; we have

$$aB_k = \frac{1}{2}(aB_k + (-1)^{k+1} B_k a) + \frac{1}{2}(aB_k + (-1)^k B_k a) = a \cdot B_k + a \wedge B_k,$$

where $a \cdot B_k$ is a $(k-1)$-vector and $a \wedge B_k$ is a $(k+1)$-vector.

Given a geometric number $g \in cl_{n,n}$, the *reverse* $g^\dagger$ of g is defined by reversing the order of all its vector products. Thus, for example, for $\alpha = 1, 2, 3$

$$e_\alpha^\dagger = (e_1 e_2 e_3)^\dagger = e_3 e_2 e_1 = e_{3,2,1} = -e_\alpha.$$

A good introduction to geometric algebras can be found in [1].

We can easily construct a $2^n \times 2^n$ real matrix representation of the geometric algebra $cl_{n,n}$. We first define the $2n$ idempotents $u_i^{\pm} = \frac{1}{2}(1 \pm e_{i,n+i})$ for $1 \le i \le n$, which we use to construct the 2^n *primitive idempotents*

$$u_{1\ 2\ \ldots n}^{\pm\pm\ldots\pm} = u_1^{\pm} u_2^{\pm} \cdots u_n^{\pm}.$$

Letting $\mathcal{U}_n^+ = u_1^+ \cdots u_n^+$, all of the other primitive idempotents can be defined succinctly by

$$\mathcal{U}^{sn(\alpha)} = e_\alpha^\dagger \mathcal{U}_n^+ e_\alpha$$

for $\alpha \in \mathcal{I}_n$ the index set for the geometric algebra cl_n.

By the *spectral basis* of the geometric algebra $cl_{n,n}$, we mean

$$\mathcal{E}_n^\dagger \mathcal{U}_n^+ \mathcal{E}_n, \tag{20}$$

where we are writing $\mathcal{E}_n^\dagger$ as a column matrix of the reverses of the basis elements of cl_n, and where the conventions of matrix row-column multiplication is respected. Given a geometric number $g \in cl_{n,n}$, the relationship between g and its matrix representation $[g]$ with respect to the spectral basis (20) is given by

$$g = \mathcal{E}_n \mathcal{U}_n^+ [g]^T \mathcal{E}_n^\dagger \iff g = [\mathcal{E}_n^\dagger]^T \mathcal{U}_n^+ [g][\mathcal{E}_n]^T, \tag{21}$$

where $[g]^T$ denotes the transpose of the matrix $[g]$. If we denote the entries in the matrix $[g]$ by $g_{\alpha\beta}$ where $\alpha, \beta \in \mathcal{I}_n$, then we also have the relationship

$$g_{\beta\alpha} \mathcal{U}_n^+ = \mathcal{U}_n^+ e_\alpha^{-1} g e_\beta \mathcal{U}_n^+. \tag{22}$$

More details of this construction can be found in [11, 12].

For example, for $n = 2$ we get the spectral basis

$$\begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_{21} \end{pmatrix} \mathcal{U}_2^+ \begin{pmatrix} 1 & e_1 & e_2 & e_{12} \end{pmatrix} = \begin{pmatrix} u_{++} & e_1 u_{-+} & e_2 u_{+-} & e_{12} u_{--} \\ e_1 u_{++} & u_{-+} & e_{12} u_{+-} & e_2 u_{--} \\ e_2 u_{++} & e_{21} u_{-+} & u_{+-} & -e_1 u_{--} \\ e_{21} u_{++} & e_2 u_{-+} & -e_1 u_{+-} & u_{--} \end{pmatrix}. \tag{23}$$

for the geometric algebra $\mathbb{G} = cl_{2,2}$. The relationship (21) for an element $g \in cl_{2,2}$ and its matrix $[g]$ is

$$g = \mathcal{E}_2 \mathcal{U}_2^+ [g]^T \mathcal{E}_2^\dagger = \begin{pmatrix} 1 & e_1 & e_2 & e_{12} \end{pmatrix} u_1^+ u_2^+ [g]^T \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_{21} \end{pmatrix}. \tag{24}$$

3 The twisted symmetric product of geometric numbers

Let $a, b \in \mathbb{G}^1$ be vectors. From the definition (1),

$$a \circ b = \frac{1}{2}(1 + a + b - ab), \text{ and } b \circ a = \frac{1}{2}(1 + a + b - ba).$$

Using (10) and the definition of the exterior product of vectors, we find that

$$a \circ b - b \circ a = \frac{1}{2}(ba - ab) = 2(b^- a^- - a^- b^-) = b \wedge a. \tag{25}$$

A similar calculation shows that for vectors $a, b, c \in \mathbb{G}^1$,

$$a \circ (b \wedge c) - (b \wedge c) \circ a = \frac{1}{2}[(b \wedge c)a - a(b \wedge c)] = (b \wedge c) \cdot a. \tag{26}$$

Note that if the vector a is orthogonal to the plane of $b \wedge c$, then $a \cdot (b \wedge c) = 0$ so that $a \circ (b \wedge c) = (b \wedge c) \circ a$ in this case.

It is interesting to consider the meaning of the *symmetric* part of the twisted symmetric product. For vectors $a, b \in \mathbb{G}^1$, we find that

$$a \circ b + b \circ a = 1 + a + b - a \cdot b \equiv 2a \dot{\circ} b, \tag{27}$$

and for the vector a and bivector $b \wedge c$, we find that

$$a \circ (b \wedge c) + (b \wedge c) \circ a = 1 + a + b \wedge c - a \wedge (b \wedge c) \equiv 2a \hat{\circ} b. \tag{28}$$

If $a, b, c \in \mathbb{G}$ are geometric numbers with the property that $ac = -ca$ and $cb = bc$. Then

$$c[a \circ b] = [(-a) \circ b]c. \tag{29}$$

If in addition $cb = -bc$, then we have

$$c[a \circ b] = [(-a) \circ (-b)]c. \tag{30}$$

These relationships are easily established directly from the definition (3) and can easily be extended to the twisted symmetric product of more arguements.

Let a and b be any geometric numbers with the property that $ab = -ba$. Then from the alternative definition of the twisted symmetric product (3), we find that $ab^+ = b^- a$ and

$$a^+ b^+ = \frac{1}{2}(1 + a)b^+ = \frac{1}{2}(b^+ + b^- a) = \frac{1}{2} b \circ a. \tag{31}$$

Compare this with the general relationship (7) for *arbitrary* elements. Under the same condition that $ab = -ab$, we also have the similar relationships

$$a^- b^+ = \frac{1}{2} b \circ (-a), \quad a^+ b^- = \frac{1}{2}(-b) \circ a, \quad \text{and} \quad a^- b^- = \frac{1}{2}(-b) \circ (-a). \tag{32}$$

We also have

$$a \circ b = 1 - (-b) \circ (-a) \qquad \Longleftrightarrow \qquad a^+ b^+ = \frac{1}{2} - b^- a^- \tag{33}$$

which follows from (31), (32) and (7), and the simple steps

$$a \circ b = \frac{1}{2} b^+ a^+ = 1 - (1 - 2b^+ a^+) = 1 - (-b) \circ (-a).$$

Identities (32) and (33) are striking because they allow us to reverse the order of the twisted symmetric product by changing the signs of its respective arguments.

For three geometric numbers a, b, c with the properties that $a^2 = 1$ and $ab = -ba$, we find that

$$(a \circ b)(a \circ c) = (a^+ + a^- b)(a^+ + a^- c) = a^+ + a^- b = a \circ b. \tag{34}$$

If, instead, we have the condition that $a^2 = 1$ and $ac = ca$, then

$$(a \circ c)(a \circ b) = (a^+ + a^- c)(a^+ + a^- b) = a^+ + a^- cb = a \circ (cb). \tag{35}$$

We see in the above relationships that the properties of the twisted symmetric product $a \circ b$ heavily depends upon the commutativity of a and b. Suppose now that we have two pairs of of commutative arguments, a_1, a_2 and b_1, b_2. Whereas by assumption $a_1 a_2 = a_2 a_1$ and $b_1 b_2 = b_2 b_1$, we need some way of conveniently expressing the commutativity or anticommutativity relationships beween them. To this end we define the *entanglement table* $en(a_1, a_2; b_1, b_2)$,

Definition 1 *If* $a_1 b_1 = -b_1 a_1, a_1 b_2 = b_2 a_1, a_2 b_1 = -b_1 a_2$ *and* $a_2 b_2 = -b_2 a_2$, *we say that the entanglement table* $en(a_1, a_2; b_1, b_2)$ *is*

	b_1	b_2
a_1	$-$	$+$
a_2	$-$	$-$

and similarly for other possible commutativity relationships between a_1, a_2 *and* b_1, b_2.

We are interested doing calculations for commutative pairs of arguments a_1, a_2 and b_1, b_2 which satisfy the following four entanglement tables:

		b_1	b_2
1)	a_1	$-$	$+$
	a_2	$-$	$-$

		b_1	b_2
2)	a_1	$+$	$-$
	a_2	$-$	$+$

		b_1	b_2
3)	a_1	$+$	$-$
	a_2	$+$	$+$

		b_1	b_2
4)	a_1	$-$	$-$
	a_2	$-$	$-$

Thus, suppose that we are given two pairs of commutative elements $a_1, a_2 \in \mathbb{G}$ and $b_1, b_2 \in \mathbb{G}$ that satisfy the entanglement Table 1). In addition, we assume that $a_i^2 = b_i^2 = 1$ for $i = 1, 2$. Then, using (11) and the basic relationships found in **Section 1.2**, we calculate

$$(a_1 \circ a_2)(b_1 \circ b_2)(a_1 \circ a_2) = [(a_1 \circ a_2)b_1(a_1 \circ a_2)] \circ [(a_1 \circ a_2)b_2(a_1 \circ a_2)]$$
$$= \big[(a_1 \circ a_2)[(-a_1) \circ (-a_2)]b_1\big] \circ \big[(a_1 \circ a_2)[a_1 \circ (-a_2)]b_2\big]$$
$$= (-a_1a_2b_1) \circ (a_1b_2) = (a_1b_2) \circ (-a_2b_1b_2).$$

A similar calculation shows that

$$(b_1 \circ b_2)(a_1 \circ a_2)(b_1 \circ b_2) = (a_1b_2) \circ (-a_2b_1b_2).$$

It follows from the above calculations that for Table 1 entanglement and square one arguments,

$$(a_1 \circ a_2)(b_1 \circ b_2)(a_1 \circ a_2) = (b_1 \circ b_2)(a_1 \circ a_2)(b_1 \circ b_2). \quad (36)$$

This is the basic relationship between adjacent 2-cyles in the symmetric group. Indeed the relationship (36) shows that the element $(a_1 \circ a_2)(b_1 \circ b_2)$ has order 3.

For Table 2 entanglement and square one arguments, we find that

$$(a_1 \circ a_2)(b_1 \circ b_2)(a_1 \circ a_2) = (a_1b_1) \circ (a_2b_2) = (b_1 \circ b_2)(a_1 \circ a_2)(b_1 \circ b_2), \quad (37)$$

so the element $(a_1 \circ a_2)(b_1 \circ b_2)$ with entanglement Table 2 will also have order 3. Calculations for Table 3 and Table 4 entanglement with square one arguments proceed in exactly the same way and give for Table 3

$$(a_1 \circ a_2)(b_1 \circ b_2)(a_1 \circ a_2) = b_1 \circ (a_2b_2) = (a_2 \circ b_1)(b_1 \circ b_2), \quad (38)$$

and for Table 4

$$(a_1 \circ a_2)(b_1 \circ b_2)(a_1 \circ a_2) = [(-a_1) \circ (-a_2)](b_1 \circ b_2). \quad (39)$$

From (38), we see directly that

$$[(a_1 \circ a_2)(b_1 \circ b_2)]^2 = a_2 \circ b_1. \quad (40)$$

For this case if $b_1 = -a_2$ then $(a_1 \circ a_2)(b_1 \circ b_2)$ will have order 2, otherwise it will have order 4. In the case of Table 4 entanglement, we have

$$[(a_1 \circ a_2)(b_1 \circ b_2)]^2 = (-a_1) \circ (-a_2), \quad (41)$$

so $(a_1 \circ a_2)(b_1 \circ b_2)$ will have order 4.

There is one very important property about higher degree twisted symmetric products that we will need. Suppose that $a_1 \circ a_2$ and $b_1 \circ b_2$ are pairs of commutative square one arguments, and let c be an element with square one and such that $ca_i = a_ic$ and $cb_i = b_ic$ for $i = 1, 2$. Then the order of the element

$(a_1 \circ a_2 \circ c)(b_1 \circ b_2 \circ c)$ will be the same as the element $(a_1 \circ a_2)(b_1 \circ b_2)$; in symbols,

$$|(a_1 \circ a_2 \circ c)(b_1 \circ b_2 \circ c)| = |(a_1 \circ a_2)(b_1 \circ b_2)|, \tag{42}$$

as easily follows from the steps

$$(a_1 \circ a_2 \circ c)(b_1 \circ b_2 \circ c) = [c^+ + (a_1 \circ a_2)c^-][c^+ + c^-(b_1 \circ b_2)] = c^+ + c^-(a_1 \circ a_2)(b_1 \circ b_2),$$

and the fact that c^+ and c^- are mutually annihiliating idempotents.

Although we are far from exhausting the many interesting identities that can be worked out, we now have a sufficient arsenal to continue our study of the symmetric groups in the geometric algebra $cl_{n,n}$.

4 Symmetric groups in geometric algebras

Our objective is to construct geometric numbers which represent the elements of the various subgroups $\mathcal{S}$ of the symmetric group S_n as a *group algebra* $\mathcal{R}$ in the geometric algebra $\mathbb{G} = cl_{n,n}$, [3, 8]. The construction naturally leads to the interpretation of $\mathcal{R}$ as a *real regular* $\mathcal{R}$*-module*, [7, p.55-56]. To accomplish this we construct elements which represent the successive entangled 2-cycles

$$(12), (13), \ldots, (1n),$$

which are the generators of the symmetric group S_n. Although there are many such representations, we consider here only elements in $cl_{n,n}$ of the form

$$g = a_1 \circ a_2 \circ \cdots \circ a_n, \tag{43}$$

where $a_1, \ldots, a_n \in cl_{n,n}$ are *mutually commuting blades* such that $a_i^2 = 1$ for $1 \le i \le n$. Since by (15), $g^2 = 1$, the element g has order 2 and is a good candidate for representing a 2-cycle, where $1 \in \mathbb{G}$ represents the group identity element. Once we have a way of representing all 2-cycles in S_n as elements of $cl_{n,n}$, we can also construct the elements of any its subgroups.

Let $\mathcal{R} = \{g_1, \ldots, g_k\}$ be a representation of a finite group all of whose elements are of the form (43), or a product of such elements in $cl_{n,n}$. Let $g \in \mathcal{S}$. By the *group character* $\chi(g)$ (with respect to its embedding in $cl_{n,n}$), we mean

$$\chi(g) = 2^n < g >_0 . \tag{44}$$

We have defined the group character function χ on $cl_{n,n}$ to agree with its matrix equivalent, since $cl_{n,n}$ is real isomorphic to the real matrix algebra $\mathcal{M}_{\mathbb{R}}(2^n \times 2^n)$. In so doing, we can directly incorporate all of the results from the rich theory of group representations that has evolved over the years since April, 1896, when F. G. Frobenius wrote to R. Dedekind about his new ideas regarding finite groups [6]. Indeed, we fully expect that a new richer representation theory can be constructed based upon geometric algebra.

4.1 The symmetric group S_4 in $cl_{4,4}$

We represent the symmetric group S_1 by $\mathcal{R}_1 = \{1\} = cl_{0,0} \subset cl_{4,4}$, where $cl_{0,0} = I\!R$ is the degenerate geometric algebra of real numbers. The symmetric group S_2 is represented by

$$\mathcal{R}_2 = gen\{e_1\}, \tag{45}$$

where $e_1 \in cl_{1,1} \subset cl_{4,4}$. Using (20), we find that

$$1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

Let us now carefully construct the representation $\mathcal{R}_4$ of S_4 in $cl_{2,2}$ from (45) given above, since it is the $k = 1$ step in our recursive construction of the representation $\mathcal{R}_{2^{k+1}}$ of $S_{2^{k+1}}$ in $cl_{k+1,k+1}$, given the representation $\mathcal{R}_{2^k}$ of S_{2^k} in $cl_{k,k}$. We use the conventions for the basis elements of $cl_{4,4}$ that we previously established in (18). All calculations can be checked using Lounesto's Clical [9].

The construction of the representation of S_{2^2} from the representation of S_2 breaks down into 3 steps.

Step 1. We map the generator(s) e_1 of S_2 into the larger geometric algebra $cl_{2,2}$ by taking the twisted symmetric product of e_1 with the element e_{62} to produce $e_1 \circ e_{62}$, which will be the new element representing the 2-cycle (12) in S_4.

Step 2. The element representing the 2-cycle (13) in S_4 is created from $e_1 \circ e_{62}$ by introducing the orthogonal transformation represented by π_2 which *permutes* the basis vectors e_1, e_2, e_5, e_6 to give

$$e_1 \to e_2 \to e_1, \quad e_5 \to e_6 \to e_5. \tag{47}$$

Thus,

$$e_2 \circ e_{51} = \pi_2(e_1 \circ e_{62}) = [\pi_2(e_1)] \circ [\pi_2(e_{61})].$$

We can construct one more entangled 2-cycle in the geometric algebra $cl_{2,2}$ to get a regular representation $\mathcal{R}_4$ of S_4, which constitutes the last Step 3 of the construction.

Step 3. We multiply the vector e_1 in (45) on the left by e_6 to get the bivector e_{61} which *anticommutes* with both e_1 and e_{62} used in Step 1 of the construction. Using this bivector, we then construct

$$e_{25} = -\pi_2(e_{61}) = -\pi_2(e_6)\pi_2(e_1) = -e_5 e_2.$$

The element that represents (14) in $cl_{2,2}$ is given by

$$e_{61} \circ [-\pi_2(e_{61})] = e_{61} \circ e_{25}. \tag{48}$$

We now have a complete set of generators for our representation $\mathcal{R}_4$ of S_4 in $cl_{2,2}$.

$$\begin{pmatrix} e_1 \circ e_{62} & \simeq & (12) \\ e_2 \circ e_{51} & \simeq & (13) \\ e_{61} \circ e_{25} & \simeq & (14) \end{pmatrix} \tag{49}$$

Indeed, the matrix representations $[e_1 \circ e_{62}], [e_2 \circ e_{51}], [e_{61} \circ e_{25}]$ of these elements given by (23) are exactly the matrices obtained by interchanging the first and second rows, the first and third rows, and the first and fourth rows of the 4×4 identity matrix, respectively. By appealing to the matrix - geometric algebra isomorphism, we have found the corresponding regular representation of S_4 in the geometric algebra $cl_{2,2}$. The reason for the minus sign that appears in step 3 will become evident later.

Let us consider in detail the structure of the representation $\mathcal{R}_4$ of S_4 that we have found, since, in turn, it will determine the structure of the faithful representations of all higher order symmetric groups that we consider. In order to facilitate and clarify our discussion, we write

$$r_1 = e_1 \circ e_{62} = a_1 \circ a_2, \; r_2 = e_2 \circ e_{51} = b_1 \circ b_2, \; r_3 = e_{61} \circ e_{25} = c_1 \circ c_2.$$

The elements $a_1, a_2, b_1, b_2, c_1, c_2 \in cl_{4,4}$ are not all independent. It can be easily checked that $c_1 = -a_1 a_2 b_1$ and $c_2 = a_1 b_1 b_2$. For convenience, we produce here an entanglement table for the elements defining r_1, r_2, r_3:

		a_1	a_2	b_1	b_2	c_1	c_2
	a_1	+	+	−	−	−	+
5)	a_2	+	+	−	+	−	−
	b_1	−	−	+	+	+	−
	b_2	−	+	+	+	−	−
	c_1	−	−	+	−	+	+
	c_2	+	−	−	−	+	+

We give here a complete table of the geometric numbers representing the 24 elements of S_4. As generators we choose the elements which represent (12), (13), and (14), respectively.

$$\begin{pmatrix} S_4 & \simeq & g \in cl_{2,2} \subset cl_{4,4} & \chi(g) \\ (1)(2)(3) & \simeq & 1 = 1 \circ 1 & 4 \\ (12) & \simeq & r_1 = a_1 \circ a_2 & 2 \\ (13) & \simeq & r_2 = b_1 \circ b_2 & 2 \\ (14) & \simeq & r_3 = c_1 \circ c_2 = (-a_2 b_2) \circ (-a_1 b_1 b_2) & 2 \\ (23) & \simeq & r_1 r_2 r_1 = c_1 \circ (-c_2) = (a_2 b_2) \circ (-a_1 a_2 b_1) & 2 \\ (24) & \simeq & r_1 r_3 r_1 = b_1 \circ (-b_2) & 2 \\ (34) & \simeq & r_2 r_3 r_2 = (-a_1) \circ (-a_2) & 2 \\ (12)(34) & \simeq & r_1 r_2 r_3 r_2 = -a_1 a_2 & 0 \\ (13)(24) & \simeq & r_2 r_1 r_3 r_1 = b_1 & 0 \\ (14)(23) & \simeq & r_3 r_1 r_2 r_1 = c_1 = -a_1 a_2 b_1 & 0 \\ (123) & \simeq & r_2 r_1 = (b_1 \circ b_2)(a_1 \circ a_2) & 1 \\ (132) & \simeq & r_1 r_2 = (a_1 \circ a_2)(b_1 \circ b_2) & 1 \\ (124) & \simeq & r_3 r_1 = (c_1 \circ c_2)(a_1 \circ a_2) & 1 \\ (142) & \simeq & r_1 r_3 = (a_1 \circ a_2)(c_1 \circ c_2) & 1 \\ (134) & \simeq & r_3 r_2 = (c_1 \circ c_2)(b_1 \circ b_2) & 1 \\ (143) & \simeq & r_2 r_3 = (b_1 \circ b_2)(c_1 \circ c_2) & 1 \\ (234) & \simeq & r_1 r_3 r_2 r_1 = -a_1 a_2 r_3 r_1 & 1 \\ (243) & \simeq & r_1 r_2 r_3 r_1 = -a_1 a_2 r_2 r_1 & 1 \\ (1234) & \simeq & r_3 r_2 r_1 = c_1 r_2 = -a_1 a_2 b_1 r_2 & 0 \\ (1243) & \simeq & r_2 r_3 r_1 = b_1 r_3 & 0 \\ (1324) & \simeq & r_3 r_1 r_2 = c_1 r_1 = -a_1 a_2 b_1 r_1 & 0 \\ (1342) & \simeq & r_1 r_3 r_2 = -a_1 a_2 r_3 & 0 \\ (1423) & \simeq & r_2 r_1 r_3 = b_1 r_1 & 0 \\ (1432) & \simeq & r_1 r_2 r_3 = -a_1 a_2 r_2 & 0 \end{pmatrix} . \tag{50}$$

Let us summarize in a table what we have accomplished so far.

$$(\,1\,) \qquad \begin{pmatrix} 1 & 62 \\ 2 & 51 \\ 61 & 25 \end{pmatrix} \tag{51}$$

In the table (51) we have abreviated the elements e_i and e_{ij} by using only their indices which serve to completely define the elements in the product. We have thus found S_2 in $cl_{1,1}$, and S_4 in $cl_{2,2}$, since only the basis elements e_1, e_2, e_5, e_6 have been used in the construction. In fact, we see that the matrices of our elements correspond exactly to the regular 4×4 matrix representation.

4.2 The geometric algebra $\mathbb{G} = cl_{4,4}$

In order to see more clearly see what is involved in this recursive construction, let us continue the construction for $k = 2$ and $k = 3$, and find the representations $\mathcal{R}_8$ of S_8 and $\mathcal{R}_{16}$ of S_{16} in $cl_{4,4}$. As a subalgebra of $cl_{4,4}$, the geometric algebra $cl_{3,3}$ is generated by the basis vectors $e_1, e_2, e_3, e_5, e_6, e_7$.

We begin by applying **Step 1** to the table (51) with the element e_{73} to produce the new table

$$\begin{pmatrix} 1 & 62 & 73 \\ 2 & 51 & 73 \\ 61 & 25 & 73 \end{pmatrix}. \tag{52}$$

For **Step 2**, we define the orthogonal transformation $\pi_3 = \pi_{k+1}$ on the basis elements of $cl_{3,3}$ $\{e_1, e_2, e_3, e_5, e_6, e_7\}$ by

$$e_1 \to e_2 \to e_3 \to e_1, \quad e_5 \to e_6 \to e_7 \to e_5. \tag{53}$$

Applying this transformation successively twice to (52), gives us the additional two copies

$$\begin{pmatrix} 2 & 73 & 51 \\ 3 & 62 & 51 \\ 72 & 36 & 51 \end{pmatrix} \simeq \begin{pmatrix} 3 & 51 & 62 \\ 1 & 73 & 62 \\ 53 & 17 & 62 \end{pmatrix}, \tag{54}$$

which represent different but equivalent representations of S_4 in $cl_{3,3}$ considered as a subalgebra of $cl_{4,4}$. Next, we paste together the distinct elements found in these 3 tables to get

$$\begin{pmatrix} 1 & 62 & 73 \\ 2 & 51 & 73 \\ 61 & 25 & 73 \\ 3 & 62 & 51 \\ 72 & 36 & 51 \\ 35 & 71 & 62 \end{pmatrix}. \tag{55}$$

Notice that we have changed the sign of the first two elements of the row $(\,53 \quad 17 \quad 62\,)$. The reason for this will be explained later. This completes Step 2.

In (55), we have constructed 6 of the 7 elements that represent the 2-cycles of S_8 in $cl_{3,3}$. We use **Step 3** to construct the last element by taking the first two elements $(\,61 \quad 25\,)$ from the last row of (51), and place a 3 in front of them, to get $(\,361 \quad 325\,)$. Step 3 is completed by applying π_3 to e_{361} to get $\pi_3(e_{361}) = e_{172}$. Notice in this case we do not change the sign of this element as we did in (48). In general, the sign for π_{k+1} in this step is $(-1)^k$. The element representing the 7^{th} 2-cycle (18) is thus found to be

$$e_{361} \circ e_{325} \circ e_{172} \simeq (18). \tag{56}$$

Thus, the table

$$\begin{pmatrix} 1 & 62 & 73 \\ 2 & 51 & 73 \\ 61 & 25 & 73 \\ 3 & 62 & 51 \\ 72 & 36 & 51 \\ 35 & 71 & 62 \\ 361 & 325 & 172 \end{pmatrix} \tag{57}$$

gives the regular representation $\mathcal{R}_8$ of S_8 in $cl_{3,3} \subset cl_{4,4}$.

We continue the construction for $k = 3$ of the representation $\mathcal{R}_{16}$ of S_{16} in $cl_{4,4}$.

Step 1. We extend the table (57) by adding a 4^{th} column consisting of the element (84), which represents e_{84} which has square 1 and commutes with all of the other elements in the table, getting

$$\begin{pmatrix} 1 & 62 & 73 & 84 \\ 2 & 51 & 73 & 84 \\ 61 & 25 & 73 & 84 \\ 3 & 62 & 51 & 84 \\ 72 & 36 & 51 & 84 \\ 35 & 71 & 62 & 84 \\ 361 & 325 & 172 & 84 \end{pmatrix} \tag{58}$$

which is a faithful but no longer regular representation of S_8 because it is no longer maximal in $cl_{4,4}$.

Step 2. Next, by applying the othogonal transformation $\pi_{k+1} = \pi_4 = (1234)(5678)$, or

$$e_1 \to e_2 \to e_3 \to e_4 \to e_1, \quad \text{and} \quad e_5 \to e_6 \to e_7 \to e_8 \to e_5 \tag{59}$$

repeatedly to the table (58), we find the additional representations of the following *distinct* 2-cycles of S_8 in $cl_{4,4}$,

$$\begin{pmatrix} 4 & 73 & 62 & 51 \\ 46 & 82 & 73 & 51 \\ 83 & 47 & 62 & 51 \\ 472 & 436 & 283 & 51 \\ 45 & 81 & 73 & 62 \\ 183 & 147 & 354 & 62 \\ 542 & 182 & 614 & 73 \end{pmatrix}. \tag{60}$$

We now paste together the above representations of the 14 distinct 2-cycles of S_8 in (58) and (60), getting the table

$$\begin{pmatrix} 1 & 62 & 73 & 84 \\ 2 & 51 & 73 & 84 \\ 61 & 25 & 73 & 84 \\ 3 & 62 & 51 & 84 \\ 72 & 36 & 51 & 84 \\ 35 & 71 & 62 & 84 \\ 361 & 325 & 172 & 84 \\ 4 & 73 & 62 & 51 \\ 46 & 82 & 73 & 51 \\ 83 & 47 & 62 & 51 \\ 472 & 436 & 283 & 51 \\ 45 & 81 & 73 & 62 \\ 183 & 147 & 354 & 62 \\ 542 & 182 & 614 & 73 \end{pmatrix}. \tag{61}$$

This completes Step 2 of the construction.

Step 3. The last additional row to the table (61) is constructed by taking the first *three* numbers $(\,361 \quad 325 \quad 172\,)$ of the last row in (58) and placing the number 8 in front of them to get $(\,8361 \quad 8325 \quad 8172\,)$. The fourth entry is obtained by applying $-\pi_4$ to the first number 8361 to get $-\pi_4(8361) = (4572)$. We thus get the 15^{th} and last row $(\,8361 \quad 8325 \quad 8172 \quad 4572\,)$. This completes the last step of the construction of the elements in $\mathcal{R}_{16}$ representing S_{16} in $cl_{4,4}$.

We give a summary of our results in the combined tables

$$
(\,1\,) \quad \begin{pmatrix} 1 & 62 \\ 2 & 51 \\ 61 & 25 \end{pmatrix} \quad \begin{pmatrix} 1 & 62 & 73 \\ 2 & 51 & 73 \\ 61 & 25 & 73 \\ 3 & 62 & 51 \\ 72 & 36 & 51 \\ 35 & 71 & 62 \\ 361 & 325 & 172 \end{pmatrix} \quad \begin{pmatrix} 1 & 62 & 73 & 84 \\ 2 & 51 & 73 & 84 \\ 61 & 25 & 73 & 84 \\ 3 & 62 & 51 & 84 \\ 72 & 36 & 51 & 84 \\ 35 & 71 & 62 & 84 \\ 361 & 325 & 172 & 84 \\ 4 & 73 & 62 & 51 \\ 46 & 82 & 73 & 51 \\ 83 & 47 & 62 & 51 \\ 472 & 436 & 283 & 51 \\ 45 & 81 & 73 & 62 \\ 183 & 147 & 354 & 62 \\ 542 & 182 & 614 & 73 \\ 8361 & 8325 & 8172 & 4572 \end{pmatrix} \tag{62}
$$

4.3 The general construction in $cl_{n,n}$

For $k = 0$ and $\mathcal{R}_1 = \{1\}$, we are given $\mathcal{R}_2 = \{1, e_1\}$, the representation of S_2 in $cl_{1,1}$ considered as a subalgebra of $cl_{n,n}$. We give here the general rule for the recursive construction of the representation $\mathcal{R}_{2^{k+1}}$ of $S_{2^{k+1}}$ in $cl_{k+1,k+1}$, given that the representation of $\mathcal{R}_{2^k}$ of S_{2^k} in $cl_{k,k}$ for $1 \leq k < n$ has been constructed, where both $cl_{k,k}$ and $cl_{k+1,k+1}$ are considered as subalgebras of $cl_{n,n}$.

Step 1 Each $r_\alpha \in \mathcal{R}_{2^k}$ is mapped into $r_\alpha \circ e_{n+k+1,k+1} \in \mathcal{R}_{2^{k+1}}$.

Step 2 The orthogonal transformation represented by $\pi_{k+1} = (1, \ldots, k+1)(n+1, \ldots, n+k+1)$ is applied repeatedly to each of the elements of $\mathcal{R}_{2^k} \circ e_{n+k+1,k+1}$ until no more distinct new elements are formed. Taken together, this will form a table of $2^{k+1} - 1$ distinct elements of $\mathcal{R}_{2^{k+1}}$.

Step 3 If k is *odd*, the last element of $\mathcal{R}_{2^{k+1}}$ is formed by placing the vector e_{n+k+1} in front of the first k elements of the last entry in $\mathcal{R}_{2^k}$. Thus, if this last entry in $\mathcal{R}_{2^k}$ is $e_{\gamma_1} \circ \cdots \circ e_{\gamma_k}$, this operation gives

$$e_{n+k+1,\gamma_1} \circ \cdots \circ e_{n+k+1,\gamma_k}.$$

The last element in $\mathcal{R}_{2^{k+1}}$ is then formed by applying $-\pi_{k+1}$ to e_{n+k+1,γ_1} to get $-\pi_{k+1}(e_{n+k+1,\gamma_1})$. The last element is then given by

$$e_{n+k+1,\gamma_1} \circ \cdots \circ e_{n+k+1,\gamma_k} \circ [-\pi_{k+1}(e_{n+k+1,\gamma_1})].$$

If k is *even*, this step is modified by placing the vector e_{k+1} in front of the first k elements, instead of e_{n+k+1}, and applying the orthogonal transformation π_{k+1} *without* the minus sign.

5 The heart of the matter

Let us formalize the regular representations of the symmetric groups that we have found in the previous section. We start by reorganizing the rows of (62), identifying the primitive idempotent $\mathcal{U}_4^+ = e_{15}^+ e_{26}^+ e_{37}^+ e_{48}^+$ and the identity element $r_0 = 1 \circ \cdots \circ 1 = 1$ of our representation $\mathcal{R}_{16}$ of the group S_{16}. Let us also explicitly define the elements r_α of our representation by writing

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_{21} \\ r_{31} \\ r_{41} \\ r_{32} \\ r_{42} \\ r_{43} \\ r_{321} \\ r_{421} \\ r_{431} \\ r_{432} \\ r_{4321} \end{pmatrix} \mathcal{U}_4^+ = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 62 & 73 & 84 \\ 2 & 51 & 73 & 84 \\ 3 & 62 & 51 & 84 \\ 4 & 73 & 62 & 51 \\ 61 & 25 & 73 & 84 \\ 35 & 71 & 62 & 84 \\ 45 & 81 & 73 & 62 \\ 72 & 36 & 51 & 84 \\ 46 & 82 & 73 & 51 \\ 83 & 47 & 62 & 51 \\ 361 & 325 & 172 & 84 \\ 542 & 182 & 614 & 73 \\ 183 & 147 & 354 & 62 \\ 472 & 436 & 283 & 51 \\ 8361 & 8325 & 8172 & 4572 \end{pmatrix} \mathcal{U}_4^+ = \begin{pmatrix} e_0 = 1 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_{21} \\ e_{31} \\ e_{41} \\ e_{32} \\ e_{42} \\ e_{43} \\ e_{321} \\ e_{421} \\ e_{431} \\ e_{432} \\ e_{4321} \end{pmatrix} \mathcal{U}_4^+. \tag{63}$$

The table (63) reflects everything we have accomplished in this paper, and also what is left to be done in order to *prove* that our construction produces a regular representation as claimed. First, we are identifying and renaming the geometric numbers which represent the entangled 2-cycles of S_{16}. Thus, for example, the first and second equalities of the ninth row of our matrix equation expresses that

$$r_{32} = e_{72} \circ e_{36} \circ e_{51} \circ e_{84}, \quad \text{and} \quad r_{32}\mathcal{U}_4^+ = e_{32}\mathcal{U}_4^+, \tag{64}$$

respectively.

Multiplying both sides of the last expression in (64) on the left by r_{32} gives

$$\mathcal{U}_4^+ = r_{32} e_{32} \mathcal{U}_4^+$$

since $r_{32}^2 = 1$, showing that r_{32} flips the first (the row containing e_0) and the ninth rows of (63). This is more elegantly stated for the representation $\mathcal{R}_{2^n}$ of S_{2^n} in $cl_{n,n}$ by saying that

$$r_\alpha \mathcal{U}_n^+ = e_\alpha \mathcal{U}_n^+ \iff \mathcal{U}_n^+ = r_\alpha e_\alpha \mathcal{U}_n^+ \tag{65}$$

for all indexes $\alpha \in \mathcal{I}_n$ which define the basis elements e_α of the geometric algebra cl_n. For a regular representation we also need to show that such an operation leaves all the other rows of (63) fixed. In other words, we must show in general that for all non-zero indices $\alpha, \beta \in \mathcal{I}_n$ and $\beta \neq \alpha$, that

$$r_\alpha e_\beta \mathcal{U}_n^+ = e_\beta \mathcal{U}_n^+ \iff r_\alpha \mathcal{U}_n^{sgn(\beta)} = r_\alpha e_\beta \mathcal{U}_n^+ e_\beta^{-1} = \mathcal{U}_n^{sgn(\beta)}. \tag{66}$$

The usual regular matrix representation of $\mathcal{S}_n$ follows immediately from (65) and (66), giving for each $\alpha \in \mathcal{I}_n$,

$$r_\alpha = r_\alpha \sum_{\beta \in \mathcal{I}_N} \mathcal{U}_n^{sn(\beta)} = r_\alpha \mathcal{U}_n^+ + \sum_{\beta \in \mathcal{I}_n, \beta \neq 0} e_\alpha^{-\delta_{\alpha,\beta}} \mathcal{U}_n^{sn(\beta)},$$

where $\delta_{\alpha,\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha,\beta} = 1$ when $\alpha = \beta$.

Examining the right side of (63), we see that each 2-cycle representative r_α can be classified into *k-vector types*, where $k \geq 1$. For example, we say that the r_α in the sixth through eleventh rows of (63) are of bivector or 2-vector type. Thus, each of the representative elements r_α of S_{16}, as given in (63), are of k-vector type for some $1 \leq k \leq 4$. Since in the Steps 1, 2, and 3, of our construction, each k-vector is related by the orthogonal transformation π_4 to each of the other representatives of the same k-vector type, we need only prove our assertions for one representative element of each k-vector type. Note also that the orthogonal transformation π_4 leaves the primitive idempotent $\mathcal{U}_4^+$ invariant, that is $\pi_4(\mathcal{U}_4^+) = \mathcal{U}_4^+$. This means that for all $\alpha \in \mathcal{I}_4$,

$$\pi_4(r_\alpha \mathcal{U}_4^+) = \pi_4(r_\alpha)\mathcal{U}_4^+ = \pi_4(e_\alpha)\mathcal{U}_4^+. \tag{67}$$

The property (67) justifies our change of the sign in the last row of the table (55). Indeed, if we adjusted the indices of standard basis (19) to reflect the order obtained in the construction process, no change of sign would be required.

More generally, let $\mathcal{R}_{2^n}$ be the representation obtained for S_{2^n} by the three Steps outlined in Section 4.1, and let π_n denote the orthogonal transformation defined in the second step. Then $\pi_n(\mathcal{U}_n^+) = \mathcal{U}_n^+$, since it only *permutes* the idempotent factors of $\mathcal{U}_n^+$ among themselves. In addition, because π_n is an orthogonal transformation, it permutes k-vector representatives among themselves so that

$$\pi_n(r_\alpha \mathcal{U}_n^+) = \pi_n(r_\alpha)\mathcal{U}_n^+ = \pi_n(e_\alpha)\mathcal{U}_n^+. \tag{68}$$

We are now ready to state and prove the only theorem of this paper.

Theorem 1 *For each n, there exist a regular representation $\mathcal{R}_{2^n}$ of S_{2^n} in the geometric algebra $cl_{n,n}$ which acts on the set of elements $\mathcal{E}_n^\dagger \mathcal{U}_n^+$. The representations $\mathcal{R}_{2^n}$ are constructed recursively using the Steps 1, 2, and 3, as explained in Section 4.3 for $cl_{n,n}$, and satisfy the relationships (65), (66), (68).*

Proof Suppose that $n > 1$, and that the representation $\mathcal{R}_{2^n}$ has been constructed according to three steps outlined in Section 4.1. By construction,

$$\mathcal{U}_n^+ = \prod_{i=1}^{n} e_{i,n+i}^+.$$

We will show, by induction on k for $1 \leq k \leq n-1$ that the first representative of each $k+1$-vector type satisfies the relationships (65), and (66). Whereas the recursive construction begins with $k=1$, we must first show that the element $r_1 = e_1 = e_1^+ + e_1^- e_1$, which occurs when $k=0$ and which has vector type 1, satisfies (65), and (66).

Thus, for $k=0$, we are given that $r_1 = e_1 \circ e_{n+2,2} \circ \cdots \circ e_{2n,n}$. By the repeated application of the absorbtion properties (16) and (17), we find that

$$r_1 \mathcal{U}_n^+ = (e_1 \circ e_{n+2,2} \circ \cdots \circ e_{2n,n}) \prod_{i=1}^{n} e_{i,n+i}^+ = e_1 \prod_{i=1}^{n} e_{i,n+i}^+ = e_1 \mathcal{U}_n^+,$$

so that the relationship (65) is satisfied for all of the 1-vector types $r_1, \ldots, r_n$. To show that (66) is satisfied, let $\beta \in \mathcal{I}_n$, $\beta \neq 1$ and $\beta \neq 0$. It follows that for some i, $1 < i \leq n$ we have $\mathcal{U}_n^{sn(i)} = e_{n+i,i}$ and $\mathcal{U}_n^{sn(i)} \mathcal{U}_n^\beta = \mathcal{U}_n^\beta$. It then easily follows from the absorbtion property (17) that

$$r_1 \mathcal{U}_n^\beta = r_1 \mathcal{U}_n^{sn(i)} \mathcal{U}_n^\beta = \mathcal{U}_n^\beta. \tag{69}$$

For $k=1$, the recursive constrution produces the element

$$r_{21} = (e_{n+2,1} \circ e_{2,n+1}) \circ e_{n+3,3} \circ \cdots \circ e_{2n,n},$$

which is the first element with 2-vector type. To show (65), we calculate with the repeated help of the absorbtion property (17), as well as (33),

$$r_{21} \mathcal{U}_n^+ = [(e_{n+2,1} \circ e_{2,n+1}) \circ e_{n+3,3} \circ \cdots \circ e_{2n,n}] \mathcal{U}_n^+$$
$$= [(e_{n+2,1} \circ e_{2,n+1}) \mathcal{U}_n^+ = (e_{2,n+1}^+ + e_{2,n+1}^- e_{n+2,1}) \mathcal{U}_n^+$$
$$= (e_{21}^+ + e_{211(n+1)}^- e_{1,n+1}^- e_{2,2,n+2,1}) \mathcal{U}_n^+ = (e_{21}^+ + e_{21}^+ e_{21}) \mathcal{U}_n^+$$
$$= (e_{21}^+ - e_{21}^-) \mathcal{U}_n^+ = e_{21} \mathcal{U}_n^+.$$

A similar calculation to (69) shows that (66) is true, i.e., that

$$r_{21} \mathcal{U}_n^{sn(\beta)} = \mathcal{U}_n^{sn(\beta)}$$

for any index $\beta \in \mathcal{I}_n$ such that $\beta \neq 0$ and $\beta \neq 21$.

Suppose now that the j-vector representative $r_{j\ldots 1}$ has the properties (65) and (66) for all positive integers j less than or equal to k where $1 \leq k < n$. Then we must prove that the same is true for the representative element $r_{k+1,k,\ldots,1}$. Recall that $r_{k+1,k,\ldots,1}$ is constructed in Step 3 from

$$r_{k,\ldots,1} = e_{\gamma_1} \circ \cdots \circ e_{\gamma_k}$$

where e_{γ_i}, for $1 \leq i \leq k$, are commuting square one k-vectors in $cl_{n,n}$ obtained in the construction process. Then, by the induction assumption, we know that

$$r_{k,\ldots,1} \mathcal{U}_n^+ = e_{k,\ldots,1} \mathcal{U}_n^+.$$

According to Step 3 given in Section 4.3,

$$r_{k+1,k,\ldots,1} = e_{n+k+1,\gamma_1} \circ \cdots \circ e_{n+k+1,\gamma_k} \circ e_{\gamma_{k+1}},$$

where $e_{\gamma_{k+1}} = (-1)^k \pi_{k+1}(e_{n+k+1,\gamma_1})$.

We now verify (65) for $r_{k+1,\ldots,1}$, getting

$$r_{k+1,\ldots,1}\mathcal{U}_n^+ = (e_{n+k+1,\gamma_1} \circ \cdots \circ e_{n+k+1,\gamma_k} \circ e_{\gamma_{k+1}}) \prod_{i=1}^{n} e_{i,n+i}^+$$

$$= [e_{\gamma_{k+1}}^+ + e_{\gamma_{k+1}}^-(e_{n+k+1,\gamma_1} \circ \cdots \circ e_{n+k+1,\gamma_k})] \prod_{i=1}^{n} e_{i,n+i}^+. \tag{70}$$

There are basically four cases of (70) that need to be verified, depending upon whether k is *even* or *odd*, and whether $e_{1\ldots k+1}^2 = \pm 1$.

Case i) k is odd and $e_{1\ldots k+1}^2 = -1$. In this case, with the help of (16) and (33), (70) simplifies to

$$= [e_{(k+1),\cdots,1}^+ + e_{(k+1),\cdots,1}^+ e_{(k+1),\cdots,1}] \prod_{i=1}^{n} e_{i,n+i}^+ = e_{(k+1),\cdots,1} \prod_{i=1}^{n} e_{i,n+i}^+,$$

since $e_{(k+1),\cdots,1}^+ e_{(k+1),\cdots,1} = -e_{(k+1),\cdots,1}^-$.

Case ii) k is even and $e_{1\ldots k+1}^2 = -1$. In this case, with the help of (16) and (33), (70) simplifies in exactly the same way as case i).

Case iii) k is odd and $e_{1\ldots k+1}^2 = 1$. In this case, with the help of (16) and (33), (70) simplifies to

$$= [e_{(k+1),\cdots,1}^+ + e_{(k+1),\cdots,1}^- e_{(k+1),\cdots,1}] \prod_{i=1}^{n} e_{i,n+i}^+ = e_{(k+1),\cdots,1} \prod_{i=1}^{n} e_{i,n+i}^+,$$

since $e_{(k+1),\cdots,1}^- e_{(k+1),\cdots,1} = -e_{(k+1),\cdots,1}^-$.

Case iv) k is even and $e_{1\ldots k+1}^2 = 1$. In this case, with the help of (16) and (33), (70) simplifies in exactly the same way as case iii).

□

Acknowledgements

The author wants to thank Dr. Andrés Ramos Ramírez, Dean of the School of Sciences, and Dr. Reyla Navarro Cruz, Chairperson of the Department of Mathematics and Physics, for their support at the Universidad de Las Américas - Puebla, and to Conacyt SNI Exp. 14587. He dedicates this article to the Memory of Pertti Lounesto, whose powerful software package Clical made this work possible.

References

[1] R. Ablamowicz, G. Sobczyk, Lectures on Clifford (Geometric) Algebras and Applications, Birkhäuser, Boston 2004.

[2] C.W. Curtis, *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer*, AMS and the London Mathematical Society, 1999.

[3] Hestenes D. and Sobczyk G., Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics, 2nd edition, Kluwer 1992.

[4] D. Hestenes, *Point Groups and Space Groups in Geometric Algebra*, In: L. Doerst, C. Doran, J. Lasenby (Eds), Applications of Geometric Algebra with Applications in Computer Science and Engineering, Birkhauser, Boston (2002). p. 3-34.

[5] D. Hestenes and J. Holt, *The Crystallographic Space Groups in Geometric Algebra*, Journal of Mathematical Physics. 48, 023514 (2007)

[6] T.Y. Lam, Representations of Finite Groups: A Hundred Years, Part I NOTICES OF THE AMS VOLUME 45, NUMBER 3, pp. 361-372

[7] G. James, M. Liebeck, Representations and Characters of Groups, 2nd Ed., Cambridge University Press, 2001.

[8] P. Lounesto. Clifford Algebras and Spinors. Cambridge University Press, Cambridge, 1997.

[9] P. Lounesto. CLICAL software paquet and user manual. Helsinki University of Technology of Mathematics, Research Report A248, 1994.

[10] J. Pozo and G. Sobczyk. Geometric Algebra in Linear Algebra and Geometry. *Acta Applicandae Mathematicae*, 71: 207–244, 2002.

[11] G. Sobczyk, Geometric Matrix Algebra, Linear Algebra and its Applications, 429 (2008) 1163-1173.

[12] G. Sobczyk. Clifford Geometric Algebras in Multilinear Algebra and Non-Euclidean Geometries. Editor: Jim Byrnes, Computational Noncommutative Algebra and Applications: NATO Science Series, 1–28, Kluwer 2004.