Vector Analysis of Spinors

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October 27, 2014

Abstract

The geometric algebra of space $\mathbb{G}_3$ is derived by extending the real number system to include three mutually anticommuting square roots of $+1$. The resulting geometric algebra is isomorphic to the algebra of complex 2×2 matrices, also known as the Pauli algebra. The so-called spinor algebra of $\mathbb{C}_2$, the language of the ubiquitous quantum mechanics, is formulated in terms of the idempotents and nilpotents of the geometric algebra $\mathbb{G}_3$, including its beautiful representation on the Riemann sphere.

Keywords: bra-ket formalism, geometric algebra, Schrödinger-Pauli equation, spinor, spacetime algebra, Minkowski spacetime, Riemann sphere.

AMS Subject Classication 2010: 15A66.

1 Introduction

Three dimensional Gibbs-Heaviside vector analysis was developed early in the 20th Century, before the development of relativity and quantum mechanics. What is still not widely appreciated is that the Gibbs-Heaviside formalism can be fortified into a far more powerful geometric algebra which serves the much more sophisticated needs of relativity theory and quantum mechanics. The associative Geometric algebra is viewed here as the natural completion of the real number system to include the concept of direction.

We assume that the real numbers can be always be extended to include new *anticommuting* square roots of $+1$ and -1. The new square roots of $+1$ represent orthogonal Euclidean vectors along *xyz*-coordinate axes, whereas new square roots of -1 represent orthogonal pseudo-Euclidean vectors along the coordinate axes of more general pseudo-Euclidean spacetimes. We shall primarily be interested in the geometric algebra $\mathbb{G}_3$ of the ordinary 3-dimensional space $\mathbb{R}^3$ of experience, but the interested reader may pursue how this geometric algebra can be *factored* into the spacetime algebra $\mathbb{G}_{1,3}$ of the pseudo-Riemannian space $\mathbb{R}^{1,3}$ of 4-dimensional Minkowski spacetime, [1, Ch.11].

One of the most important concepts in quantum mechanics is the concept of spin, and the treatment of spin has led to many important mathematical developments, starting with the Pauli and Dirac matrices in the early development of quantum mechanics, to the development of the differential forms, geometric algebras, and other more specialized formalisms, such as the twistor formalism of Roger Penrose [2, Ch.33]. We show here how the geometric algebra $\mathbb{G}_3$ of 3-dimensional Euclidean space $\mathbb{R}^3$ has all the algebraic tools necessary to give a clear geometrical picture of the relationship between a classical 2-component *spinor* in the complex plane, and a point on the Riemann sphere obtained by stereographic projection from the South Pole.

So let's get started.

2 Geometric algebra of space

We extend the real number system $\mathbb{R}$ to include three new *anticommuting* square roots $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $+1$, which we identify as unit vectors along the x- y- and z-axis of Euclidean space $\mathbb{R}^3$. Thus,

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1, \quad \text{and} \quad \mathbf{e}_{jk} := \mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j = -\mathbf{e}_{kj}$$

for $1 \le j < k \le 3$. We assume that the associative and distributive laws of multiplication of real numbers remain valid in our geometrically extended number system, and give the new quantities $I := \mathbf{e}_{23}$, $J := \mathbf{e}_{13}$ and $K := \mathbf{e}_{12}$ the geometric interpretation of *directed plane segments*, or *bivectors*, parallel to the respective yz-, xz- and xy-planes. Every unit bivector is the generator of rotations in the vector plane of that bivector, and this property generalizes to bivectors of the n-dimensional Euclidean space $\mathbb{R}^n$. We leave it for the reader to check that I,J,K satisfy exactly the same rules as Hamilton's famous quaternions, but now endowed with the geometric interpretation of oriented bivectors, rather than Hamilton's original interpretation of these quantities as the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Whereas the unit bivectors I,J,K satisfy $I^2 = J^2 = K^2 = -1$, the new quantity $i := \mathbf{e}_{123} := \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is a unit *trivector*, or *directed volume* element. We easily calculate, with the help of the associative and anticommutitive properties,

$$i^2 = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1^2\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1^2\mathbf{e}_2^2\mathbf{e}_3^2 = -1,$$

so the unit trivector i has same square minus one as do the bivectors I,J,K. The geometric numbers of 3-dimensional space are pictured in Figure 1.

The important Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for $\theta \in \mathbb{R}$ and the unit trivector $i = \mathbf{e}_{123}$, depends only upon the algebraic property that $i^2 = -1$, and so is equally valid for the unit bivectors I,J,K. For the unit vectors $\mathbf{e}_k$, we have the *hyperbolic Euler identities*

$$e^{\mathbf{e}_k\phi} = \cosh\phi + \mathbf{e}_k\sinh\phi$$

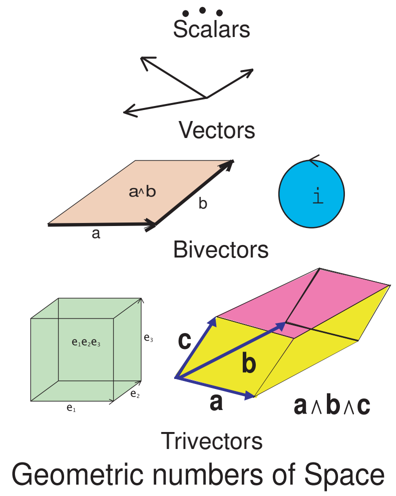

Figure 1: Geometric numbers of space

for $\phi \in \mathbb{R}$ and $k = 1,2,3$. All of these identities are special cases of the general algebraic definition of the exponential function

$$e^X \equiv \sum_{n=0}^{\infty} \frac{X^n}{n!} = \cosh X + \sinh X,$$

[1, Chp. 2] and [3].

The *standard basis* of the $2^3 = 8$ dimensional geometric algebra $\mathbb{G}_3$, with respect to the *coordinate frame* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Euclidean space $\mathbb{R}^3$, is

$$\mathbb{G}_3 := \mathbb{G}(\mathbb{R}^3) = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}. \tag{1}$$

Alternatively, we can obtain the geometric algebra $\mathbb{G}_3$ by its representation as a 2×2 matrix algebra $Mat_{\mathbb{C}}(2)$ over the complex numbers. We define the mutually anihiliating idempotents $u_\pm := \frac{1}{2}(1 \pm \mathbf{e}_3)$, and note the fundamental relationships

$$u_+ u_- = 0, \quad u_+ + u_- = 1, \quad u_+ - u_- = \mathbf{e}_3, \quad \text{and} \quad \mathbf{e}_1 u_+ = u_- \mathbf{e}_1. \tag{2}$$

Since the unit trivector i commutes with all the elements (is in the center) of $\mathbb{G}_3$, the *spectral basis* of $\mathbb{G}_3$, over the *fomally* complex numbers $\mathbb{C} = span_{\mathbb{R}}\{1, i\}$, is specified by

$$\mathbb{G}_3 = span\{\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix}\} = span\{\begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}\}. \tag{3}$$

The relationship between the *standard basis* (1) and the spectral basis (3) is directly expressed by

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_+ \\ \mathbf{e}_1 u_+ \\ \mathbf{e}_1 u_- \\ u_- \end{pmatrix}. \tag{4}$$

For example, using the relationships (2), the spectral basis (3), and the fact that

$$\mathbf{e}_2 = \mathbf{e}_{123}\mathbf{e}_1\mathbf{e}_3 = i\,\mathbf{e}_1(u_+ - u_-),$$

the famous *Pauli matrices* of the coordinate frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are simply obtained, getting

$$[\mathbf{e}_1] := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ [\mathbf{e}_2] := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ [\mathbf{e}_3] := -i[\mathbf{e}_1][\mathbf{e}_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Indeed, the matrix representation $[g] \in Mat_{\mathbb{C}}(2)$ of any geometric number $g \in \mathbb{G}_3$ is

$$g = (\,1 \quad \mathbf{e}_1\,)\,u_+[g]\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix}. \tag{6}$$

For example, the unit vector $\mathbf{e}_2$ of the Pauli matrix $[\mathbf{e}_2]$ in (5), is specified by

$$\mathbf{e}_2 = (\,1 \quad \mathbf{e}_1\,)\,u_+\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = i\mathbf{e}_1 u_+ - i\,\mathbf{e}_1 u_- = i\,\mathbf{e}_1(u_+ - u_-)$$

in agreement with (4). The proof of the isomorphism of the complex matrix algebra $Mat_{\mathbb{C}}(2)$ and the geometric algebra $\mathbb{G}_3$ is left to the reader. For a further discussion, see [1, p.79].

Any geometric number $g \in \mathbb{G}_3$ can be written in the form

$$g = \sum_{k=0}^{3} \alpha_k \mathbf{e}_k \tag{7}$$

where $\mathbf{e}_0 := 1$, $\alpha_k = a_k + i\,b_k$ for $a_k, b_k \in \mathbb{R}$, $i = \mathbf{e}_{123}$, and where $0 \le k \le 3$, giving $2^3 = 8$ degrees of freedom. The conjugation, known as the *reverse* $g^\dagger$ of the geometric number g, is defined by reversing the orders of all the products of the vectors that make up g, giving

$$g^\dagger := \sum_{k=0}^{3} \overline{\alpha}_k \mathbf{e}_k. \tag{8}$$

In particular, writing $g = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$, the sum of a real number $s \in \mathbb{G}_3^0$, a vector $\mathbf{v} \in \mathbb{G}_3^1$, a bivector $\mathbf{B} \in \mathbb{G}_3^2$ and a trivector $\mathbf{T} \in \mathbb{G}_3^3$, $g^\dagger = s + \mathbf{v} - \mathbf{B} - \mathbf{T}$.

Two other conjugations are widely used in geometric algebra. The *grade inversion* is obtained by replacing each vector in a product by its negative. It corresponds to an inversion in the origin, otherwise known as a *parity inversion*. For the geometric number g, given in (7), the grade inversion is

$$g^- := \overline{\alpha}_0 - \sum_{k=1}^{3} \overline{\alpha}_k \mathbf{e}_k. \tag{9}$$

When g is written as $g = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$, the grade inversion $g^- = s - \mathbf{v} + \mathbf{B} - \mathbf{T}$. The *Clifford conjugation* g^* of the geometric number $g \in \mathbb{G}_3$, defined by

$$g^* := (g^-)^\dagger = \alpha_0 - \sum_{k=1}^{3} \alpha_k \mathbf{e}_k = s - \mathbf{v} - \mathbf{B} + \mathbf{T}, \tag{10}$$

is just the inversion of g followed by the reversion.

All other products in the geometric algebra are defined in terms of the geometric product. For example, given vectors $\mathbf{a}, \mathbf{b} \in \mathbb{G}_3^1 \equiv \mathbb{R}^3$,

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \in \mathbb{G}_3^{0+2}, \tag{11}$$

where $\mathbf{a} \cdot \mathbf{b} := \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \in \mathbb{R}$ is the symmetric *inner product*, and $\mathbf{a} \wedge \mathbf{b} := \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$ is the antisymmetric *outer product* of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. The outer product satisfies $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$, expressing the duality relationship between the standard Gibbs-Heaviside cross product $\mathbf{a} \times \mathbf{b}$ and the outer product $\mathbf{a} \wedge \mathbf{b} \in \mathbb{G}_3^2$. A great advantage of the geometric algebra $\mathbb{G}_3$ over the Gibbs-Heaviside vector algebra is the cancellation property

$$\mathbf{ab} = \mathbf{ac} \iff \mathbf{a}^2\mathbf{b} = \mathbf{a}^2\mathbf{c} \iff \mathbf{b} = \mathbf{c},$$

provided $\mathbf{a}^2 = |\mathbf{a}|^2 \neq 0$. The equation $\mathbf{ab} = \mathbf{ac}$ forces equality of *both* the scalar and bivector parts of (11).

The triple products $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$ and $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ of three vectors are also important. Similar to (11), we write

$$\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) + \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}), \tag{12}$$

where in this case

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) := \frac{1}{2}\big(\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) - (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\big) = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \in \mathbb{G}_3^1,$$

and

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) := \frac{1}{2}\big(\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\big) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})i \in \mathbb{G}_3^3.$$

We refer the reader back to Figure 1 for a picture of the bivector $\mathbf{a} \wedge \mathbf{b}$ and the trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$.

A much more detailed treatment of $\mathbb{G}_3$ is given in [1, Chp.3], and in [4] I explore the close geometric relationship that exists between geometric algebras and their matrix counterparts. Geometric algebra has been has been extensively developed by many authors over the last 40 years as a new foundation for much of mathematics and physics. See, for example, [1, 5, 6, 7].

3 Idempotents and the Riemann sphere

An *idempotent* $s \in \mathbb{G}_3$ has the defining property $s^2 = s$. Other than 0 and $+1$, no other idempotents exist in the real or complex number systems. As we show below, the most general idempotent in $\mathbb{G}_3$ has the form $s = \frac{1}{2}(1 + \mathbf{m} + i\mathbf{n})$ for $\mathbf{m}, \mathbf{n} \in \mathbb{G}_3^1$, where $\mathbf{m}^2 - \mathbf{n}^2 = 1$ and $\mathbf{m} \cdot \mathbf{n} = 0$. In many respects, idempotents have similar properties to the eigenvectors of a linear operator.

Let $g = \alpha + \mathbf{m} + i\mathbf{n}$ be a general non-zero geometric number for $\mathbf{m}, \mathbf{n} \in \mathbb{G}_3^1$ and $\alpha \in \mathbb{G}_3^{0+3}$. In order for g to be an idempotent, we must have

$$g^2 = \alpha^2 + \mathbf{m}^2 - \mathbf{n}^2 + 2i(\mathbf{m} \cdot \mathbf{n}) + 2\alpha(\mathbf{m} + i\mathbf{n}) = \alpha + \mathbf{m} + i\,\mathbf{n} = g.$$

Equating complex scalar and complex vector parts, gives

$$\alpha^2+\mathbf{m}^2+2i(\mathbf{m}\cdot\mathbf{n})-\mathbf{n}^2=\alpha, \quad \text{and} \quad 2\alpha(\mathbf{m}+i\mathbf{n})=\mathbf{m}+i\mathbf{n},$$

from which it follows that

$$\alpha=\frac{1}{2}, \quad \mathbf{m}\cdot\mathbf{n}=0, \quad \text{and} \quad \mathbf{m}^2-\mathbf{n}^2=\frac{1}{4}.$$

Taking out a factor of $\frac{1}{2}$, we conclude that the most general idempotent $s\in\mathbb{G}_3$ has the form

$$s=\frac{1}{2}(1+\mathbf{m}+i\mathbf{n}), \quad \text{where} \quad \mathbf{m}^2-\mathbf{n}^2=1, \tag{13}$$

as mentioned above.

Let us explore the structure of a general idempotent $s=\frac{1}{2}(1+\mathbf{m}+i\mathbf{n})\in\mathbb{G}_3$. Factoring out the vector $\mathbf{m}:=|\mathbf{m}|\hat{\mathbf{m}}$, we get

$$s=\mathbf{m}\Big(\frac{1}{2}\big(1+\frac{\hat{\mathbf{m}}+i\hat{\mathbf{m}}\mathbf{n}}{|\mathbf{m}|}\big)\Big)=\mathbf{m}\hat{\mathbf{b}}_+, \tag{14}$$

where $\hat{\mathbf{b}}_+=\frac{1}{2}\big(1+\frac{\hat{\mathbf{m}}+i\hat{\mathbf{m}}\mathbf{n}}{|\mathbf{m}|}\big)$ for the unit vector $\hat{\mathbf{b}}=\frac{\hat{\mathbf{m}}+i\hat{\mathbf{m}}\mathbf{n}}{|\mathbf{m}|}\in\mathbb{G}_3^1$. With a little more manipulation, we find that

$$s=s^2=(\mathbf{m}\hat{\mathbf{b}}_+)^2=\mathbf{m}\hat{\mathbf{b}}_+\mathbf{m}\hat{\mathbf{b}}_+=\mathbf{m}^2\hat{\mathbf{m}}\mathbf{b}_+\hat{\mathbf{m}}\hat{\mathbf{b}}_+=\mathbf{m}^2\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+, \tag{15}$$

where $\hat{\mathbf{a}}_+:=\hat{\mathbf{m}}\mathbf{b}_+\hat{\mathbf{m}}$. Since

$$\hat{\mathbf{m}}\hat{\mathbf{b}}_+\hat{\mathbf{m}}=(-i\hat{\mathbf{m}})\hat{\mathbf{b}}_+(i\hat{\mathbf{m}}), \tag{16}$$

this means that the parallel component of the vector $\hat{\mathbf{b}}$ in the plane of the bivector $i\hat{\mathbf{m}}$ is being rotated through π radians (180 degrees) to obtain the unit vector $\hat{\mathbf{a}}$.

Let us further analyse properties of the idempotent $s=\frac{1}{2}(1+\mathbf{m}+i\mathbf{n})$ given in (13) and (14). Since $\mathbf{m}^2-\mathbf{n}^2=1$, we can write

$$\mathbf{m}+i\mathbf{n}=\hat{\mathbf{m}}\cosh\phi+i\hat{\mathbf{n}}\sinh\phi=\hat{\mathbf{m}}e^{\phi i\hat{\mathbf{m}}\hat{\mathbf{n}}}=e^{-\frac{1}{2}\phi i\hat{\mathbf{m}}\hat{\mathbf{n}}}\hat{\mathbf{m}}e^{\frac{1}{2}\phi i\hat{\mathbf{m}}\hat{\mathbf{n}}}, \tag{17}$$

where $\cosh\phi:=|\mathbf{m}|$, and $\sinh\phi:=|\mathbf{n}|$ for some $0\le\phi<\infty$. The relation (17) suggests that the complex unit vector $\mathbf{m}+i\mathbf{n}$ can be interpreted as being the *Lorentz boost* of the unit vector $\hat{\mathbf{m}}\in\mathbb{G}_3^1$ through the *velocity*

$$\mathbf{v}/c=\tanh(\phi i\hat{\mathbf{m}}\hat{\mathbf{n}})=-\hat{\mathbf{m}}\times\hat{\mathbf{n}}\tanh\phi. \tag{18}$$

We call $\mathbf{v}/c$ the *spin velocity* associated with the idempotent s. The spin velocity $\mathbf{v}=0$ when $\hat{\mathbf{a}}=\hat{\mathbf{m}}=\hat{\mathbf{b}}$, and the spin velocity $\mathbf{v}\to c$ as $\hat{\mathbf{m}}\to\hat{\mathbf{b}}_\perp$ and $\hat{\mathbf{a}}\to-\hat{\mathbf{a}}$, where $\hat{\mathbf{b}}_\perp$ is any unit vector perpendicular to $\hat{\mathbf{b}}$.

There is a very important property of *simple idempotents* of the form $\hat{\mathbf{a}}_\pm=\frac{1}{2}(1\pm\hat{\mathbf{a}})$, where $\hat{\mathbf{a}}\in\mathbb{G}_3^1$. Let $\hat{\mathbf{b}}_+=\frac{1}{2}(1+\hat{\mathbf{b}})$ be a second simple idempotent. Then

$$\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\hat{\mathbf{a}}_+=\frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}_+. \tag{19}$$

This property is easily established with the help of (11),

$$\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+\hat{\mathbf{a}}_+ = \frac{1}{4}(1+\hat{\mathbf{a}})(1+\hat{\mathbf{b}})\hat{\mathbf{a}}_+ = \frac{1}{4}(1+\hat{\mathbf{a}}+\hat{\mathbf{b}}+\hat{\mathbf{a}}\hat{\mathbf{b}})\hat{\mathbf{a}}_+$$

$$= \frac{1}{4}(1+\hat{\mathbf{a}}+\hat{\mathbf{b}}-\hat{\mathbf{b}}\hat{\mathbf{a}}+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}_+ = \frac{1}{4}(2+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}_+ = \frac{1}{2}(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}_+.$$

Since $s = \mathbf{m}^2\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+$ in (15) is an idempotent, it easily follows from (19) that

$$\mathbf{m}^2 = \frac{2}{1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}}. \tag{20}$$

Another consequence of (19) that easily follows is

$$\hat{\mathbf{a}}_+\hat{\mathbf{b}}\hat{\mathbf{a}}_+ = (\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{a}}_+. \tag{21}$$

As an example of a general idempotent (13), consider $s = (1+z\mathbf{e}_1)u_+$ where $z = \frac{\alpha_1}{\alpha_0} = x+iy$ for any $\alpha_0, \alpha_1 \in \mathbb{G}_3^{0+3}$, and $\alpha_0 \neq 0$. It follows from (13), (14), and (15) that for

$$\mathbf{m} = 2\langle s\rangle_1 = \frac{z+\bar{z}}{2}\mathbf{e}_1 + \frac{z-\bar{z}}{2i}\mathbf{e}_2 + \mathbf{e}_3 = x\mathbf{e}_1 + y\mathbf{e}_2 + \mathbf{e}_3 \in \mathbb{G}_3^1,$$

or in terms of α_0 and α_1,

$$\mathbf{m} = \frac{1}{2\alpha_0\overline{\alpha}_0}\Big((\alpha_1\overline{\alpha}_0+\alpha_0\overline{\alpha}_1)\mathbf{e}_1 + (\alpha_0\overline{\alpha}_1-\alpha_1\overline{\alpha}_0)\mathbf{e}_2 + 2\alpha_0\overline{\alpha}_0\mathbf{e}_3\Big), \tag{22}$$

which we use to calculate

$$s = \frac{1}{2}(1+\mathbf{m}+i\mathbf{n}) = \mathbf{m}^2\hat{\mathbf{a}}_+u_+ = \mathbf{m}u_+, \tag{23}$$

$$\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = \frac{1}{1+z\bar{z}}\Big((z+\bar{z})\mathbf{e}_1 - i(z-\bar{z})\mathbf{e}_2 + (1-z\bar{z})\mathbf{e}_3\Big)$$

$$= \frac{\Big((\overline{\alpha}_0\alpha_1+\alpha_0\overline{\alpha}_1)\mathbf{e}_1 + i(\overline{\alpha}_1\alpha_0-\alpha_1\overline{\alpha}_0)\mathbf{e}_2 + (\overline{\alpha}_0\alpha_0-\alpha_1\overline{\alpha}_1)\mathbf{e}_3\Big)}{\overline{\alpha}_0\alpha_0+\alpha_1\overline{\alpha}_1}, \tag{24}$$

and

$$\mathbf{m}^2 = 1+z\bar{z} = 1+\frac{\alpha_1\overline{\alpha}_1}{\alpha_0\overline{\alpha}_0} = \frac{\alpha_0\overline{\alpha}_0+\alpha_1\overline{\alpha}_1}{\alpha_0\overline{\alpha}_0}. \tag{25}$$

The above ideas can be related very simply to the *Riemann sphere*. The compact Riemann sphere is defined to be the projection of the xy-plane onto the the unit sphere S_2, centered at the origin, in $\mathbb{R}^3$. The stereographic projection from the south pole at the point $-\mathbf{e}_3$, is defined in terms of the projection of $\mathbf{m}$, given in (22), onto the xy-plane,

$$\mathbf{x} := P_{xy}(\mathbf{m}) = \frac{z+\bar{z}}{2}\mathbf{e}_1 + \frac{z-\bar{z}}{2i}\mathbf{e}_2 = x\mathbf{e}_1 + y\mathbf{e}_2,$$

that corresponds to the point $\hat{\mathbf{a}} \in S_2$ defined in (24). To check our calculations, we see that

$$\mathbf{x} = t(\hat{\mathbf{a}}+\mathbf{e}_3) - \mathbf{e}_3 \tag{26}$$

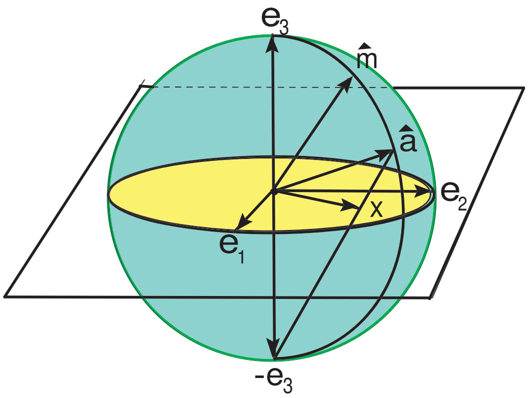

Figure 2: Stereographic Projection from the South Pole to the xy-plane.

for $t = (1+z\bar{z})/2$, so $\mathbf{x}$ is on the ray passing through the south pole and the point $\hat{\mathbf{a}} \in S_2$. Conversely, given the point $\hat{\mathbf{a}} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in S_2$, we find that

$$z = \frac{a_1 + ia_3}{1+a_3} = \frac{\alpha_1}{\alpha_0},$$

as can be checked using (24) and (26). See Figure 2. Stereographic projection is just one example of conformal mappings, which have important generalizations to higher dimensions [8].

Using (23), the quantity

$$|\alpha\rangle := \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+ = \sqrt{2}\alpha_0 s = \sqrt{2}\alpha_0\mathbf{m}u_+ = \sqrt{2}\alpha_0\mathbf{m}^2\hat{\mathbf{a}}_+u_+ \tag{27}$$

for $\alpha_0, \alpha_1 \in \mathbb{G}_3^{0+3}$, defines what I call a *geometric ket-spinor*. Whereas the spinor $|\alpha\rangle$ is defined for all $\alpha_0, \alpha_1 \in \mathbb{G}_3^{0+3}$, the idempotent s given in (23) is only defined when $\alpha_0 \neq 0$. However, by a simple trick, even this restriction can be removed, as we will see in the next section.

4 Properties of spinors

Classically, the *Pauli spinor* was introduced by Wolfgang Pauli (1900-1958) to incorporate spin into the Schrodinger equation for an electron. A Pauli spinor is defined to be a *column vector* $|\alpha\rangle_p := \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$ in the *complex* 2-*dimensional Euclidean space* $\mathbb{C}^2$. The notation $|\alpha\rangle_p$, called a *ket-vector*, is due to Dirac. The corresponding *bra-vector* is the *complex conjugate transpose*

$$\langle\alpha|_p := \overline{|\alpha\rangle}_p^T = (\,\overline{\alpha}_0 \quad \overline{\alpha}_1\,)$$

of the ket-vector. The *Euclidean norm* on $\mathbb{C}^2$ is defined by taking the two together to form the *bra-ket*

$$||\alpha\rangle_p|^2 = \langle\alpha|\alpha\rangle_p := (\,\overline{\alpha}_0 \quad \overline{\alpha}_1\,)\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = \overline{\alpha}_0\alpha_0 + \overline{\alpha}_1\alpha_1. \tag{28}$$

A spinor $|\alpha\rangle_p$ is said to be *normalized* if $\langle\alpha|\alpha\rangle_p = 1$.

The complex 2-dimensional space, with the norm as defined above, behaves like a 4-dimensional real Euclidean space. More generally, for distinct ket-vectors $|\alpha\rangle_p$ and $|\beta\rangle_p$, the *sesquilinear inner product* is defined by

$$\langle\alpha|\beta\rangle_p := (\,\overline{\alpha}_0 \quad \overline{\alpha}_1\,)\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1 \in \mathbb{C}. \tag{29}$$

We replace the Pauli spinor with a corresponding element of a *minimal left ideal*, and the geometric number (3), (6) that it represents

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix}\alpha_0 & 0\\ \alpha_1 & 0\end{pmatrix} \quad \longleftrightarrow \quad (\alpha_0 + \alpha_1\mathbf{e}_1)u_+. \tag{30}$$

We then define the spinor $|\alpha\rangle := \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+$, using the same Dirac ket-notation but without the subscript "p". The bra-vector is then introduced as the geometric algebra *reverse* of the ket-spinor,

$$\langle\alpha| := |\alpha\rangle^\dagger = \sqrt{2}u_+(\overline{\alpha}_0 + \overline{\alpha}_1\mathbf{e}_1).$$

For the bra-ket inner product of the ket-spinors $|\alpha\rangle$ and $|\beta\rangle$, we first form the geometric product

$$\langle\alpha||\beta\rangle = 2(\overline{\alpha}_0 + \overline{\alpha}_1\mathbf{e}_1)(\beta_0 + \beta_1\mathbf{e}_1)u_+ = 2(\overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1)u_+,$$

and then take the scalar and 3-vector parts

$$\langle\alpha|\beta\rangle := \Big\langle \langle\alpha||\beta\rangle \Big\rangle_{0+3} = (\overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1). \tag{31}$$

The extra factor of $\sqrt{2}$ was introduced into the definition of a ket-spinor in order to to eliminate the unwanted factor of 2 that would otherwise occur in the definition of the inner product (31).

We see that the spinor inner products (29) and (31) agree with each other, but whereas the Pauli ket-vector is a complex 2-component column matrix, the corresponding object in $\mathbb{G}_3$ is the geometric ket-spinor given in (30). Spinor spaces of a left ideal of a matrix algebra were first considered in the 1930's by G. Juvet and F. Sauter [7, p.148]. The advantage enjoyed by the geometric ket-spinor over the Pauli ket-vector is that the former inherits the unique algebraic properties of a geometric number, in addition to a comprehensive geometric significance. It is often remarked that the complex numbers play a special role in mathematics and physics because of their many almost *magical* properties [2, p.67]. Geometric algebra takes away some of the magic by providing a comprehensive geometric interpretation to the quantities involved.

A spinor $|\alpha\rangle$ with *norm* $\rho := \sqrt{\langle\alpha|\alpha\rangle}$ is said to be *normalized* if

$$\rho^2 = \langle\alpha|\alpha\rangle = \alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1 = 1,$$

from which it follows, using (25) and (27), that any non-zero spinor $|\alpha\rangle$ can be written in the perspicuous cannoical form

$$|\alpha\rangle = \sqrt{2}\rho e^{i\theta}\hat{\mathbf{m}}u_+ \quad \longleftrightarrow \quad \langle\alpha| = \sqrt{2}\rho e^{-i\theta}u_+\hat{\mathbf{m}}, \tag{32}$$

where $e^{i\theta} := \frac{\alpha_0}{\sqrt{\alpha_0\overline{\alpha}_0}}$. Many non-trivial properties of spinors can be easily derived from this form. As a starter, from (19) and (32), we calculate the *ket-bra* geometric product of the normalized geometric ket-spinor $|\alpha\rangle$, getting

$$\frac{1}{2}|\alpha\rangle\langle\alpha| = \hat{\mathbf{m}}u_+u_+\hat{\mathbf{m}} = \hat{\mathbf{m}}u_+\hat{\mathbf{m}} = \hat{\mathbf{a}}_+. \tag{33}$$

From (32), other important cannonical forms for a non zero spinor $|\alpha\rangle$ are quickly established. In particular, using the fact that $\mathbf{e}_3u_+ = u_+$, from

$$|\alpha\rangle = \sqrt{2}\rho e^{i\theta}\hat{\mathbf{m}}u_+ = \sqrt{2}\rho e^{i\theta}\hat{\mathbf{m}}\mathbf{e}_3u_+ = \sqrt{2}\rho\hat{\mathbf{m}}\mathbf{e}_3e^{i\mathbf{e}_3\theta}u_+,$$

it follows that

$$|\alpha\rangle = \sqrt{2}\rho e^{i\theta}\hat{\mathbf{m}}u_+ = \sqrt{2}\rho e^{i(\theta+\hat{\mathbf{v}}\phi)}u_+ = \sqrt{2}\rho e^{i\hat{\mathbf{v}}\phi}e^{i\mathbf{e}_3\theta}u_+ = \sqrt{2}\rho e^{i\hat{\mathbf{c}}\omega}u_+, \tag{34}$$

where $e^{i\hat{\mathbf{v}}\phi} := \hat{\mathbf{m}}\mathbf{e}_3$ and $e^{i\hat{\mathbf{c}}\omega} := e^{i\hat{\mathbf{v}}\phi}e^{i\mathbf{e}_3\theta}$. Of course, all these new variables

$$\theta, \phi, \omega \in \mathbb{R} \quad \text{and} \quad \hat{\mathbf{a}}, \hat{\mathbf{m}}, \hat{\mathbf{v}}, \hat{\mathbf{c}} \in \mathbb{G}_3^1,$$

have to be related back to the non-zero spinor $|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+$, where

$$\alpha_0 = x_0 + iy_0 \in \mathbb{G}_3^{0+3}, \quad \text{and} \quad \alpha_1 = x_1 + iy_1 \in \mathbb{G}_3^{0+3},$$

for $x_0, y_0, x_1, y_1 \in \mathbb{R}$, which we now do.

We first find, by using (22) and (24), that

$$\mathbf{m} = \frac{x_0x_1 + y_0y_1}{x_0^2 + y_0^2}\mathbf{e}_1 + \frac{x_0y_1 - x_1y_0}{x_0^2 + y_0^2}\mathbf{e}_2 + \mathbf{e}_3, \tag{35}$$

so that

$$\hat{\mathbf{m}} = \frac{x_0x_1 + y_0y_1}{\sqrt{x_0^2 + y_0^2}}\mathbf{e}_1 + \frac{x_0y_1 - x_1y_0}{\sqrt{x_0^2 + y_0^2}}\mathbf{e}_2 + \sqrt{x_0^2 + y_0^2}\mathbf{e}_3 \tag{36}$$

from which it follows that $\hat{\mathbf{v}} = \frac{\hat{\mathbf{m}}\times\mathbf{e}_3}{|\hat{\mathbf{m}}\times\mathbf{e}_3|}$, where

$$\cos\phi = \hat{\mathbf{m}}\cdot\mathbf{e}_3 = \sqrt{x_0^2 + y_0^2}, \quad \text{and} \quad \sin\phi = |\hat{\mathbf{m}}\times\mathbf{e}_3| = \sqrt{1 - x_0^2 - y_0^2},$$

and where $0 < \phi < \frac{\pi}{2}$. We also have

$$\cos\theta = \frac{Re\,\alpha_0}{\sqrt{\alpha_0\overline{\alpha}_0}} = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}, \quad \text{and} \quad \sin\theta = \frac{-i\,Im\,\alpha_0}{\sqrt{\alpha_0\overline{\alpha}_0}} = \frac{y_0}{\sqrt{x_0^2 + y_0^2}},$$

where $0 \le \theta < 2\pi$. We also use (24) to compute

$$\hat{\mathbf{a}} = 2(x_0x_1 + y_0y_1)\mathbf{e}_1 + 2(x_0y_1 - y_1y_0)\mathbf{e}_2 + (x_0^2 + y_0^2 - x_1^2 - y_1^2)\mathbf{e}_3. \tag{37}$$

Using the immediately preceeding results, and that $\hat{\mathbf{v}}\mathbf{e}_3 = i\big(\frac{(\hat{\mathbf{m}}\cdot\mathbf{e}_3)\mathbf{e}_3 - \hat{\mathbf{m}}}{\sin\phi}\big)$, we now calculate

$$\begin{aligned} e^{i\hat{\mathbf{c}}\omega} = e^{i\hat{\mathbf{v}}\phi}e^{i\mathbf{e}_3\theta} &= (\cos\phi + i\hat{\mathbf{v}}\sin\phi)(\cos\theta + i\mathbf{e}_3\sin\theta) \\ &= \cos\theta\cos\phi + i\big(\hat{\mathbf{v}}\cos\theta\sin\phi + \mathbf{e}_3\cos\phi\sin\theta\big) - \hat{\mathbf{v}}\mathbf{e}_3\sin\phi\sin\theta \\ &= \cos\theta\cos\phi + i\big(\hat{\mathbf{v}}\cos\theta\sin\phi + \hat{\mathbf{m}}\sin\theta\big), \end{aligned}$$

and finally,

$$e^{i\hat{\mathbf{c}}\omega} = e^{i\hat{\mathbf{v}}\phi}e^{i\mathbf{e}_3\theta} = \cos\omega + i\hat{\mathbf{c}}\sin\omega \tag{38}$$

where

$$\hat{\mathbf{c}} = \frac{\hat{\mathbf{m}}\times\mathbf{e}_3\cos\theta + \hat{\mathbf{m}}\sin\theta}{\sqrt{1-\cos^2\theta\cos^2\phi}} = \frac{y_1\mathbf{e}_1 - x_1\mathbf{e}_2 + y_0\mathbf{e}_3}{\sqrt{1-x_0^2}},$$

and

$$\cos\omega = \hat{\mathbf{c}}\cdot\mathbf{e}_3 = x_0, \quad \text{and} \quad \sin\omega = |\hat{\mathbf{c}}\times\mathbf{e}_3| = \sqrt{1-x_0^2}$$

for $0 \le \omega < \pi$. All of these straight forward calculations are made easy with Mathematica!

Let us take a cross section of the Riemann sphere, shown in Figure 2, to get a better understanding of the nature of a spinor. Whereas any cross section through the north and south poles would do, we choose the great circle obained by taking the intersection of the xz-plane ($y = 0$) with the Riemann sphere, see Figure 3. As the point $\hat{\mathbf{m}}$ is moved along this great circle, the point $\hat{\mathbf{a}}$ moves in such a way that $\hat{\mathbf{m}}$ is always at the midpoint of the arc joining the points $\mathbf{e}_3$ at the north pole and the point $\hat{\mathbf{a}}$. When the point $\hat{\mathbf{m}} = \mathbf{e}_3$, the three points coincide. Choosing $\alpha_0 = x_0 = 1$, and $\alpha_1 = x_1$ the geometric ket-spinor which determines *both* the points $\hat{\mathbf{a}}$ and $\hat{\mathbf{m}}$ is $|\alpha\rangle = \sqrt{2}(1 + x_1\,\mathbf{e}_1)u_+$, corresponding to the Pauli ket-vector $\begin{pmatrix} 1 \\ x_1 \end{pmatrix}$. Using (36) and (37), the unit vectors

$$\hat{\mathbf{a}} = \frac{1}{1+x_1^2}\Big(2x_1\,\mathbf{e}_1 + (1-x_1^2)\mathbf{e}_3\Big), \quad \text{and} \quad \hat{\mathbf{m}} = \frac{1}{1+x_1^2}(x_1\,\mathbf{e}_1 + \mathbf{e}_3),$$

and $\hat{\mathbf{a}}$ is on the ray eminating from the South Pole at $-\mathbf{e}_3$. This ray crosses the x-axis at the point $x_1\mathbf{e}_1$, which also determines, by (35), the point $\mathbf{m} = x_1\,\mathbf{e}_1 + \mathbf{e}_3$.

Let us now rotate the unit vector starting at the point $\hat{\mathbf{m}}$, counterclockwise past the North Pole at $\mathbf{e}_3$, until it reaches the point $\hat{\mathbf{m}}_\perp$ perpendicular to the unit vector $\hat{\mathbf{m}}$. The unit vectors $\hat{\mathbf{m}}$ and $\hat{\mathbf{m}}_\perp$ are specified by

$$\hat{\mathbf{m}} = \frac{1}{1+x_1^2}(x_1\,\mathbf{e}_1 + \mathbf{e}_3), \quad \hat{\mathbf{m}}_\perp = \frac{1}{1+{x_1'}^2}(x_1'\,\mathbf{e}_1 + \mathbf{e}_3),$$

where x_1' is to be determined so that $\hat{\mathbf{m}}\cdot\hat{\mathbf{m}}_\perp = 0$. This occurs in the northern hemisphere when $x_1' = -\frac{1}{x_1}$, and uniquely determines the point

$$\frac{1}{1+{x_1'}^2}\Big(2x_1'\,\mathbf{e}_1 + (1-{x_1'}^2)\mathbf{e}_3\Big) = \frac{1}{1+\frac{1}{x_1^2}}\Big(-\frac{2}{x_1}\,\mathbf{e}_1 + (1-\frac{1}{x_1^2})\mathbf{e}_3\Big) = -\hat{\mathbf{a}}.$$

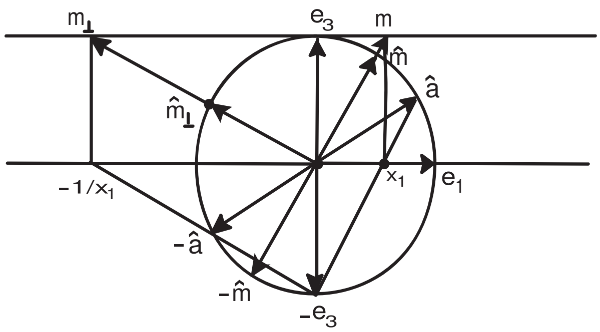

Figure 3: Cross section of the unit sphere in the xz-plane.

Thus, when the unit vector $\hat{\mathbf{m}}$ is turned counterclockwise through 90 degrees, the unit vector $\hat{\mathbf{a}}$ is rotated through 180 degrees into $-\hat{\mathbf{a}}$. The corresponding geometric ket-spinor to both $\hat{\mathbf{m}}_\perp$ and $-\hat{\mathbf{a}}$, is $|\beta\rangle = \sqrt{2}(1 - \frac{1}{x_1}\mathbf{e}_1)u_+$. The details of this construction are shown in Figure 3.

If we continue rotating $\hat{\mathbf{m}}$ counterclockwise to a point where it crosses the equator into the southern hemisphere, the ket-vector corresponding $\hat{\mathbf{m}}$ and $\hat{\mathbf{a}}$ will be $-\begin{pmatrix} 1 & x_1' \end{pmatrix}^T$, in accordance with the fact that

$$(-\hat{\mathbf{m}})\mathbf{e}_3(-\hat{\mathbf{m}}) = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = \hat{\mathbf{a}}.$$

Continuing rotating $\hat{\mathbf{m}}$ counterclockwise will return $\hat{\mathbf{a}}$ to its original position, but the Pauli ket-vector representing it will now be $-\begin{pmatrix} 1 & x_1 \end{pmatrix}^T$. Because the rotation (24) is tethered to the point $\mathbf{e}_3$, it is only after rotating $\hat{\mathbf{m}}$ through another 180 degrees that it will return to its original position. This *double covering* behaviour will only occur when $\hat{\mathbf{m}}$ is rotated on great circles passing through the North and South poles. For example, when $\hat{\mathbf{m}}$ is rotated on the circle at a latitude of 45-degrees North, the point $\hat{\mathbf{a}}$ will follow an orbit around the equator directly beneath $\hat{\mathbf{m}}$, and hence one complete rotation of $\hat{\mathbf{m}}$ will product one complete rotation of $\hat{\mathbf{a}}$, bringing its ket-vector back to its original value without attaching a minus sign.

In quantum mechanics, a normalized spinor represents the *state* of a particle which evolves in time by a unitary transformation. This should not be surprising in so far as that, by (34), any normalised spinor $|\alpha\rangle$ can be expressed in the cannocal form

$$|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+ = \sqrt{2}e^{i\hat{\mathbf{c}}\omega}u_+,$$

where $e^{i\hat{\mathbf{c}}\omega}$ is a *unitary transformation*, since $e^{i\hat{\mathbf{c}}\omega}\left(e^{i\hat{\mathbf{c}}\omega}\right)^\dagger = e^{i\hat{\mathbf{c}}\omega}e^{-i\hat{\mathbf{c}}\omega} = 1$. I will return to these ideas shortly, after I discuss *Cartan spinors* and *spinor operators*.

5 Cartan spinors and spinor operators

Elie Cartan defines a 2-*component spinor* $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ to represent a null complex vector

$$N = \mathbf{r} + i\mathbf{s} = \sum_{k=1}^{3} (r_k + i s_k)\mathbf{e}_k = \sum_{k=1}^{3} z_k \mathbf{e}_k, \text{ where } z_1^2 + z_2^2 + z_3^2 = 0,$$

for $z_k \in \mathbb{G}_3^{0+3}$, [9, p.41]. We find that

$$N = \Big(\sqrt{2}(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+\Big)\mathbf{e}_1\Big(\sqrt{2}(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+\Big)^*$$

$$= 2(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+(\alpha_0 - \alpha_1 \mathbf{e}_1)\mathbf{e}_1 = (\alpha_0^2 - \alpha_1^2)\mathbf{e}_1 + (\alpha_0^2 + \alpha_1^2)i\,\mathbf{e}_2 - 2\alpha_0\alpha_1\mathbf{e}_3. \quad (39)$$

Following Cartan, we solve these equations for α_0, α_1 in terms of z_k for $k = 1,2,3$, to get

$$\alpha_0 = \pm\sqrt{\frac{z_1 - iz_2}{2}} \quad \text{and} \quad \alpha_1 = \pm i\sqrt{\frac{z_1 + iz_2}{2}}.$$

Using (32) and (39), we can solve for N directly in terms of the idempotents $\hat{\mathbf{a}}_\pm$ and the vector $\hat{\mathbf{m}}\mathbf{e}_1\hat{\mathbf{m}}$. We find that

$$N = -2\rho^2 e^{2i\theta}\hat{\mathbf{a}}_+\hat{\mathbf{m}}\mathbf{e}_1\hat{\mathbf{m}} = -2\rho^2 e^{2i\theta}\hat{\mathbf{m}}\mathbf{e}_1\hat{\mathbf{m}}\hat{\mathbf{a}}_-. \quad (40)$$

We can also represent the null complex vector N explicitly in terms of the unit vector $\hat{\mathbf{m}}$ and the null vector $\mathbf{e}_1 u_- = \mathbf{e}_1 + i\mathbf{e}_2$, or in terms of $\hat{\mathbf{a}}_+$ and a reflection of the vector $e^{\mathbf{e}_{12}\theta}\mathbf{e}_1$ in the plane of $i\hat{\mathbf{m}}$,

$$N = -\rho^2 e^{2i\theta}\hat{\mathbf{m}}(\mathbf{e}_1 + i\,\mathbf{e}_2)\hat{\mathbf{m}} = -2\rho^2\hat{\mathbf{a}}_+\hat{\mathbf{m}}\,e^{2\mathbf{e}_{12}\theta}\mathbf{e}_1\hat{\mathbf{m}}. \quad (41)$$

Whereas $\hat{\mathbf{a}}$, in (24), is a *rotation* of the unit vector $\mathbf{e}_3$ in the plane of $i\hat{\mathbf{m}}$, the null vector N is a *reflection* of the complex null vector $\mathbf{e}_1 + i\,\mathbf{e}_2$ in the plane $i\hat{\mathbf{m}}$.

The *spinor operator* of a ket-spinor $|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1 \mathbf{e}_1)u_+$ is defined by

$$\psi := \frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\alpha\rangle^-\big) = (\alpha_0 + \alpha_1 \mathbf{e}_1)u_+ + (\overline{\alpha}_0 - \overline{\alpha}_1 \mathbf{e}_1)u_- \quad (42)$$

or by using the canonical form (34),

$$\psi = \frac{1}{\sqrt{2}}\big(|\alpha\rangle + |\alpha\rangle^-\big) = \rho e^{i\hat{\mathbf{c}}\omega}u_+ + \rho e^{i\hat{\mathbf{c}}\omega}u_- = \rho e^{i\hat{\mathbf{c}}\omega}. \quad (43)$$

The matrix $[\psi]$ of the spinor operator ψ, in the spectral basis (3), is given by

$$[\psi] = \begin{pmatrix} \alpha_0 & -\overline{\alpha}_1 \\ \alpha_1 & \overline{\alpha}_0 \end{pmatrix} = \rho[e^{i\hat{\mathbf{c}}\omega}].$$

If $|\alpha\rangle$ is a normalized spinor, then its spinor operator satisfies

$$\det[\psi] = [e^{i\hat{\mathbf{c}}\omega}] = \det\begin{pmatrix} \alpha_0 & -\overline{\alpha}_1 \\ \alpha_1 & \overline{\alpha}_0 \end{pmatrix} = \alpha_0\overline{\alpha}_0 + \alpha_1\overline{\alpha}_1 = 1,$$

and

$$[\psi\psi^\dagger] = [\psi][\psi]^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\psi = e^{i\hat{\mathbf{c}}\omega}$ defines a unitary transformation in the group $SU(2)$. Note also by using (34), it is easy to show that

$$\psi\mathbf{e}_3\psi^\dagger = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}} = \hat{\mathbf{a}}, \text{ and } \psi u_+\psi^\dagger = \hat{\mathbf{m}}u_+\hat{\mathbf{m}} = \hat{\mathbf{a}}_+. \tag{44}$$

The close relationship (43) and (44) between geometric ket-spinors and spinor operators, has been used by D. Hestenes to develop an elaborate theory of the electron in terms of the latter, [10].

6 The magic of quantum mechanics

Quantum mechanics, even more than Einstein's great theories of relativity, have transformed our understanding of the microuniverse and made possible the marvelous technology that we all depend upon in our daily lives. More and more scientists are realizing that our revolutionary understanding of the microscopic World must somehow be united with the great theories of cosmology based upon relativity theory. Whereas we will not discuss the famous paradoxes of quantum mechanics, we can concisely write down some of the basic rules upon which the foundation of this great edifice rests.

A geometric number $S \in \mathbb{G}_3$ is an *observable* if $S^\dagger = S$. From (8), we see that $S = s_0 + \mathbf{s}$ where $s_0 \in \mathbb{R}$ is a real number and $\mathbf{s} = \sum_{k=1}^3 s_k\mathbf{e}_k$ is a vector in $\mathbb{G}_3^1 \equiv \mathbb{R}^3$. Generally, in quantum mechanics an observer is identified with an Hermitian operator. Indeed, using (5) and (7), we find that

$$S = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+[s_0 + \mathbf{s}] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+[S] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} \tag{45}$$

where $[S] = \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix}$ is an Hermitian matrix.

We can immediately write down the eigenvalues and eigenvectors of the matrix $[S]$ by appealing to properties of the corresponding geometric number S. We first express S in terms of its natural spectral basis $\hat{\mathbf{s}}_\pm = \frac{1}{2}(1 \pm \hat{\mathbf{s}})$, getting

$$S = S(\hat{\mathbf{s}}_+ + \hat{\mathbf{s}}_-) = (s_0 + |\mathbf{s}|)\hat{\mathbf{s}}_+ + (s_0 - |\mathbf{s}|)\hat{\mathbf{s}}_-, \tag{46}$$

where $\mathbf{s} = |\mathbf{s}|\hat{\mathbf{s}}$ for $|\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2}$. The *eigenvalues* of the geometric number S, and its matrix $[S]$, are $s_0 \pm |\mathbf{s}|$, and we say that $\hat{\mathbf{s}}_\pm$ are its corresponding *eigenpotents*, respectively. The standard eigenvectors of the matrix $[S]$ can be retrived by taking any non-zero columns of the matrices $[\hat{\mathbf{s}}_\pm]$, respectively. See [1, 11, 12, 13] for more details regarding the construction and use of the spectral basis of a linear operator.

Having found the eigenvalues and eigenpotents of the observable S, we can immediately write down the corresponding *eigenvalues* and *eigenspinors* of S simply by multiplying the equation (46) on the right by u_+, getting

$$Su_+ = S(\hat{\mathbf{s}}_+u_+ + \hat{\mathbf{s}}_-u+) = (s_0 + |\mathbf{s}|)\hat{\mathbf{s}}_+u_+ + (s_0 - |\mathbf{s}|)\hat{\mathbf{s}}_-u_+.$$

Expressed as a spinor equation, this says that

$$|S\rangle = (s_0 + |\mathbf{s}|)|\hat{\mathbf{s}}_+\rangle + (s_0 - |\mathbf{s}|)|\hat{\mathbf{s}}_-\rangle,$$

where

$$|S\rangle := \sqrt{2}Su_+, \quad |\hat{\mathbf{s}}_+\rangle := \sqrt{2}\hat{\mathbf{s}}_+u_+, \quad |\hat{\mathbf{s}}_-\rangle := \sqrt{2}\hat{\mathbf{s}}_-u_+.$$

With the help of (46), we now can write down the eigenspinors of the observable S

$$S|\hat{\mathbf{s}}_+\rangle = (s_0 + |\mathbf{s}|)|\hat{\mathbf{s}}_+\rangle, \quad S|\hat{\mathbf{s}}_-\rangle = (s_0 - |\mathbf{s}|)|\hat{\mathbf{s}}_+\rangle. \tag{47}$$

The *expected value* of an observable $S = s_0 + \mathbf{s}$ measured in the state $|\alpha\rangle = \sqrt{2}\alpha_0\mathbf{m}u_+$ is defined by

$$\langle S\rangle := \langle\alpha|S|\alpha\rangle_{0+3} = s_0 + \mathbf{s}\cdot\hat{\mathbf{a}},$$

where $\hat{\mathbf{a}} = \hat{\mathbf{m}}\mathbf{e}_3\hat{\mathbf{m}}$ as before. The expectated value of the observable S is used to calculate the *standard deviation*

$$\sigma_S^2 := \langle\alpha|\big(S - \langle S\rangle\big)^2|\alpha\rangle = |\mathbf{s}|^2\langle\alpha|\big(\hat{\mathbf{s}} - \hat{\mathbf{s}}\cdot\hat{\mathbf{a}}\big)^2|\alpha\rangle = (\mathbf{s}\times\hat{\mathbf{a}})^2$$

of the expected value of the observable measured in that state.

We shall now state and prove a special case of Heisenburg's famous *uncertainty principle* as a simple vector analysis identity relating areas in $\mathbb{R}^3$.

Uncertainty Principle 6.1 *Given the spinor state $|\alpha\rangle = \sqrt{2}\mathbf{m}^2\hat{\mathbf{a}}_+u_+$, and the two observables $S = s_0 + \mathbf{s}$, and $T = t_0 + \mathbf{t}$, the standard deviations σ_S and σ_T of measuring the expected values of these observables in the state $|\alpha\rangle$ satisfy the vector identity*

$$(\mathbf{s}\times\hat{\mathbf{a}})^2(\mathbf{t}\times\hat{\mathbf{a}})^2 = |(\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}})|^2 + |(\mathbf{s}\times\mathbf{t})\cdot\hat{\mathbf{a}}|^2,$$

which directly implies that

$$\sigma_S^2\sigma_T^2 \geq \langle\mathbf{s}\times\mathbf{t}\rangle^2,$$

since the expected value $<\mathbf{s}\times\mathbf{t}> = (\mathbf{s}\times\mathbf{t})\cdot\hat{\mathbf{a}}$.

Proof: We include the proof to show off the power of the geometric algebra $\mathbb{G}_3$ over the standard Gibbs-Heaviside vector algebra, at the same time noting that the former fully encompases the later.

$$(\mathbf{s}\times\hat{\mathbf{a}})^2(\mathbf{t}\times\hat{\mathbf{a}})^2 = (\mathbf{s}\times\hat{\mathbf{a}})\Big((\mathbf{s}\times\hat{\mathbf{a}})(\mathbf{t}\times\hat{\mathbf{a}})\Big)(\mathbf{t}\times\hat{\mathbf{a}})$$

$$= (\mathbf{s}\times\hat{\mathbf{a}})\Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}}) + (\mathbf{s}\times\hat{\mathbf{a}})\wedge(\mathbf{t}\times\hat{\mathbf{a}})\Big)(\mathbf{t}\times\hat{\mathbf{a}})$$

$$= \Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}}) + (\mathbf{s}\times\hat{\mathbf{a}})\wedge(\mathbf{t}\times\hat{\mathbf{a}})\Big)\Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}}) - (\mathbf{s}\times\hat{\mathbf{a}})\wedge(\mathbf{t}\times\hat{\mathbf{a}})\Big)$$

$$= \Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}})\Big)^2 - \Big((\mathbf{s}\times\hat{\mathbf{a}})\wedge(\mathbf{t}\times\hat{\mathbf{a}})\Big)^2 = \Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}})\Big)^2 + \Big((\mathbf{s}\times\hat{\mathbf{a}})\times(\mathbf{t}\times\hat{\mathbf{a}})\Big)^2$$

$$= \Big((\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{t}\times\hat{\mathbf{a}})\Big)^2 + \Big((\mathbf{s}\times\mathbf{t})\cdot\hat{\mathbf{a}}\Big)^2.$$

□

It is interesting to note in the **Uncertainty Principle**, it is not enough that the observables S and T commute for $\sigma_S\sigma_T = 0$. If $S = s_0 + \mathbf{s}$ and $T = t_0 + \mathbf{s}$ and $s_0 \neq t_0$, $ST = TS$, but $\sigma_S\sigma_T = |(\mathbf{s}\times\hat{\mathbf{a}})\cdot(\mathbf{s}\times\hat{\mathbf{a}})| = 0$ only when $\mathbf{s}\times\hat{\mathbf{a}} = 0$. I have never seen this matter discussed in standard textbooks on quantum mechanics, probably because they have enough trouble as it is in just establishing the inequality $\sigma_S\sigma_T \geq |\langle \mathbf{s}\times\mathbf{t}\rangle|$ for the observables S and T, [14, p.108].

Up until now we have only considered 2-component spinors in the equivalent guise of the elements in the minimal left ideal $\{\mathbb{G}_3 u_+\}$ of $\mathbb{G}_3$, generated by the primitive idempotent $u_+ \in \mathbb{G}_3$. Much more generally, a *spinor state* Ψ is an element of a finite or infinite dimensional Hilbert space $\mathscr{H}$, and an *observable* S is represented by a *self-adjoint operator* on $\mathscr{H}$.[1] In the finite dimensional case a self-adjoint operator is just a Hermitian operator. The interested reader may want to delve into the much more profound theory of the role of self-adjoint operators on a Hilbert space in quantum mechanics, [15, 16].

In order that a normalized state $\Psi \in \mathscr{H}$ represents an experimentally measurable quantity of an observable S, it must be an eigenvector of S, meaning that $S\Psi = \alpha\Psi$ for some $\alpha \in \mathbb{C}$, and it must satisfy *Schrödinger's equation*

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{48}$$

where the *Hamiltonian operator* H is obtained from the classical Hamiltonian equation

$$H_C = \frac{1}{2}m\mathbf{v}^2 + V = \frac{1}{2m}\mathbf{p}^2 + V \tag{49}$$

for the *total energy* of a particle with mass m, velocity $\mathbf{v}$, momentum $\mathbf{p} = m\mathbf{v}$, and potential energy V, by way of the substitution rule

$$\mathbf{p} \to \frac{\hbar}{i}\nabla, \quad H_C \to i\hbar\frac{\partial}{\partial t},$$

Making this subsitution into (49) gives the quantum mechanical Hamiltonian operator equation

$$i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 + V = H \tag{50}$$

where $\hbar$ is Planck's constant, $\nabla = \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z}$ is the standard gradient operator, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. When the operator equation (50) is applied to the wave function Ψ, we get the Schrödinger equation (48) for $H = -\frac{\hbar^2}{2m}\nabla^2 + V$.

In general, the Hamiltonian $H = H(\mathbf{x}, t)$ is a function of both the position $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ of the particle, and time t. If Ψ sastisfies Shrödinger's equation (48), then according to the Born interpretation of the wave function Ψ, the probablity of finding the particle in the infinitesimal volume $|d^3\mathbf{x}| = dxdydz$ at the point $\mathbf{x}$ and time t

[1] The general spinor Ψ is not to be confused with the spinor operator ψ discussed in Section 5.

is $|d^3\mathbf{x}||\Psi| = |d^3\mathbf{x}|\sqrt{\Psi^\dagger\Psi}$. The Hamiltonian H for position is an important example of a self-adjoint operator in the infinite dimensional Hilbert space $\mathscr{L}^2$ of square integrable functions on $\mathbb{R}^3$ which has a *continuous spectrum* [14, p.50,101,106].

Let us return to our study of an observable $S = s_0 + \mathbf{s}$ whose possible ket-spinor states are characterised by points (α_0, α_1) in the finite dimensional Hilbert space $\mathbb{C}^2$. In our equivalent vantage point, our ket-spinor $|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+$ lives in $\mathbb{G}_3$. One of the magical properties of quantum mechanics is that if a measurment is taken of an observerable S in the normalized ket-state,

$$|\alpha\rangle \in \{\mathbb{G}_3 u_+\},$$

whose evolution satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = \big(-\frac{\hbar^2}{2m}\nabla^2 + V\big)|\alpha\rangle = H|\alpha\rangle,$$

the *probability* of finding the observable S in the normalized ket-state $|\beta\rangle$ is $|\langle\alpha|\beta\rangle|^2$, and the *outcome* of that measurment will leave it in the state $|\beta\rangle$.

Calculating $|\langle\alpha|\beta\rangle|^2$, we find that

$$\langle\alpha||\beta\rangle = 2\overline{\alpha}_0\beta_0\mathbf{m}^2\mathbf{n}^2u_+\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+u_+ = 2\frac{1}{\alpha_0\overline{\beta}_0}\mathbf{m}^2\mathbf{n}^2u_+\hat{\mathbf{a}}_+\hat{\mathbf{b}}_+u_+,$$

so that by associativity of the geometric product,

$$|\langle\alpha|\beta\rangle|^2 = \Big\langle\Big(\langle\beta||\alpha\rangle\Big)\Big(\langle\alpha||\beta\rangle\Big)\Big\rangle_{0+3} = \Big\langle\langle\beta|\Big(|\alpha\rangle\langle\alpha|\Big)|\beta\rangle\Big\rangle_{0+3}$$

$$= 2\langle\beta|\hat{\mathbf{a}}_+|\beta\rangle_{0+3} = \frac{1}{2}(1 + \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}).$$

From this we see that the probability of finding the particle in the state $|\beta\rangle = |\alpha\rangle$ is 1, whereas the probability of finding the particle in the state $|\beta\rangle = \sqrt{2}\alpha_0\mathbf{m}^2\hat{\mathbf{a}}_-u_+$ is 0.

Note that the expectation values of the observable $S = s_0 + \mathbf{s}$, with respect to the normalized eigen ket-spinors $\frac{|\mathbf{s}_\pm\rangle}{\sqrt{\langle\mathbf{s}_\pm|\mathbf{s}_\pm\rangle}} = \frac{\sqrt{2}\hat{\mathbf{s}}_\pm u_+}{\sqrt{\langle\mathbf{s}_\pm|\mathbf{s}_\pm\rangle}}$, are

$$\frac{\langle\mathbf{s}_\pm|S|\mathbf{s}_\pm\rangle}{\langle\mathbf{s}_\pm|\mathbf{s}_\pm\rangle} = s_0 \pm |\mathbf{s}|,$$

respectively, as would be expected.

The quantum mechanics of 2-component spinors in $\mathbb{C}^2$, or their equivalent as elements of the minimal ideal $\{\mathbb{G}_3u_+\}$ in $\mathbb{G}_3$, apply to a whole host of problems where *spin* $\frac{1}{2}$ particles are involved, as well as many other problems in elementary particle physics. Much effort has been devoted to the development of a *quantum computer*, built upon the notion of the *superposition* of the *quantum bits*

$$|0\rangle := \sqrt{2}u_+, \quad \text{and} \quad |1\rangle := \sqrt{2}\mathbf{e}_1u_+,$$

defined by the ket-spinor

$$|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1\mathbf{e}_1)u_+ = \alpha_0|0\rangle + \alpha_1|1\rangle.$$

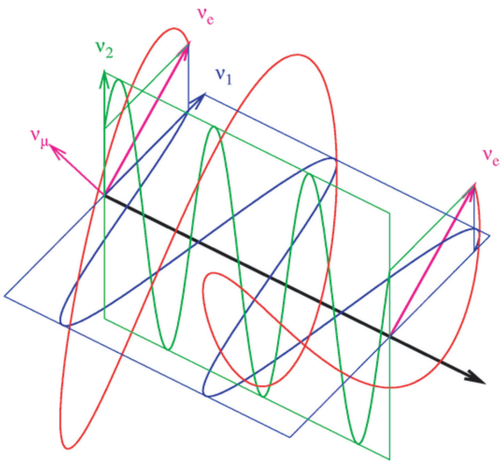

Figure 4: Neutrino oscillation between two flavor states, blue and green, and the superposition of those states, orange.

See [17] for a discussion of quantum bits using the spinor operator approach.

There has also recently been much excitement about the quantum phenomenum of *neutrino oscillation*, where a neutrino changes its *flavor* between three different types, *electron neutrinos* υ_ε, *muon neutrinos* υ_μ, and *tau neutrinos* υ_τ. This variation in flavor implies that neutrinos have *mass*, and therefore travel at *less* than the speed of light, rather than at the speed of light as was previously thought. The observable for a neutrino with three flavors would require a 3×3 Hermitian matrix, but in the cases of atmospheric neutrinos, or neutrinos propagating in a vacume, it has been found the oscillations are largely between only the electron and muon neutrino states, and therefore a 2 component spinor model is sufficient.

The general solution to the Schrödinger equation

$$i\hbar\frac{\partial|\alpha\rangle}{dt}=H|\alpha\rangle$$

for a time-independent Hamiltonian $H=s_0+\mathbf{s}$ of the form (45), (46), is particularly simple,

$$\begin{aligned}|\alpha\rangle &= \sqrt{2}e^{-\frac{iH}{\hbar}t}u_+ = \sqrt{2}e^{-i\frac{s_0}{\hbar}t}e^{-i\frac{|\mathbf{s}|\hat{\mathbf{s}}}{\hbar}t}u_+\\ &= \sqrt{2}e^{-i\frac{s_0}{\hbar}t}\big(\cos\frac{|\mathbf{s}|t}{\hbar}+i\hat{\mathbf{s}}\sin\frac{|\mathbf{s}|t}{\hbar}\big)u_+.\end{aligned}$$

Since $\mathbf{e}_3u_+=u_+$,

$$\big(\cos\frac{|\mathbf{s}|t}{\hbar}+i\hat{\mathbf{s}}\sin\frac{|\mathbf{s}|t}{\hbar}\big)u_+=\big(\mathbf{e}_3\cos\frac{|\mathbf{s}|t}{\hbar}+i\hat{\mathbf{s}}\mathbf{e}_3\sin\frac{|\mathbf{s}|t}{\hbar}\big)u_+,$$

where $i\hat{\mathbf{s}}\mathbf{e}_3=\hat{\mathbf{s}}\cdot(\mathbf{e}_1\wedge\mathbf{e}_2)=\mathbf{e}_3\times\hat{\mathbf{s}}$ is a unit vector in the xy-plane.

If we denote the electron neutrino by the state $|\upsilon_\varepsilon\rangle := |0\rangle$ and the muon neutrino by $|\upsilon_\mu\rangle := |i\hat{\mathbf{s}}\mathbf{e}_3\rangle$, then on the great circle of the Riemann sphere with the axis $\mathbf{e}_3 \times \hat{\mathbf{s}}$, the electron neutrino at the North Pole at time $t = 0$, will evolve into a muon neutrino at the South Pole at time $t = \frac{\pi\hbar}{2|\mathbf{s}|}$, and then back again, [18, 19, 20]. See Figure 4.

Acknowledgement

I thank Professor Melina Bork (UDLAP) for many helpful discussions about quantum mechanics, and the Universidad de Las Americas-Puebla (UDLAP) for many years of support.

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