ABSTRACT

SOBCZYK G.E., Polish Academy of Science, Current Adr: 109 Hillcrest Ave., Clemson, S.C. 29631. Geometric Structures in a Certain Banach Algebra. (125 pages. August 1974)

Let $\{\mathcal{L}(\mathcal{B}),\circ\}$ be the algebra of bounded linear operators on a Banach algebra $\{\mathcal{B},*\}$. By a biderivation in $\mathcal{B}$ we mean a transformation $\chi:\mathcal{L}(\mathcal{B})\to\mathcal{L}(\mathcal{B})$, which satisfies for all $F,G\in\mathcal{L}(\mathcal{B})$ and all $A,B\in\mathcal{B}$,

1. $\chi[F+G](A) = \chi[F](A) + \chi[G](A)$
2. $\chi[F\circ G](A) = \chi[F]\circ G(A) + F\circ\chi[G](A)$
3. $\chi[F(A)+G(B)] = \chi[F](A) + \chi[G](B)$
4. $\chi[F(A)*G(B)] = \chi[F](A)*G(B) + F(A)*\chi[G](B)$.

This work is primarily a study of biderivations called geometric structures in a certain real Banach algebra called geometric algebra. Based on this new concept new characterizations are given of such concepts from differential geometry as Integrability, Riemannian Curvature, Fields, and the Lie Bracket of Fields. In addition cogent new formulations of the Weyl Projective and Conformal Tensors are given which have rich geometric significance.

The introductory part of this work develops geometric algebra (which is an infinite dimensional Clifford algebra) into a powerful new tool for studying multilinear algebra. Examples of the utility of these methods are provided, including a direct algebraic way of inverting a non-singular linear operator and a simple new proof of the Cayley-Hamilton theorem.

GEOMETRIC STRUCTURES

IN A CERTAIN BANACH ALGEBRA

"Beauty is the first test ___
there is no permanent place in
the world for ugly mathematics."
___ G.H. Hardy

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CONTENTS

APPENDICES

0. Introduction

In this paper we characterize the local properties of a differentiable manifold by introducing the concept of a geometric structure. This concept makes it possible to study the local properties of a manifold in the framework of linear algebra. That this is indeed possible was noted by R. Sikorski in [23] and [24], where he does this in the language of modules. Unlike Sikorski, our setting is a certain Banach algebra called geometric algebra. A geometric algebra is a real graded linear space provided with a Clifford, or geometric product which has comprehensive geometric significance. The term 'geometric algebra' was first used by Clifford himself in [4].

Presently Clifford algebra finds a rather restricted use among mathematicians and physicists in spinor representations and the classification of orthogonal groups, and in the study of certain invariants of quadratic forms. See [22], [1], [20], and [19]. That Clifford algebra has in fact a much wider range of application was first recognized by D. Hestenes, who conceived it as a comprehensive language for the expression of geometrical ideas. In [13] Hestenes shows how the basic ideas of calculus find a direct expression in geometric algebra. In [12] and [14] he further demonstrates the conceptual and symbolic clarity basic equations of physics attain when expressed in this language. More recently in [15] he shows how the language leads to a new understanding of the geometrical significance of the Dirac equation. Additional references to the works of Hestenes along these lines can be found in [15]. The author in [25], and then jointly with Hestenes in [16], [17], and [18], continued the mathematical development of this language.

The present work is the natural outgrowth of the work begun in [13] and [25], and continued in [16], [17], and [18], but goes considerably beyond it. For instance, multivector differentiation is studied in its full generality for the first time, and the concept of a geometric structure is entirely new.

There are four chapters in part I:

Chapter one sets down the algebraic framework which in this work will be refered to as geometric algebra. One of the most striking algebraic features of this language is its powerful, and geometrically significant cancellation property. The projection of the whole geometric algebra onto a finite dimensional geometric subalgebra is discussed, and is of central importance to the ideas presented in this work. In addition various notational conventions are established.

Chapter two is concerned with topological aspects. In particular it is noted that we are dealing with a Banach algebra which is also a real Hilbert space. When we wish to emphasize topological properties of a geometric algebra, we shall sometimes refer to it as a geometric space. Additional notation and terminology are introduced for the study of differantiable functions of a multivector variable.

Chapter three is concerned with differentiation in a geometric space, in terms of which are defined the tangent map, the generalized gradient, and the adjoint map of a differentiable function of a multivector variable. The exact relationship between this language and the currently popular tensors and differential forms begins to become clear.

Chapter four narrows attention to the Banach algebra of bounded linear operators of the geometric space into itself. This limitation guarantees the existence of bounded adjoints and makes possible a clean exposition, which is yet sufficiently general to exibit the full power of what we call the theory of geometric structures, studied in part II. The close relationship between a linear map and its adjoint is studied, and it is shown how a non-singular linear map, called an outermorphism can be inverted directly in terms of its adjoint. In addition a new and simple proof of the Cayley-Hamilton theorem is given, which further reflects the richness of our algebraic framework. The methods used are also of particular interest because they suggest generalization to infinite dimensions.

Part II consists of six chapters, and its setting is the Banach algebra of bounded operators defined in chapter four.

In chapter five this Banach algebra is considered as a ring of bounded operators, and is used to generate a polynomial ring consisting of polynomials of bounded operators. A geometric structure is then defined in terms of these rings. Briefly, a geometric structure is a derivation on the polynomial ring which is at the same time a derivation on the ring of bounded operators. Integrable P-structures are then defined, and the problem of extending a structure which is given on a finite geometric subalgebra is discussed.

Chapter six shows how the notions of shape and curvature of a structure naturally arise in studying derivations of the projection operator. The implication of this is that integrability conditions of a manifold can be algebraically characterized by derivations of the projection operator.

In chapter seven intrinsic structure is defined in terms of the given (extrinsic) structure. Although this could be turned around, doing so would result in a much more complicated theory in which the shape operator, which involves a normal component, would have to be given up. However, the intrinsic structure does have important properties of its own, and these are considered in this chapter.

In chapter eight scalar valued linear mappings called forms are studied. Each such form is the unique representation of a multivector called the body or field of a form. Since multivectors are multiplied by the rules of geometric algebra, there is no reason to define any operations, such as an exterior product, or an exterior derivative, on forms. The gradient of a field is studied, and a generalized Lie bracket of fields is considered.

In chapter nine general properties of two structures which are related by an outermorphism are studied. Special attention is given to how the divergence and Lie brackets of related fields transform between related structures.

Chapter ten considers structures which are projectively or conformally related. Particularly significant is the simple algebraic way in which this relatedness is expressed, with the noticable absence of differential equations. Simple computations of the equivalents of the Weyl Projective and Conformal tensors are given. These serve to illustrate the full utility and directness of this theory, dealing only in terms of the relevant geometric quantities themselves.

There are two appendices to this work.

Appendix A discusses basic formulas for differentiation which have been used in this work. The notion of the signature of a multivector is defined, and tables of such signatures are given for $n = 2, 3$, and 4.

Appendix B suggests how a differentiable manifold can be defined in terms of geometric structures. Roughly a manifold is a connected point set in geometric 1-space, with an attached geometric structure at each point. A brief discussion of integration on manifolds is included to give some idea of the general theory of integration which is possible.

A list of symbols used, and where they first appear, is attached after appendix B.

This work is necessarily incomplete, but it is hoped that it throws sufficient light on a virtually unused language, whose full generality and conceptual clarity has for years lain hidden from view.

A few more remarks about references are necessary. Most helpful in the beginning stages of this work was J.H.C. Gerretsen's discussion of the Weyl Projective and Conformal Tensors in [6]. Insight into the different ways in which the notion of curvature of a manifold can arise was provided by M. Spivac's very readable books [26]. References [3] and [21] were most helpful in establishing the functional analytic framework of this work. The remaining references [2], [5], [7], [8], [9], [10], [11], [27], and [28] were used for the most part in making comparisons with other approaches.

PART I

BASICS

1. Algebraic Framework

Let $\mathcal{G} = \sum_{r \geq 0} \mathcal{G}^r$ be a real graded linear space with a Clifford geometric product satisfying for $A, B, C \in \mathcal{G}$,

$$(AB)C = A(BC), \quad \text{the associative law,} \tag{1.1}$$

and

$$\begin{aligned} A(B+C) &= AB+AC, \\ (B+C)A &= BA+CA, \end{aligned} \quad \text{the distributives laws.} \tag{1.2}$$

Elements of $\mathcal{G}$ are called multivectors, and in this work will always be denoted by capital letters $A, B, C,$.

Each multivector A can always be written as

$$A = \sum_{r \geq 0} A_r \quad , \tag{1.3a}$$

where

$$A_r \equiv \langle A \rangle_r \in \mathcal{G}^r \quad . \tag{b}$$

Definition (1.3b) introduces the operation $\langle \ \rangle_r$ of taking the r-vector part of a multivector. Elements of $\mathcal{G}^r$ are called r-vectors, and in this work will always be denoted by capital letters $A_r, B_r, C_r, \cdots,$ subscripted by r, the degree of the multivectors. The terms scalars, vectors, and bivectors are reserved for elements of $\mathcal{G}^0$, $\mathcal{G}^1$, and $\mathcal{G}^2$, respectively. In addition we will use the special symbolism $\alpha, \beta, \gamma, \cdots,$ and $a, b, c, \cdots,$ when refering exclusively to scalars or vectors respectively.

Two vectors $a, b,$ are said to be orthogonal if

$$ab = -ba \quad . \tag{1.4}$$

If

$$A_r = a_1 a_2 \cdots a_r \; , \tag{1.5}$$

where the vectors a_i are mutually orthogonal, then $A_r \in \mathcal{G}^r$ and is said to be a simple r-vector.

The geometric algebra $\mathcal{G}$ has the additional distinguishing features:

$$\mathcal{G}^0 \equiv \{\sigma \mid \sigma \text{ is a real number}\} \quad . \tag{1.6}$$

If $a \in \mathcal{G}^1$; then

$$aa = a^2 \geq 0 \text{ , and } a^2 = 0 \text{ iff } a = 0 \; . \tag{1.7}$$

Finally if $A_r \in \mathcal{G}^r$, then

$$A_r = \sum_i A_{r_i} \quad , \tag{1.8}$$

where each A_{r_i} is a simple r-vector. In words (1.8) says that each r-vector can be written as a sum of simple r-vectors.

a) Outer and Inner Products

The geometric algebra $\mathcal{G}$ is not an exterior algebra, but an exterior, or outer product can be defined in terms of the geometric product. For $A_r \in \mathcal{G}^r$, and $B_s \in \mathcal{G}^s$, define $A_r \wedge B_s \in \mathcal{G}^{r+s}$ by

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s} \quad . \tag{1.9}$$

Complementary to the outer product is an interior or inner product defined by

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|} \in \mathcal{G}^{|r-s|} \, . \tag{1.10}$$

It is natural to extend the definitions of the inner and outer products distributively to all of $\mathcal{G}$, and thus they will satisfy for all $A,B,C \in \mathcal{G}$,

$$A \cdot (B+C) = A \cdot B + A \cdot C \tag{1.11a}$$

and

$$A \wedge (B+C) = A \wedge B + A \wedge C \, . \tag{b}$$

Note that for $\alpha \in \mathcal{G}^0$, and $B \in \mathcal{G}$,

$$\alpha \cdot B = \alpha B = \alpha \wedge B \, , \tag{1.12}$$

ie., the inner, outer, and geometric products are in this case equivalent.

In terms of the outer product, we now introduce the <u>principle of multivector decomposition</u>: Let $\mathcal{J}(A)$ be a proposition about multivectors $A \in \mathcal{G}$. If

i) $\mathcal{J}(A)$ is true for all scalars and 1-vectors in $\mathcal{G}$.

ii) $\mathcal{J}(A+B)$ is true whenever $\mathcal{J}(A)$ and $\mathcal{J}(B)$ are true. (1.13)

iii) $\mathcal{J}(a \wedge A)$ is true for each vector $a \in \mathcal{G}^1$ whenever $\mathcal{J}(A)$ is true.

Then $\mathcal{J}(A)$ is true for all $A \in \mathcal{G}$. This rule is a simple consequence of (1.3a) and (1.8), and is helpful in establishing general properties about multivectors.

Finally we define an <u>absolute inner product</u> of multivectors A and B by

$$A \odot B = \langle AB \rangle_0 \, , \tag{1.14a}$$

and an <u>abreviated inner product</u> of A and B by

$$A{:}B = A \cdot B - \langle A \rangle_0 B - A \langle B \rangle_0 + \langle A \rangle_0 \langle B \rangle_0 \, . \tag{b}$$

The absolute and abreviated inner products are introduced because they greatly simplify the expression of general properties of multivectors. In order to avoid later possible confusion, we immediately give the close relationships between the three kinds of inner products.

For $\alpha \in \mathcal{G}^0$ and $A, B \in \mathcal{G}$,

$$\alpha{:}B = 0 \,, \tag{1.15a}$$

and

$$A{:}B = A \cdot B \,, \quad \text{if} \quad \langle A \rangle_0 = 0 = \langle B \rangle_0 \,. \qquad \text{b)}$$

$$A \odot B = \sum_{r \geq 0} \langle A \rangle_r \cdot \langle B \rangle_r = \sum_{r \geq 0} A_r \cdot B_r \,, \tag{1.16a}$$

and

$$\langle A \rangle_r \odot B = \langle A \rangle_r \cdot \langle B \rangle_r = A \odot \langle B \rangle_r \,. \qquad \text{b)}$$

The relationship (1.15) is a trivial consequence of the definition (1.14b), and (1.16) is a consequence of the definition (1.14a) and the more general algebraic identity (1.21) given in the next section.

b) Distinquishing Algebraic Identities

The following list of identities should establish the complementary roles played by the inner and outer products. Those identities, and others which can be derived directly from them, will be used throughout this work. The proofs of these identities can be established by inductive arguments on the degree of the multivectors, but they will not be given here. Some of the proofs can be found in [12].

For a vector $a \in \mathcal{G}^1$, and a multivector $B \in \mathcal{G}$,

$$aB = a{:}B + a \wedge B \,, \tag{1.17a}$$

$$a : B_r = \frac{1}{2}\left[aB_r + (-1)^{r+1} B_r a \right] \text{ , and} \qquad \text{b)}$$

$$a \wedge B_r = \frac{1}{2}\left[aB_r + (-1)^{r} B_r a \right] \text{ .} \qquad \text{c)}$$

For $a \in \mathcal{G}^1$, $B_r \in \mathcal{G}^r$, and $C \in \mathcal{G}$,

$$a : (B_r \wedge C) = (a : B_r) \wedge C + (-1)^r B_r \wedge (a : C) \text{ , and} \qquad (1.18a)$$

$$B_r \cdot (a \wedge C_s) = (B_r : a) \cdot C_s + (-1)^r a \wedge (B_r \cdot C_s) \text{ , for } r \le s \text{ .} \qquad \text{b)}$$

Finally,

$$(A_r \wedge B_s) \cdot C_t = A_r \cdot (B_s \cdot C_t) \text{ , for } r+s \le t \text{ ,} \qquad (1.19a)$$

and for the vectors a_i and b_j ,

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_r) \cdot (b_r \wedge b_{r-1} \wedge \cdots \wedge b_1) = \det(a_i \cdot b_j) \text{ ,} \qquad \text{b)}$$

where "det" means "determinant".

Commuting the factors of inner and outer products can be accomplished by using the rules

$$A_r \cdot B_s = (-1)^{r(s-1)} B_s \cdot A_r \text{ , and} \qquad (1.20a)$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r \text{ .} \qquad \text{b)}$$

The geometric product of an r-vector A_r and an s-vector B_s can be decomposed into various multivector parts by using the identity

$$A_r B_s = A_r : B_s + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s-2} + A_r \wedge B_s \text{ .} \qquad (1.21)$$

Identity (1.17a) is a special case of this when $r=1$. In the next section we discuss the special case when $r=2$.

c) The Commutator Product

Let B_2 be a bivector, and $A_r = \langle A \rangle_r$. Then using (1.21) and (1.20) we find that for each r,

$$B_2 A_r = B_2 : A_r + \langle B_2 A_r \rangle_r + B_2 \wedge A_r ,$$

and (1.22)

$$A_r B_2 = B_2 : A_r - \langle B_2 A_r \rangle_r + B_2 \wedge A_r$$

from which it follows that

$$[B_2, A_r] \equiv B_2 A_r - A_r B_2 = 2 \langle B_2 A_r \rangle_r . \tag{1.23}$$

More generally we define

$$[B , A] \equiv BA - AB \tag{1.24}$$

to be the <u>commutator product</u> of B and A.

From the easily established distributive rule

$$[B , AC] = [B , A] C + A [B , C] , \tag{1.25a}$$

with the help of (1.23) and (1.22), follows the non-trivial special cases

$$[B_2 , A \cdot C] = [B_2 , A] \cdot C + A \cdot [B_2 , C] , \qquad b)$$

and

$$[B_2 , A \wedge C] = [B_2 , A] \wedge C + A \wedge [B_2 , C] . \qquad c)$$

The commutator product is not associative, but does satisfy

$$[A, [B,C]] + [C, [A,B]] + [B, [C,A]] = 0 , \tag{1.26}$$

which is known as the Jacobi identity.

Equation (1.23) implies that the space $\mathcal{G}^2$ of bivectors is closed under the commutator product. It follows that under the commutator product the bivectors make up a Lie Algebra, which is, as is well known, the Lie algebra of rotations. The so-called structure equations for this Lie algebra can be written in the form

$$\frac{1}{2}\left[a\wedge b\ ,\ c\wedge d\right] = (b\cdot c)\ a\wedge d - (b\cdot d)\ a\wedge c + (a\cdot d)\ b\wedge c - (a\cdot c)\ a\wedge d \quad , \tag{1.27}$$

where a, b, c, d are vectors in $\mathcal{G}^1$. Equation (1.27) is easily derived with the help of (1.25c), (1.23) and (1.18a).

d) Reversal and Norm Operations

Suppose that A_r is a simple r-vector, and can be factored into $A_r = a_1 a_2 \cdots a_r$. Then we say that

$$A_r^{\dagger} = a_r a_{r-1} \cdots a_1 \tag{1.28}$$

is the <u>reversal</u> of A_r, and that

$$|A_r| \equiv \sqrt{A_r A_r^{\dagger}} = \sqrt{A_r \odot A_r^{\dagger}} \tag{1.29}$$

is the <u>magnitude</u> of A_r. In particular for a scalar α, $\alpha \equiv \alpha^{\dagger}$ and $|\alpha| = \sqrt{\alpha^2}$, and for a vector b, $b^{\dagger} \equiv b$ and $|b| = \sqrt{b^2}$, as would be expected. Both the reversal and the magnitude of a simple r-vector are independent of the particular factoring into vectors.

Since, as a consequence of (1.8), any multivector A can be written as a sum of simple multivectors, it is natural to define the reversal $A^{\dagger}$ of A to be the sum of the reversals of all its simple parts. (new paragraph) In terms of the absolute inner product (1.14a), and the reversal operation, it is now possible to define the <u>norm</u> of an arbitrary element $A \in \mathcal{G}$ by the equation

$$\|A\|^2 = A \odot A^\dagger \ . \tag{1.30}$$

It can be taken as a final axiom that the geometric algebra $\mathcal{G}$ is both closed and complete under the norm operation. Ie.,

$$\|A\| < \infty \ , \quad \text{for each } A \in \mathcal{G} \ , \tag{1.31a}$$

and if $\{{}_kA \mid k>0\}$ is a Cauchy sequence in $\mathcal{G}$, with respect to the given norm, then there exists an $A \in \mathcal{G}$ such that

$$\lim_{k\to\infty} \|{}_kA - A\| = 0 \ . \qquad \text{b)}$$

Note that the index k is being used on the left of A, since when it is used on the right it denotes the degree of the multivector.

The most important properties of the reversal operation are now given, and can easily be established from the definition (1.28) and (1.20b). For $A,B \in \mathcal{G}$,

$$\langle A\rangle_r^\dagger = \langle A^\dagger\rangle_r = (-1)^{\frac{1}{2}r(r-1)}\langle A\rangle_r \tag{1.32a}$$

$$(AB)^\dagger = B^\dagger A^\dagger \ , \text{ and } \ (A+B)^\dagger = A^\dagger + B^\dagger \ , \qquad \text{b)}$$

$$(A^\dagger)^\dagger = A \quad , \text{ and} \qquad \text{c)}$$

$$\langle AB_rC\rangle_r = \langle C^\dagger B_r A^\dagger\rangle_r \ . \qquad \text{d)}$$

The properties (1.32a,b,c) establish the reversal operation as an involution of the geometric algebra $\mathcal{G}$. Property (1.32d) is a simple but useful consequence of (1.32a,b).

Finally we summarize important properties of the norm operation, which also can be easily verified. If A_r is simple, then the magnitude and norm of A_r are identical.

Ie.,

$$\| A_r \| = | A_r | \quad . \tag{1.33}$$

The norm is positive definite: For $A \in \mathcal{G}$,

$$\| A \| \geq 0 \text{ , and } \| A \| = 0 \text{ iff } A = 0 \text{ .} \tag{1.34a}$$

For $A, B \in \mathcal{G}$,

$\| A+B \| \leq \| A \| + \| B \|$; and   b)

$\| AB \| \leq \| A \| \| B \|$. in $\mathcal{E}_4$ $A = E_1 + IE_2 = B$, $A^2 = AB = B^2 = -2E_3$; $\| A \| =$ c)

Furthermore,

$\| AB \| = \| A \| \| B \|$ if A or B is simple.   d)

The properties (1.34) guarantee that the geometric algebra $\mathcal{G}$ is a real Banach algebra (see [21; 2]), since we have already assumed that $\mathcal{G}$ is complete (1.31). In addition since the norm is given by the absolute inner product (1.30), $\mathcal{G}$ is a real Hilbert space. The topological properties of $\mathcal{G}$ will be discussed in chapter 2.

e) The Inverse Operation and Cancellation

If A_r is a simple non-zero r-vector, then it has an inverse. It is given by

$$A_r^{-1} \equiv | A_r |^{-2} A_r^{\dagger} \tag{1.35}$$

More generally a multivector B is said to have an <u>Inverse</u> B^{-1} , provided

$$BB^{-1} = 1 = B^{-1}B \quad . \tag{1.36}$$

The associative law (1.1) guarantees that when inverses exist, they are unique.

We can now give the very powerful and geometrically significant cancellation law of geometric algebra: If A is a multivector for which A^{-1} exists, then

$$AB = AC \quad \text{iff} \quad B = C \quad . \tag{1.37}$$

That this identity is by no means trivial should be clear from identity (1.21). In fact this law is a most distinquishing feature of this language, and we challenge any interested reader who doubts this statement to formulate its equivalent in his choice of languages.

f) Finite Subalgebras and Projections

The geometric algebra need not be finite dimensional, but to each unit simple n-vector $I \in \mathcal{G}^n$ there corresponds a unique 2^n-dimensional subalgebra $\mathcal{D} = \mathcal{D}(I)$ defined by

$$\mathcal{D} = \{ A \in \mathcal{G} \mid I^{\dagger} \cdot (I \cdot A) = A \} \quad . \tag{1.38}$$

The finite geometric algebra $\mathcal{D}$ is assigned the orientation of I, and I is called the pseudoscalar of $\mathcal{D}$. The algebra $\mathcal{D}$ can be expressed as a finite sum of graded spaces $\mathcal{D}^k = \mathcal{D}^k(I)$. Thus we write

$$\mathcal{D} = \sum_{k=0}^{n} \mathcal{D}^k \quad , \tag{1.39a}$$

where

$$\mathcal{D}^k = \{ A_k \in \mathcal{G}^k \mid I^{\dagger} \cdot (I \cdot A_k) = A_k \} \quad , \tag{b}$$

and $\dim[\mathcal{D}^k] = \binom{n}{k}$. If we define the projection operation onto the algebra $\mathcal{D}$ by

$$P(A) = I^{\dagger} \cdot (I \cdot A) \quad , \text{ for each } A \in \mathcal{G} \, , \tag{1.40}$$

then (1.38) and (1.39b) can be expressed more simply by

$P[\mathcal{G}] = \mathcal{D}$, and $P[\mathcal{G}^k] = \mathcal{D}^k$.

The role played by the projection operator P in this work is of central importance. the following are its most important properties. Let $A, B \in \mathcal{G}$. Then

$$P^2(A) = P(A) , \tag{1.41a}$$

$$P(A+B) = P(A) + P(B) , \qquad b)$$

$$P(A \wedge B) = P(A) \wedge P(B) , \text{ and} \qquad c)$$

$$P(AB) = A\,P(B) , \text{ if } P(A) = A . \qquad d)$$

The properties (1.41) follow easily from the definition (1.40) and algebraic identities.

Whereas P is the projection operation onto the finite subalgebra $\mathcal{D} = \mathcal{D}(I)$, it is also expedient to have a projection operator Q onto the whole geometric algebra $\mathcal{G}$. Thus we define

$$Q(A) = A \text{ for all } A \in \mathcal{G} . \tag{1.42}$$

Clearly Q is the identity mapping on $\mathcal{G}$. It is also clear that Q satisfies the properties (1.41), but note that it isn't possible to give an algebraic definition like (1.40) for Q, unless $\mathcal{G}$ is itself finite dimensional. This is because a pseudoscalar exists only for finite geometric algebras. However, in the next section we shall express Q as the limit of a sequence $\{P_n\}$ of finite projection operators.

g) Basis, Reciprocal Basis, and Summation Conventions

We first restrict our attention to a finite subalgebra

$\mathfrak{D} = \mathfrak{D}(I)$ of $\mathcal{G}$. Let $\{a_i \mid 1 \leq i \leq n\}$ be a basis of $\mathfrak{D}^1$ (recalling definition (1.39)). Then immediately an r-vector basis $\{a_{i_{\bar{r}}} \mid 1 \leq i_{\bar{r}} \leq n\}$ can be written down for $\mathfrak{D}^r$. It is defined by

$$a_{i_{\bar{r}}} = a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r} \quad , \text{where } 1 \leq i_{\bar{r}} \leq n \tag{1.43}$$

by which we mean $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Of course the $\dim \{a_{i_{\bar{r}}}\} = \binom{n}{r}$.

Reciprocal to the basis $\{a_i\}$ is the basis $\{a^j\}$, which is uniquely determined by the relations

$$a_i \cdot a^j = \delta_i^j \quad \text{for } 1 \leq i, j \leq n . \tag{1.44}$$

The reciprocal r-basis of $\{a_{i_{\bar{r}}}\}$ is then $\{a^{j_{\bar{r}}}\}$, where

$$a^{j_{\bar{r}}} \equiv a^{j_r} \wedge a^{j_{r-1}} \wedge \cdots \wedge a^{j_1} \quad , \tag{1.45a}$$

and it satisfies the relations

$$a_{i_{\bar{r}}} \cdot a^{j_{\bar{r}}} = \delta_{i_{\bar{r}}}^{j_{\bar{r}}} \quad . \tag{b}$$

The exact meaning of the symbol $\delta_{i_{\bar{r}}}^{j_{\bar{r}}}$ should be evident from its usage in (1.45b) .

Now let $B_r \in \mathfrak{D}^r$. The general summation convention used in this paper can be clearly illustrated by writing B_r in terms of the basis $a_{i_{\bar{r}}}$. Thus

$$B_r \cdot a^{i_{\bar{r}}} \, a_{i_{\bar{r}}} \equiv \sum_{1 \leq i_{\bar{r}} \leq n} \left(B_r \cdot a^{i_{\bar{r}}}\right) a_{i_{\bar{r}}} = B_r \quad . \tag{1.46a}$$

For $r = 1$, this reduces to the usual summation convention

$$b \cdot a^i a_i = \sum_{1 \leq i \leq n} (b \cdot a^i) a_i = b \quad . \qquad b)$$

We now turn our attention to the whole geometric algebra $\mathcal{G}$. If we assume, and we will always do so in this work, that $\mathcal{G}^1$ is a (real) separable Hilbert space, then we are guaranteed the existence of a countable maximal orthonormal set $\{p_i \mid 1 \leq i < \infty\}$. (See [3; 354] for details about Hilbert spaces used here.) We shall call the set $\{p_i\}$ a standard basis of $\mathcal{G}^1$. The basis $\{p_i\}$ orients the space $\mathcal{G}$, and since it is orthonormal, it is identical to its reciprocal basis $\{p^j\}$. The standard basis' $\{p_i\}$ and $\{p^j\}$ can now be used to generate the standard r-basis

$$\{p_{i_{\bar{r}}} \mid 1 \leq i_{\bar{r}} < \infty\} \quad , \qquad (1.47a)$$

and reciprocal r-basis

$$\{p^{j_{\bar{r}}} \mid 1 \leq j_{\bar{r}} < \infty\} \, , \qquad b)$$

of each grade $\mathcal{G}^r$ of $\mathcal{G}$, in exactly the same way as in the finite dimensional case just discussed. In terms of these basis' each $B_r \in \mathcal{G}^r$ can be written

$$B_r = \sum_{1 \leq i_{\bar{r}} < \infty} (B_r \cdot p^{i_{\bar{r}}}) p_{i_{\bar{r}}} \equiv B_r \cdot p^{i_{\bar{r}}} p_{i_{\bar{r}}} \quad , \qquad (1.48)$$

where in this case we are summing over a possible infinite number of indicies.

We have already noted that the basis $\{p_i\}$ is identical to its reciprocal basis $\{p^j\}$. In the case that we are given an arbitrary countable basis $\{a_i\}$ of $\mathcal{G}^1$, it may happen that a reciprocal basis to $\{a_i\}$ does not exist. This case will be discussed in section 3d).

Finally the <u>standard sequence of finite subalgebras</u> $\{\mathcal{D}(I_n)\}$, with respect to the standard basis $\{p_n\}$, is Generated by specifying the sequence

$$I_1 = p_1 , \ I_2 = p_1 p_2 , \ldots , \ I_n = p_1 p_2 \cdots p_n , \ldots \qquad (1.49)$$

of simple multivectors. It can be seen that for each n the pseudoscalar I_n of $\mathcal{D}(I_n)$ inherits the orientation of the standard basis $\{p_n\}$. An algebraic way of saying this is that

$$I_{n-1}^{\dagger} I_n = p_n , \text{ for each } n > 1 . \qquad (1.50)$$

We shall denote by P_n the projection operator onto $\mathcal{D}(I_n)$, for $n \geq 1$, and say that $\{P_n\}$ is the <u>standard sequence of finite projection operators</u>. It is now possible to express the projection operator Q, defined in the last section, by

$$Q = \lim_{n \to \infty} P_n , \qquad (1.51a)$$

since for each $A \in \mathcal{G}$,

$$\lim_{n \to \infty} \| P_n(A) - A \| = 0 . \qquad b)$$

2. Topological Framework

The norm (1.30) defined on $\mathcal{G}$ makes it meaningful to talk about the topology on $\mathcal{G}$ with respect to this norm. We shall call the geometric algebra $\mathcal{G}$ a <u>geometric space</u> when we wish to emphasize its topological properties.

Each grade $\mathcal{G}^k$ of $\mathcal{G}$ is a real separable Hilbert space with the norm (1.30) given by the absolute inner product (1.13), when restricted of course to $\mathcal{G}^k$. A countable orthonormal basis for each $\mathcal{G}^k$ is given by (1.47). The whole geometric space $\mathcal{G}$ is the topological sum of the real separable Hilbert spaces $\mathcal{G}^k$, with the additional property that

$$\bigcap_{k=0}^{\infty} \mathcal{G}^k = 0 \text{, the zero element of } \mathcal{G}. \qquad (2.1)$$

Thus all of the "zero points" of the spaces $\mathcal{G}^k$ have been identified with the zero point of $\mathcal{G}$. The geometric space $\mathcal{G}$ is itself a real separable Hilbert space with the norm (1.30) given by the absolute inner product (1.13). A countable orthonormal basis for $\mathcal{G}$ is given by taking the countable union of the countable basis' given for each $\mathcal{G}^k$ in (1.47).

a) Continuoss Functions

Let $Z' = F(Z)$ be a function whose domain and range are in $\mathcal{G}$. The function F is said to be continuous at a point of its domain, provided it is continuous with respect to the norm of $\mathcal{G}$. Functions of several multivector variables will also be considered. All the usual properties of continuous functions hold for continuous functions of multivector variables.

Two important functions defined and continuous on all of $\mathcal{G}\times\mathcal{G}$ are

$$F({}_1Z, {}_2Z) = {}_1Z + {}_2Z \quad \text{, and} \tag{2.2a}$$

$$G({}_1Z, {}_2Z) = {}_1Z\,{}_2Z \quad . \tag{b}$$

The continuity of F and G is guaranteed by the properties (1.34) of the norm.

Let $\mathcal{C}(Z)$ denote the set of all continuous functions F at the point $Z \in \mathcal{G}$. The range and domain of F will always be assumed to be in $\mathcal{G}$. The set $\mathcal{C}(Z)$ is a very large set of functions, so it is useful to signal out smaller subsets of $\mathcal{C}(Z)$ which have additional properties.

Call $\mathcal{C}^{(r,s)}(Z)$ the set of (r,s)-homogeneous continuous functions at Z. A function $F \in \mathcal{C}^{(r,s)}(Z)$ if and only if $F \in \mathcal{C}(Z)$ and

$$F(\langle Z\rangle_r) = \langle F(Z)\rangle_s \quad . \tag{2.3a}$$

We also require that the property (2.3a) be valid for all points contained in some open set containing the point Z. We will further say that F is λ-homogeneous at Z if

$$F \in \mathcal{C}^{(r,r+\lambda)}(Z) \quad \text{for each } r \text{ such that } r+\lambda \geq 0 \; . \tag{b}$$

In the case that $\lambda = 0$, we will also say that F is homogeneous at Z.

Another important subset of $\mathcal{C}(Z)$ is the set $\mathcal{C}_P(Z)$ of continuous P-functions at Z, where P is the projection operator onto the finite geometric subspace $\mathcal{D}$. We will say that $F \in \mathcal{C}_P(Z)$ if and only if $F \in \mathcal{C}(Z)$ and

$$F \equiv FP \tag{2.4a}$$

at all points in some open set containing the point Z. If in addition to (2.4a),

$$F \equiv P'F \qquad \text{b)}$$

at all points in some open set containing the point Z, we will say that $F \in {}_{P'}\mathcal{C}_P(Z)$, where P' is the projection operator onto a second finite geometric subspace $\mathfrak{D}'$. The set ${}_{P'}\mathcal{C}_P(Z)$ is called the set of continuous (P',P)-functions at Z.

Of course we can also talk about the set ${}_{P'}\mathcal{C}_P^{(r,s)}(Z)$ of (r,s)-homogeneous continuous (P',P)-functions at the point Z, and various other combinations of the above notations.

In the next section we discuss the subset of functions of $\mathcal{C}(Z)$ which are linear.

b) Linear Functions

A function $F : \mathcal{G} \to \mathcal{G}$ is said to be linear provided for all $A, B \in \mathcal{G}$, and $\alpha, \beta \in \mathcal{G}^0$,

$$F(\alpha A + \beta B) = \alpha F(A) + \beta F(B) \qquad (2.5)$$

If in addition F is continuous at some point Z, then F is a continuous linear operator on $\mathcal{G}$. Let $\mathcal{L}$ denote the set of all such operators on $\mathcal{G}$. It follows that $\mathcal{L}$ is a subset of $\mathcal{C}(Z)$ for each $Z \in \mathcal{G}$.

It is natural to make $\mathcal{L}$ into a normed linear space by defining

$$\|F\| = \sup_{A \in \mathcal{G}} \frac{\|F(A)\|}{\|A\|} \quad \text{for each } F \in \mathcal{L}. \qquad (2.6)$$

There is an extensive literature on the study of normed linear spaces of continuous (and hence bounded) linear operators on a Hilbert space. A basic property of this theory, which will be much used in this work, is that a linear operator F on a Hilbert space $\mathcal{G}$ is bounded if and only if it has a linear <u>adjoint</u> $F^{\dagger}$ defined by

$$F(A) \odot B = A \odot F^{\dagger}(B) \quad \text{for all } A,B \in \mathcal{G} \,. \tag{2.7}$$

Furthermore when $F^{\dagger}$ exists, it is unique and also bounded. See [3; 355] and [21; 205].

A function $F : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ given by $F(A,B)$, is said to be <u>bilinear</u> on $\mathcal{G}$ if F is linear in both of the variables A and B. By ${}^{2}\mathcal{L}$ we shall mean the normed linear space of all continuous (in each variable) bilinear functions on $\mathcal{G}$. The norm on ${}^{2}\mathcal{L}$ is given by

$$\|F\| = \sup_{A,B \in \mathcal{G}} \frac{\|F(A,B)\|}{\|A\|\|B\|} \,. \tag{2.8}$$

By the adjoint of a bilinear function $F \in {}^{2}\mathcal{L}$, we shall mean the unique function $F^{\dagger} \in {}^{2}\mathcal{L}$ which satisfies for all $A,B,C \in \mathcal{G}$,

$$F(A,B) \odot C = A \odot F^{\dagger}(B,C) \,. \tag{2.9}$$

It is natural to extend the notation used to talk about subsets of continuous functions to subsets of linear functions as well. Thus by $\mathcal{L}_P^{(r,s)}$ we mean the normed subspace of (r,s)-homogeneous, bounded, linear P-functions of $\mathcal{G}$. In addition, we shall use the symbol $\mathcal{L}^{o}$ to denote the important subset of $\mathcal{L}$ consisting of all scalar valued bounded linear <u>forms</u> on $\mathcal{G}$.

c) Decomposition of Multivector Variables

A variable $Z \in \mathcal{G}$ can always be decomposed into its homogeneous multivector parts by

$$Z = \sum_{k \geq 0} \langle Z \rangle_k \equiv Z_0 + Z_1 + Z_2 + \cdots \quad , \qquad (2.10a)$$

where it will always be understood that $Z_k \equiv \langle Z \rangle_k$. In addition the special names

$$\zeta \equiv Z_0 \quad , \text{ and } \quad z = Z_1 \quad , \qquad b)$$

will be adopted for the scalar and vector parts respectively. An illustration of the usage of this notation is provided by considering an (r,s)-homogeneous function F . We can express the property (2.3a) of F by writing

$$Z'_s = F(Z_r) \equiv \langle F(\langle Z \rangle_r) \rangle_s \quad .$$

The primed s-variable Z'_s is the value of F at the r-vector variable Z_r .

Operating on both sides of (2.10a) with P gives the corresponding decomposition of the variable $V \equiv P(Z)$, of the finite (2^n-dimensional) geometric subspace $\mathcal{D} \subset \mathcal{G}$. Thus

$$V = \sum_{0 \leq k \leq n} \langle V \rangle_k = V_0 + V_1 + \cdots + V_n \quad . \qquad (2.11a)$$

Again we adopt the special notation

$$\tau \equiv V_0 \quad \text{and} \quad v \equiv V_1 \quad , \qquad b)$$

for the scalar and vector parts respectively. In the space $\mathcal{D}$, the variable V_k has $\binom{n}{k}$ degrees of freedom.

If a t-vector variable $Z_t \in \mathcal{G}^t$, or $V_t \in \mathcal{D}^t$, can be written as

$$Z_t = Z_r \wedge Z_s \quad , \text{ or } \quad V_t = V_r \wedge V_s \quad , \tag{2.12}$$

then it is said to be decomposable into the outer product of an r- and s-vector variables. If Z_k is a simple k-vector variable, then it can be written as the outer product of k vector variables. In particular, if for the k vector variables $z_1, z_2, \cdots, z_k \in \mathcal{G}^1$,

$$Z_k = z_{\bar{k}} \equiv \frac{1}{k!} z_1 \wedge \cdots \wedge z_k \quad , \tag{2.13a}$$

we say that the variable Z_k can be simplicially decomposed into the simplicial variable $z_{\bar{k}}$. Similarily we will write

$$v_{\bar{r}} \equiv \frac{1}{r!} v_1 \wedge \cdots \wedge v_r = P(z_{\bar{r}}) \quad , \qquad \text{b)}$$

for the simplicial variable of $\mathcal{D}^r$. Also note the conventions

$$Z_o = z_{\bar{o}} = \mathfrak{Z} \quad , \text{ and } \quad V_o = v_{\bar{o}} = \gamma \; . \qquad \text{c)}$$

3. Differentiation

a) The Tangent Map

Suppose that a function F is defined in some open set containing the point $Z \in \mathcal{G}$. We will say that F is differentiable at Z provided there exist a linear transformation $\dot{F}(A)$ on $\mathcal{G}$ with the property that for each $A \in \mathcal{G}$,

$$\lim_{t \to 0} \left\| \frac{F(Z + tA) - F(Z)}{t} - \dot{F}(A) \right\| = 0 \quad . \tag{3.1a}$$

Classically $\dot{F}(A)$, when it exists, is known as the Frechet derivative of F at Z, but we shall call $\dot{F}(A)$ the Tangent map of F at Z. Several notations will be used to represent the tangent map of F; each has its respective advantages. They are

$$\dot{F}(A) = \dot{F}(Z;\, A) = A \odot \nabla_Z F(Z) = A \odot \nabla F \, . \qquad \text{b)}$$

The notation $\dot{F}(Z;\, A)$ emphasizes that the tangent map is at the point Z. The notation $A \odot \nabla_Z F(Z)$ expresses the tangent map as an A-(directional) derivative of F at Z.

Two well known properties of Fréchet derivatives on a normed linear space will now be given. See [3; 264] for details. The tangent map $\dot{F}(A)$ when it exists is unique, and it is bounded if and only if F is continuous at Z. It follows that if $\dot{F}(A)$ exists, and is a (P', P)- map, then $\dot{F}(A)$ is bounded, and $F(Z)$ is continuous at Z.

In this work we shall only consider functions which are continuous and differentiable at Z . This guarantees that $\underline{f} \in \mathcal{L}$, the set of bounded linear operators on $\mathcal{G}$.

b) The Gradient

The differential operator ∇_Z is called the <u>gradient</u> with respect to the multivector variable Z . It is characterized by the following two properties:

i) ∇_Z has the algebraic properties of a multivector in $\mathcal{G}$.

ii) $A \odot \nabla_Z$ is the A-derivative operator defined by (3.1). (3.2)

Property (3.2i) permits us to decompose ∇_Z into its homogeneous gradient parts. Thus in analogy to (2.8) we write

$$\nabla_Z = \sum_{r=0} \nabla_{Z_r} = \nabla_{Z_0} + \nabla_{Z_1} + \nabla_{Z_2} + \cdots \quad . \tag{3.3a}$$

For the scalar and vector gradient parts of ∇_Z we will also write

$$\nabla_{Z_0} = \nabla_{\mathfrak{z}} = \frac{d}{d\mathfrak{z}} \quad \text{and} \quad \nabla_{Z_1} = \nabla_z \tag{b}$$

respectivly. We will call ∇_{Z_r} the gradient with respect to the r-vector variable Z_r , as would be expected. From (1.16) we find that for each $A_r \in \mathcal{G}^r$,

$$A_r \odot \nabla_Z = A_r \cdot \nabla_{Z_r} \quad . \tag{3.4}$$

Since ∇_{Z_r} behaves algebraically like an r-vector, we can use (1.48) to express it in terms of the standard r-vector basis of $\mathcal{G}^r$. Thus we have

$$\nabla_{Z_r} = P^{\bar{r}}\, P_{\bar{r}} \cdot \nabla_{Z_r} \quad , \tag{3.5}$$

where of course we are using the summation convention discussed in section 1g).

By projecting both sides of the operator equation (3.3a) with the finite projection operator P , we get the corresponding decomposition of ∇_V , the gradient with respect to the multivector variable $V = P(Z)$ of the finite geometric subspace $\mathcal{V}$. We shall call ∇_V the <u>P-gradient</u>. Thus in anology to (2.11) , we have

$$\nabla_V = \sum_{0 \le r \le n} \nabla_{V_r} = \nabla_{V_0} + \nabla_{V_1} + \cdots + \nabla_{V_n} \quad . \tag{3.6a}$$

We will also write

$$\nabla_{V_0} = \nabla_\Upsilon = \frac{d}{d\Upsilon} \quad , \text{and} \quad \nabla_{V_1} = \nabla_v \tag{b}$$

for the scalar and vector parts of ∇_V . Note that $\frac{d}{d\Upsilon}$ is the ordinary scalar differential operator with respect to the scalar variable Υ , and ∇_v is the classical vector gradient in an n-dimensional vector space.

For the remainder of this section let $F \in \mathcal{C}(Z)$ be a differentiable function at Z , and have the tangent map $\dot{F} \in \mathcal{L}$.

If

$$\nabla F \equiv \nabla_Z F(Z) \tag{3.7a}$$

exists, F is said to have the gradient ∇F at the point Z . If $\mathcal{G}$ is finite dimensional, a differentiable function will always have a gradient. But this is not the case otherwise, as the simple example (A.2a) of appendix A. shows.

When the gradient ∇F exists, it is always equal to the gradient of its tangent map. Ie.,

$$\nabla_Z F(Z) = \nabla_A A \odot \nabla_Z F(Z) = \nabla_A \dot{F}(A) \quad , \qquad b)$$

where A is considered to be the multivector variable of the linear map $\dot{F}$. Property (3.7b) is an application of the operator identity

$$\nabla_Z \equiv \nabla_A A \odot \nabla_Z \qquad (3.8)$$

which will be seen in section e) to be a simple consequence of the chain rule for differentiation. The identity (3.8) can also be directly established by using the definition (3.1) to show its equivalence to (3.5).

Finally note that the finite P-gradient

$$\nabla_V \dot{F}(V) \equiv \nabla_V V \odot \nabla_Z F(Z) = \nabla_A P(A) \odot \nabla_Z F(Z) \qquad (3.9)$$

of the tangent map $\dot{F}$ will always exist. This makes it possible to express $\nabla_Z F(Z)$, when it does exist, by

$$\nabla_Z F(Z) = \lim_{n\to\infty} \nabla_A P_n(A) \odot \nabla_Z F(Z) = \lim_{n\to\infty} \nabla_{{}_nV} \dot{F}({}_nV) \, , \qquad (3.10)$$

where in this case P_n and ${}_nV \equiv P_n(A)$ are the finite projection operators and multivector variables of the standard geometric spaces $\mathcal{D}(I_n)$, defined at the end of section 1g). Equation (3.10) is best understood with the help of (1.51).

c) The Adjoint Map

Let $F \in \mathcal{C}(Z)$ be differentiable at the point Z and have the tangent map $\dot{F} \in \mathcal{L}$.

The adjoint of a bounded linear operator in $\mathcal{L}$ has already been defined in chapter 2., by means of the algebraic equation (2.7). We now give a direct analytic definition of the adjoint map $\dot{F}^{\dagger}$ of the function $F(Z)$ at the point Z. For each $B \in \mathcal{G}$, define

$$\dot{F}^{\dagger}(B) = \nabla_Z F(Z) \odot B = \nabla_C \dot{F}(C) \odot B \tag{3.11a}$$

Because of the identity, which follows from (3.11a) and (3.1),

$$A \odot \dot{F}^{\dagger}(B) = A \odot \nabla_C \dot{F}(C) \odot B = \dot{F}(A) \odot B$$

for all $A, B \in \mathcal{G}$, we are guaranteed that $\dot{F}^{\dagger}$ is indeed the unique algebraic adjoint of F defined by equation (2.7). This also guarantees that $\dot{F}^{\dagger} \in \mathcal{L}$ whenever $\dot{F}$ is. When we wish to emphasize that $\dot{F}^{\dagger}(B)$ is the adjoint map of the function F at the point Z, we will use the notation

$$\dot{F}^{\dagger}(Z;\, B) = \dot{F}^{\dagger}(B) \,. \tag{b}$$

We now define the P-adjoint map of F to be

$$\dot{F}^{P}(B) = \nabla_A \dot{F}(P(A)) \odot B = \nabla_V \dot{F}(V) \odot B \,. \tag{3.12}$$

It is obvious that if the tangent map $\dot{F}$ of F is a P-map, ie., if $\dot{F}(P(A)) = \dot{F}(A)$ for each $A \in \mathcal{G}$, then $\dot{F}^{P} = \dot{F}^{\dagger}$. In terms of the finite projection operators P_n of the standard geometric subspaces $\mathcal{D}(I_n)$, we can express the adjoint $\dot{F}^{\dagger}$ of F by

$$\dot{F}^{\dagger}(B) = \lim_{n \to \infty} \nabla_A \dot{F}(P_n(A)) \odot B = \lim_{n \to \infty} \dot{F}^{P_n}(B) \,. \tag{3.13}$$

The expression (3.13) should be compared with (3.10).

Suppose now that $\nabla_Z F(Z)$ exists. We have already seen how it is possible to express $\nabla_Z F(Z)$ in terms of the gradient of the tangent map $\dot{F}$ (Recall (3.7b)). It is also possible to express it as the left gradient of the adjoint map $\dot{F}^{\dagger}$, by

$$\nabla_Z F(Z) = \nabla_Z F(Z) \odot B \; \nabla_B = \dot{F}^{\dagger}(B) \nabla_B \quad , \tag{3.14}$$

where the gradient ∇_B , in this case, is assumed to operate to the left.

d) Second order Linearizations

Again let $F \in \mathcal{C}(Z)$ and have the tangent map $\dot{F} \in \mathcal{L}$.

If $\dot{F}$ is itself continuous and differentiable at Z , then it too will have a tangent map, which we will denote by $\ddot{F} \in {}^2\mathcal{L}$, where ${}^2\mathcal{L}$ is the normed linear space of bounded bilinear operators on $\mathcal{G}$ defined in section 2b). We will call $\ddot{F}$ the <u>2-tangent map</u> of F at Z , and say that F is 2-differentiable at Z . (For details of the theory of higher order derivatives in a Hilbert space, see [3; 174-86].) Several notations will be used for $\ddot{F}$; each has its respective advantages. They are

$$\ddot{F}_B(A) = \ddot{F}(Z;\, A, B) = B \odot \nabla_Z \, A \odot \nabla_Z \, F(Z) \ . \tag{3.15}$$

A <u>2-adjoint map</u> for a 2-differentiable function F can also be defined. It is given by

$$\ddot{F}_B^{\dagger}(C) \equiv \nabla_D \, \ddot{F}_B(D) \odot C = \ddot{F}^{\dagger}(Z; B, C) \quad . \tag{3.16}$$

To see that this agrees with the unique algebraic adjoint of $\ddot{F} \in {}^2\mathcal{L}$, defined by the equation (2.9) of chapter 2, it is only necessary to observe that (3.16) implies that for all A, B, C,

$$A \odot \ddot{F}_B^{\dagger}(C) = A \odot \nabla_D \, \ddot{F}_B(D) \odot C = \ddot{F}_B(A) \odot C \quad .$$

It is possible, of course, to define higher order linearizations of F', but this will not serve our purposes here.

e) Differentiation Rules

In this section all functions will be assumed to be continuous and 1 or 2-differentiable as required.

First note that the operations of differentiation given here could all be restated entirely in terms of derivatives with respect to scalar variables. To see this it is only necessary to observe that

$$A \odot \nabla_Z F(Z) \equiv \left[\frac{d}{dt} F(Z + tA)\right]_{t=0} . \qquad (3.17)$$

Relation (3.17) follows immediately from definition (3.1). Hence all the familiar rules for scalar differentiation will carry over to this setting with little need of comment. However several will be given here to show how they dress themselves in this language.

The Leibnitz product rule for A-derivatives can be stated

$$A \odot \nabla_Z F(Z)G(Z) = \left[A \odot \nabla_Z F(Z)\right] G(Z) + F(Z)\left[A \odot \nabla_Z G(Z)\right], \quad (3.18a)$$

and for the gradient, by

$$\nabla_Z F(Z)G(Z) = \nabla_{\dot{Z}} F(\dot{Z})\, G(Z) + \nabla_{\dot{Z}} F(Z)\, G(\dot{Z}) . \qquad b)$$

Notice the usage of dots in (3.18b) to indicate what variable is being differentiated. This is necessary because it is not in general possible to commute the order of non-scalar multivectors. Rule (3.18a) is equivalent to the Leibnitz rule for differentiating functions of a single scalar variable. Rule

(3.18b) is a direct consequence of differentiating both sides of (3.18a) by ∇_A .

The familiar rule that "partial derivatives commute" is equivalent to

$$\ddot{F}_A(B) \equiv A\odot\nabla B\odot\nabla F = B\odot\nabla A\odot\nabla F \equiv \ddot{F}_B(A) \qquad (3.19)$$

in this language, and can be proved in the analogous way.

Finally we give the equivalent of the "chain rule" in this language. Let $Z' = F(Z)$ and $G(Z')$ be given. The composition $G_\circ F$ of G and F is differentiated as follows:

$$A\odot\nabla_Z G[F(Z)] = [A\odot\nabla_Z F(Z)]\odot\nabla_{Z'}G(Z') \quad . \qquad (3.20a)$$

Rule (3.20a) can be stated as an operator identity by

$$A\odot\nabla_Z = \dot{F}(A)\odot\nabla_{Z'} = A\odot\dot{F}^\dagger(\nabla_{Z'}) \quad , \qquad b)$$

or in terms only of gradients by

$$\nabla_Z = \dot{F}^\dagger(\nabla_{Z'}) \quad . \qquad c)$$

Note that the last equality in b) above just expresses the algebraic relationship (2.7) between a linear map and its adjoint, for $\nabla_{Z'}$ is assumed not to differentiate $\dot{F}^\dagger$. Rule c) follows from rule b) by differentiating both sides of b) with ∇_A .

f) Decomposition

As an important application of the rules of the preceeding section, we use them to show how differentiation with respect to an s-vector variable can be decomposed into differentiation with respect to an r- and an (s-r)-vector variable, for $s > r$.

To accomplish this, suppose that

$$Z'_s = F(Z_r) = Z_r \wedge Z_{s-r} \; .$$

Then applying (3.20a) with $A = A_r$, and using (1.16b), we get

$$A_r \cdot \nabla_{Z_r} G(Z_r \wedge Z_{s-r}) = (A_r \wedge Z_{s-r}) \cdot \nabla_{Z'_s} G(Z'_s) \; . \qquad (3.21a)$$

Differentiating both sides of this equation by ∇_{A_r} gives

$$\nabla_{Z_r} G(Z_r \wedge Z_{s-r}) = Z_{s-r} \cdot \nabla_{Z'_s} G(Z'_s) \; , \qquad b)$$

with the help of the algebraic identity (1.19a). Finally differentiating both sides of (3.21b) by $\nabla_{Z_{s-r}}$, and using the Leibnitz rule (3.18b) and the differentiation formula (A.2b) from appendix A, we find that

$$\nabla_{Z_{s-r}} \wedge \nabla_{Z_r} G(Z_r \wedge Z_{s-r}) = \binom{s}{r} \nabla_{Z'_s} G(Z'_s) + \nabla_{\dot{Z}_{s-r}} \wedge (\dot{Z}_{s-r} \cdot \nabla_{\ddot{Z}'_s}) G(\ddot{Z}'_s) \; , \qquad (3.21c)$$

where again we are using dots over the variable which is to be differentiated. If G is a linear function, (3.21c) simplifies to

$$\nabla_{Z_{s-r}} \wedge \nabla_{Z_r} G(Z_r \wedge Z_{s-r}) = \binom{s}{r} \nabla_{Z'_s} G(Z'_s) \; , \qquad d)$$

since differentiating a linear function twice gives 0.

Continuing the above process as far as possible leads to the <u>simplicial decomposition of differentiation</u>. In terms of simplicial variable $z_{\bar{r}}$, previously defined by (2.13a), relations (3.21b,c) become

$$\nabla_{z_1} G(z_{\bar{s}}) = \frac{1}{s} z_{\overline{s-1}} \cdot \nabla_{Z'_s} G(Z'_s) \; , \qquad (3.22a)$$

and

$$\nabla_{\bar{s}}\, G(z_{\bar{s}}) = \nabla_{\dot{Z}_s}\, G(\dot{Z}_s) + \frac{1}{s}\, \dot{\nabla}_{\overline{s-1}} \wedge \left[z_{\overline{s-1}} \cdot \nabla_{\dot{Z}_s} \right] G(\dot{Z}_s) \quad , \qquad b)$$

where

$$\nabla_{\bar{s}} \equiv \nabla_{z_{\bar{s}}} \equiv \nabla_{z_s} \wedge \nabla_{z_{s-1}} \wedge \cdots \wedge \nabla_{z_1} \qquad (3.23a)$$

is defined to be the <u>simplicial s-gradient</u> with respect to the simplicial variable $z_{\bar{s}}$. Note also the conventions

$$\nabla_{\bar{0}} \equiv \frac{d}{d\zeta} \quad \text{and} \quad \nabla_{\bar{1}} = \nabla_z \quad . \qquad b)$$

Finally when G is linear, (3.22b) simplifies to

$$\nabla_{\bar{s}}\, G(z_{\bar{s}}) = \nabla_{\dot{Z}_s}\, G(\dot{Z}_s) \quad . \qquad (3.24)$$

This shows that for linear functions, simplicial differentiation is equivalent to s-vector differentiation. Simplicial differentiation was first defined and studied in [25].

g) Functions of Several Vector Variables

Suppose that $F(z_1, z_2, \cdots, z_r)$ is a $\mathcal{G}$-valued function which is continuous and differentiable in each of its vector variables $z_i \in \mathcal{G}^1$ at the "point" $(z_1, z_2, \cdots, z_r)$. It is natural to define tangent and adjoint maps for F at the point $(z_1, z_2, \cdots, z_r)$ by

$$\dot{F}(A) \equiv A \odot \nabla_{\bar{r}}\, F(z_1, \cdots, z_r) \quad \text{for each } A \in \mathcal{G} , \qquad (3.25a)$$

and

$$\dot{F}^{\dagger}(B) \equiv \nabla_{\bar{r}}\, F(z_1, \cdots, z_r) \odot B \quad \text{for each } B \in \mathcal{G} . \qquad b)$$

These definitions are similar to the definitions (3.1) and

(3.11) respectively, except that r-simplicial differentiation replaces multivector differentiation. This change, however, does not affect the basic relationship (2.7) between a tangent map and its adjoint, as is seen in the following identity:

$$\dot{F}(A)\odot B = A\odot\nabla_{\bar{r}} F(z_1,\cdots,z_r)\odot B = A\odot\dot{F}^{\dagger}(B) \quad . \tag{3.26}$$

To gain further insight into the nature of these maps we consider a special case. Let

$$F(z_1,\cdots,z_r) = f^1(z_1)\, f^2(z_2)\cdots f^r(z_r) \quad , \tag{3.27}$$

where each $f^i(z_i)$ is a continuous differentiable vector-valued function at the point z_i, with the tangent map $\dot{f}^i$. Then with the help of (1.19b) we find that

$$\dot{F}(a_1\wedge\cdots\wedge a_r) = \sum_{\sigma_{\bar{r}}} \operatorname{sgn}(\sigma_{\bar{r}})\, \dot{f}^1(a_{\sigma_1})\wedge\cdots\wedge\dot{f}^r(a_{\sigma_r}) , \tag{3.28}$$

where $\sigma_{\bar{r}}$ is a permutation of the indicies $1,\cdots,r$,and $\operatorname{sgn}(\sigma_{\bar{r}}) = \pm 1$ is its sign. From this it is clear that $\dot{F}$ is the antisymmetry operator on the r tangent maps $\dot{f}^i$, $1\le i\le r$.

Equation (3.28) simplifies considerably in the special case when each $f^i = f$, and each $z_i = z$, for $i = 1,\cdots,r$. In this case (3.28) becomes

$$\dot{F}(a_1\wedge\cdots\wedge a_r) = r!\,\dot{f}(a_1)\wedge\cdots\wedge\dot{f}(a_r) \quad , \tag{3.29a}$$

and the adjoint map is

$$\dot{F}^{\dagger}(a_1\wedge\cdots\wedge a_r) = r!\,\dot{f}^{\dagger}(a_1)\wedge\cdots\wedge\dot{f}^{\dagger}(a_r) \quad . \tag{b}$$

Because (3.29) is important in the study of linear maps, as will be seen in the next chapter, we introduce the special notation

$$\dot{f}_{\bar{r}}(A_r) \equiv \frac{1}{r!}\dot{F}(A_r) = \dot{f}(a_1)\wedge\dots\wedge\dot{f}(a_r) \quad , \qquad (3.30a)$$

and

$$\dot{f}^{\dagger}_{\bar{r}}(A_r) \equiv \frac{1}{r!}\dot{F}^{\dagger}(A_r) = \dot{f}^{\dagger}(a_1)\wedge\dots\wedge\dot{f}^{\dagger}(a_r) \quad , \qquad b)$$

where $A_r = a_1\wedge\dots\wedge a_r$. We are also implicitly assuming the conventions

$$\dot{f}_{\bar{1}}(a) \equiv \dot{f}(a) \;, \text{ and } \dot{f}_{\bar{0}}(\alpha) \equiv \alpha \;,$$
$$\dot{f}^{\dagger}_{\bar{1}}(a) \equiv \dot{f}^{\dagger}(a) \;, \text{ and } \dot{f}^{\dagger}_{\bar{0}}(\alpha) \equiv \alpha \;, \qquad c)$$

for all vectors $a \in \mathcal{G}^1$, and all scalars $\alpha \in \mathcal{G}^0$.

4. Linear Analysis

All the information about how a differentiable function behaves at a point in $\mathcal{G}$ is contained in the behavior of its tangent and adjoint maps, and higher order linearizations at that point. We would therefore do well to further understand the behavior of linear functions on $\mathcal{G}$. In this chapter, then, we shall only be interested in bounded linear functions $F \in \mathcal{L}$, and their bounded linear adjoints $F^{\dagger} \in \mathcal{L}$, as defined in section 2b).

A trivial but important consequence of definitions (3.1) and (3.11a) is that for a linear function,

$$\dot{F}(Z) = F(Z) \quad , \text{ and } \quad \dot{F}^{\dagger}(Z) = F^{\dagger}(Z) \quad . \tag{4.1}$$

Ie., the tangent and adjoint maps of a linear function are identically the linear function and its adjoint respectively. Thus for linear functions no further distinction will be made (by the use of dots) between them. Note also that by (3.24), no distinction need be made between simplicial and multivector differentiation of linear maps.

a) The Gradiant of a Linear Map

We have already observed that the gradient of a linear map may not exist. In this section let $F \in \mathcal{L}$ be a map for for which $\nabla_{\bar{F}}\, F(z_{\bar{F}})$ does exist. The following two identities, which are <u>termwise</u> equivalent, are an easy consequence of the basic algebraic identity (1.21), the linearity of F, and the antisymmetry of the outer product.

$$\nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\bar{r}}:F(z_{\bar{r}}) + \nabla_{\overline{r-1}}\cdot\left[\nabla_1 \wedge F(z_1 \wedge z_{\overline{r-1}})\right] + \cdots + \nabla_r\cdot\left[\nabla_{\overline{r-1}} \wedge F(z_{\overline{r-1}} \wedge z_r)\right] + \nabla_{\bar{r}} \wedge F(z_{\bar{r}}) \tag{4.2a}$$

and

$$\nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\bar{r}}:F(z_{\bar{r}}) + \nabla_r \wedge\left[\nabla_{\overline{r-1}}\cdot F(z_{\overline{r-1}} \wedge z_r)\right] + \cdots + \nabla_{\overline{r-1}} \wedge\left[\nabla_1\cdot F(z_1 \wedge z_{\overline{r-1}})\right] + \nabla_{\bar{r}} \wedge F(z_{\bar{r}}) \; . \tag{b}$$

Examining (4.2), it is easy to see that

$$\nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\bar{r}}\cdot F(z_{\bar{r}}) \quad \text{if} \quad \nabla_1 \wedge F(z_1 \wedge z_{\overline{r-1}}) = 0 \; , \tag{4.3a}$$

and

$$\nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\bar{r}} \wedge F(z_{\bar{r}}) \quad \text{if} \quad \nabla_1 : F(z_1 \wedge z_{\overline{r-1}}) = 0 \; . \tag{b}$$

However, the conditions given in (4.3) are only sufficient to guarantee the conclusions.

We shall call the condition that

$$\nabla_z \wedge F(z \wedge A_{r-1}) = 0 \quad \text{for all} \quad A_{r-1} \in \mathcal{G}^{r-1} \; , \tag{4.4}$$

The <u>Bianchi condition</u> because, as we shall later see, it implies the so called Bianchi identities of the curvature tensor. Furthermore, we will say that F is <u>completely symmetric</u> if it satisfies the Bianchi condition (4.4).

We conclude this section with two useful identities:

$$A_s \cdot \nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\overline{r-s}} F(z_{\overline{r-s}} \wedge A_s) \quad \text{for } s \le r \; , \tag{4.5a}$$

and if F is homogeneous and completely symmetric, then

$$A_s \cdot \nabla_{\bar{r}} F(z_{\bar{r}}) = \nabla_{\bar{r}} F(z_{\bar{r}}) \cdot A_s \quad \text{for all} \quad r,s \; . \tag{b}$$

The identity a) is a direct application of (3.21b) to the linear map F , and b) is a consequence of (4.3a).

b) Relationship between a Linear Map and its Adjoint

Let $F \in \mathcal{L}^{(r,r)}$, the set of bounded (r,r)-homogeneous linear maps. The following identity, which can be proved by induction on $r \geq 0$, and the algebraic identities (1.18a) and (1.19a), gives a direct relationship between an (r,r)-homogeneous map and its adjoint.

$$\begin{aligned} A_r \cdot \left[\nabla_{\bar{r}} \wedge F(z_{\bar{r}})\right] = \left(A_r \cdot \nabla_{\bar{1}}\right) \cdot \left[\nabla_{\overline{r-1}} \wedge F\left(z_{\overline{r-1}} \wedge z_{\bar{1}}\right)\right] \\ - \left(A_r \cdot \nabla_{\bar{2}}\right) \cdot \left[\nabla_{\overline{r-2}} \wedge F\left(z_{\overline{r-2}} \wedge z_{\bar{2}}\right)\right] + \cdots \\ + (-1)^r \left(A_r \cdot \nabla_{\overline{r-1}}\right) \cdot \left[\nabla_1 \wedge F\left(z_1 \wedge z_{\overline{r-1}}\right)\right] \\ + (-1)^{r+1}\left[F(A_r) - F^{\dagger}(A_r)\right] . \end{aligned} \quad (4.6)$$

In the special case that F satisfies the Bianchi condition (4.4), the identity (4.6) gives

$$F(A_r) = F^{\dagger}(A_r) \quad \text{for all} \quad A_r \in \mathcal{G}^r , \quad (4.7a)$$

which can also be obtained from (4.5b). Thus we see that the Bianchi condition (4.4) is sufficient to guarantee that an (r,r)-homogeneous linear map is <u>symmetric</u>.

A simple but important example of a symmetrix linear map, which is also completely symmetric, is the finite projection operator P . This means that

$$P(A) = P^{\dagger}(A) \quad \text{for all} \quad A \in \mathcal{G} , \quad \text{b)}$$

and also that

$$\nabla_z \wedge P(z \wedge A) = \nabla_v \wedge v \wedge P(A) = 0 \quad \text{for all} \quad A \in \mathcal{G}. \qquad c)$$

The identity c) is equivalent to the property that

$$\nabla_z \wedge P(z) = \nabla_v \wedge v = 0 \quad . \qquad d)$$

c) Outermorphisms

A homogeneous map $f \in \mathcal{L}$ is said to be an <u>outermorphism</u> if it preserves the outer product. Thus if

$$f(A \wedge B) = f(A) \wedge f(B) \quad \text{for all} \quad A, B \in \mathcal{G}. \qquad (4.8)$$

then f is an outermorphism. From (4.8) it is clear that an outermorphism is completely determined by its values on $\mathcal{G}^1$. Conversely if a bounded linear map f on $\mathcal{G}^1$ is given, it can always be extended uniquely to an outermorphism on $\mathcal{G}$ by defining

$$f(A) = \sum_{r \geq 0} f_{\bar{r}}(\langle A \rangle_r) \quad , \qquad (4.9)$$

where $f_{\bar{r}}$ has already been defined in (3.30) of chapter 3, and has the desired property (4.8). Thus no distinction in symbolism will be made between a linear map on $\mathcal{G}^1$, and its extension to all of $\mathcal{G}$ given by (4.9).

In a similar way, by using $f^{\dagger}_{\bar{r}}$, the adjoint $f^{\dagger}$ of f is extended to the unique adjoint outermorphism on all of $\mathcal{G}$. Again no symbolic distinction will be made between the adjoint of a linear map on $\mathcal{G}^1$, and the extended adjoint outermorphism on all of $\mathcal{G}$.

The important general algebraic relationship between an outermorphism and its adjoint is now given. It can be established directly from the definitions of f and $f^{\dagger}$, and the algebraic identity (1.19a). Thus for $A_r \in \mathcal{G}^r$, $B_s \in \mathcal{G}^s$,

$$f(A_r)\cdot B_s = f\left[A_r\cdot f^{\dagger}(B_s)\right] \quad \text{for } r \geq s \,, \tag{4.10a}$$

and

$$A_r\cdot f^{\dagger}(B_s) = f^{\dagger}\left[f(A_r)\cdot B_s\right] \quad \text{for } r \leq s \,. \qquad \text{b)}$$

Note that in the case $r=s$, both parts of the above relationship reduce to

$$f(A_r)\cdot B_r = A_r\cdot f^{\dagger}(B_r) \quad , \qquad \text{c)}$$

the basic relationship between a linear homogeneous map and its adjoint. In section e) the more general (4.10A,b) will be used to algebraically invert a non-singular map.

The trivial identity

$$a\cdot\left[\nabla_1 \wedge f(z_1)\right] = f(a) - f^{\dagger}(a) \quad , \tag{4.11}$$

has already been encountered in the more general (4.6), and implies that the Bianchi condition (4.4) is equivalent to $f = f^{\dagger}$. This means that a symmetric outermorphism is always completely symmetric.

To finish this section, outermorphisms of linear maps which are the sum or product of two linear maps on $\mathcal{G}^1$ are given. The proofs are simple calculations using the definition (4.9) and algebraic identities, but they will not be given here.

If $h(a) = f(a) + g(a)$ for all $a \in \mathcal{G}^1$, then

$$h(A_r) = \sum_{s=0}^{r} A_r\cdot\left(\nabla_{\bar{s}} \wedge \nabla_{\overline{r-s}}\right) f(z_{\overline{r-s}}) \wedge g(z_{\bar{s}}) \quad , \tag{4.12a}$$

and if $h(a) = fg(a)$ for all $a \in \mathcal{G}^1$, then

$$h(A_r) = f\left[g(A_r)\right] \text{, and } h^{\dagger}(A_r) = g^{\dagger}\left[f^{\dagger}(A_r)\right] . \qquad \text{b)}$$

The relationship a) will be used when discussing the characteristic polynomial. Relationship b) has already been encountered in the form of the chain rule (3.20a).

Outermorphisms were first defined and studied in [25], and details of proofs omitted here can be found there.

d) Existence of a Reciprocal Basis

As indicated in section 1g), a given basis $\{a_i\}$ of $\mathcal{G}^1$ may not have a reciprocal basis $\{a^j\}$. In this section we give a necessary and sufficient condition for a basis $\{a_i\}$ to have a reciprocal basis $\{a^j\}$, and indicate how a reciprocal r-vector basis can be constructed using outermorphisms.

An outermorphism f is uniquely determined by its values on any basis of $\mathcal{G}^1$. For the given basis $\{a_i\}$ we define a unique outermorphism f by specifying that

$$f(a_i) = p_i \quad \text{for each } i \geq 1 , \qquad (4.13a)$$

where $\{p_i\}$ is the standard basis of $\mathcal{G}^1$ defined in sec-1g). The basis $\{a_i\}$ will have a reciprocal basis $\{a^j\}$ if and only if the outermorphism f is bounded, and hence has a bounded adjoint $f^{\dagger}$. In this case the reciprocal basis is given by

$$a^j \equiv f^{\dagger}(p^j) \quad \text{for each } j \geq 1 , \qquad \text{b)}$$

as is easily verified by the identity

$$a_i \cdot a^j = a_i \cdot f^\dagger(p^j) = f(a_i) \cdot p^j = p_i \cdot p^j \quad .$$

Similarly, the reciprocal r-basis $\{a^{j_{\bar{r}}}\}$ is given by

$$a^{j_{\bar{r}}} = f^\dagger\left(p^{j_{\bar{r}}}\right) \quad \text{for each} \quad 1 \leq j_{\bar{r}} < \infty \quad , \qquad \text{c)}$$

as can also be easily verified.

e) Non-Singular Outermorphisms

Let $f \in \mathcal{L}$ be an outermorphism. The outermorphism f is said to be P non-singular if

$$f(I) \neq 0 \text{ , where } I = I_n \text{ is the pseudoscalar of } \mathcal{G}, \qquad (4.14a)$$

and non-singular if

$$f(A) \neq 0 \quad \text{for each non-zero} \quad A \in \mathcal{G} \text{ .} \qquad \text{b)}$$

Noting the identity

$$\nabla_{V_n} f(V_n) = I^\dagger I \cdot \nabla_{V_n} f(V_n) = I^\dagger f(I) \equiv J_f(I) \quad , \qquad (4.15a)$$

which is a consequence of (3.5), it is clear that $f(I) \neq 0$ is equivalent to $\nabla_{V_n} f(V_n) \neq 0$. The multivector $J_f(I)$ is a generalization of the Jacobian of a mapping, for in the case that f is a (P,P)-map (recall definition (2.4b)),

$$\nabla_{V_n} f(V_n) = I^\dagger \cdot f(I) = \det(f) \text{ .} \qquad \text{b)}$$

Note also that

$$\| \nabla_{V_n} f(V_n) \| = \| f(I) \| = \| f \|^n \text{ .} \qquad \text{c)}$$

The following formulas are the formulas (3.12) and (3.13) stated for an outermorphism $f \in \mathcal{L}$. They are given again here for easy reference. The P-adjoint of f is given by

$$f^P(A) = \nabla_V f(V) \odot A \quad , \tag{4.16a}$$

and for increasing P_n has the limit

$$f^\dagger(A) = \nabla_Z f(Z) \odot A = \lim_{n \to \infty} f^{P_n}(A) \quad , \tag{b}$$

where $\{P_n\}$ is the standard sequence of projection operators.

Now let $\mathfrak{D} = \mathfrak{D}(I)$ and $\mathfrak{D}' = \mathfrak{D}'(I')$ be two finite sub-algebras of $\mathcal{G}$. We will say that $\mathfrak{D}$ and $\mathfrak{D}'$ are f-related if f is a P non-singular outermorphism with the property that

$$I' = |f(I)|^{-1} f(I) \quad . \tag{4.17a}$$

If $\mathfrak{D}$ and $\mathfrak{D}'$ are f-related, then the "chain rule" (3.20c) relating the finite gradient operators of $\mathfrak{D}$ and $\mathfrak{D}'$ is

$$\nabla_V = \nabla_V f(V) \odot \nabla'_{V'} = f^P(\nabla'_{V'}) \quad . \tag{b}$$

For the remainder of this section we shall assume that $\mathfrak{D}$ and $\mathfrak{D}'$ are f-related.

The identity

$$I^\dagger f(I) = \nabla_{V_n} f(V_n) = f^P(V'_n) \nabla'_{V'} = f^P(I') I'^\dagger \quad , \tag{4.18a}$$

which is a consequence of (4.15a) and (3.14), suggests the close relationship that exists between f and its P-adjoint f^P. We have already seen in section 3c) that if f is a (P,P)-map then $f^P \geq f^\dagger$, and it now follows from (4.15b) and (4.18a), that for (P,P)-maps

$$\det(f) = \det\left(f^{\dagger}\right) \quad . \qquad \text{b)}$$

Note also that taking the norm of the identity (4.18a), and using (4.15a), gives

$$\left|f(I)\right| = \left\|\nabla_{\bar{n}}\, f(v_{\bar{n}})\right\| = \left|f^{P}(I')\right| = \left|J_{f}\right| \quad . \qquad \text{c)}$$

We now show how f can be inverted in terms of its P-adjoint. This is accomplished in the following three equivalent steps. Let $A \in \mathfrak{D}$, then

$$A' \equiv f(A) \quad , \qquad (4.19a)$$

$$f^{P}(A'I') = f^{P}\left[f(A)\cdot I'\right] = \left|f(I)\right| AI \quad , \qquad \text{b)}$$

$$f^{-1}(A') = \left|f^{P}(I')\right|^{-1} f^{P}(A'I')\, I^{\dagger} = A \qquad \text{c)}$$

The fundamental property (4.10b) is used in step b) and (4.18c) is used in step c). When f is a (P,P)-map, (4.19c) simplifies to

$$f^{-1}(A') = \frac{f^{\dagger}(A'I)I^{\dagger}}{\det(f)} \quad \text{for all } A' \in \mathfrak{D} \quad , \qquad \text{d)}$$

but this formula is still more general than the closely related formula for inverting a linear operator in matrix theory, since the domain of f is all of $\mathfrak{D}$, and not only $\mathfrak{D}^{1}$.

More generally, the relationship (4.16b) suggests that it is possible to invert non-singular bounded outermorphisms in terms of their bounded adjoints.

f) Characteristic Polynomials

The exposition of this section follows the one first given in [25]. It is also closely related to, but much simpler than the one given in [8; 163].

Let $f \in \mathcal{L}$ be an outermorphism, and let P be the finite projection operator of some geometric subalgebra $\mathcal{D}(I_n)$. For each scalar λ we define the outermorphism h_λ by specifying that

$$h_\lambda(v) = f(v) - \lambda P(v) \quad \text{for all } v \in \mathcal{D}^1 \quad . \qquad (4.20a)$$

(Note that P is the identity operator on $\mathcal{D}^1$). Recalling (4.12a), we find for each $V_r \in \mathcal{D}^r$ that

$$h_\lambda(V_r) = \sum_{k=0}^{r} (-1)^{r-k} V_r \cdot (\nabla_{\bar{k}} \wedge \nabla_{\overline{r-k}})\, v_{\overline{r-k}} \wedge f(v_{\bar{k}})\, \lambda^{r-k} \qquad b)$$

Differentiating the expression b) by ∇_{V_r}, and using the differentiation formula (A.2c) given in appendix A, we calculate

$$\Psi_P^r(\lambda) \equiv \nabla_{V_r} h_\lambda(V_r) = \sum_{k=0}^{r} (-1)^{r-k} \nabla_{\bar{k}}\, f(v_{\bar{k}})\, \lambda^{r-k} \qquad (4.21a)$$

for $r = 0, \cdots, n$. The polynomials $\Psi_P^r(\lambda)$ are called the <u>generalized characteristic polynomials</u> of f with respect to the projection operator P. Note that [illegible] they are, in general, polynomials in λ with multivector coifficients. For the remainder of this section, however, we shall only consider $\Psi(\lambda) \equiv \Psi_P^n(\lambda)$, when f is a (P,P)-map. In this case, $\Psi(\lambda)$ is the usual characteristic polynomial of the linear map $f : \mathcal{D}^1 \to \mathcal{D}^1$. From (4.21a), it follows that

$$\Psi(\lambda) = \sum_{k=0}^{n} (-1)^{n-k}\, \nabla_{\bar{k}} \cdot f(v_{\bar{k}})\, \lambda^{n-k} \quad , \qquad b)$$

and from (4.15b) that

$$\Psi(\lambda) = \nabla_{V_n} h_\lambda(V_n) = \det(h_\lambda) \quad , \qquad c)$$

as would be expected.

We now prove the well-known Cayley-Hamilton theorem that

$$\Psi[f](v) = 0 \quad \text{for each } v \in \mathfrak{D}^1 , \tag{4.22}$$

ie., every linear map f on $\mathfrak{D}^1$ satisfies its characteristic equation. Of course for (4.22) to have meaning, we must agree to the conventions

$$f^0(v) \equiv P(v) = v \quad \text{for all } v \in \mathfrak{D}^1 , \tag{4.23a}$$

and recursively,

$$f^k(v) = f\left[f^{k-1}(v)\right] \quad \text{for} \quad v \in \mathfrak{D}^1 \quad \text{and} \quad k \geq 1 . \tag{b}$$

The Cayley-Hamilton theorem is established by showing the $(n+1)^{st}$-term of (4.22) is the negative of the sum of the other n terms. Thus, using basic differential and algebraic identities, we find that

$$\begin{aligned}\nabla_{\bar{n}}\cdot f_{\bar{n}} \; v = v\cdot\nabla_{\bar{n}} \; f_{\bar{n}} &= \nabla_{\overline{n-1}}\cdot\left[f_{\overline{n-1}}\wedge f(v)\right] \\ &= \nabla_{\overline{n-1}}\cdot f_{\overline{n-1}} \; f(v) - \nabla_{\overline{n-1}}\cdot\left[f_{\overline{n-2}}\wedge f(v)\right] \; f_{n-1} \\ &= \cdots \\ &= \nabla_{\overline{n-1}}\cdot f_{\overline{n-1}} \; f(v) - \nabla_{\overline{n-2}}\cdot f_{\overline{n-2}} \; f^2(v) + \cdots \\ &\quad + (-1)^{n-1} f^n(v) \quad .\end{aligned}$$

Another familiar result from matrix theory is that the characteristic coifficients of (4.21b) can be entirely expressed in terms of the <u>traces</u>

$$\mathrm{Tr}(f^r) \equiv \nabla\cdot f^r = \nabla_v\cdot f^r(v) \quad . \tag{4.24}$$

by using the recursive formula

$$\nabla_{\bar{r}} \cdot f(v_{\bar{r}}) = \frac{1}{r} \sum_{k=1}^{r} \nabla_{\overline{r-k}} \cdot f(v_{\overline{r-k}}) \quad \nabla \cdot f^k \quad \text{for} \quad r \geq 1 \; . \qquad (4.25)$$

This is directly established in the steps

$$\begin{aligned} \nabla_{\bar{r}} \cdot f_{\bar{r}} &= \frac{1}{r} \left(\nabla_{\overline{r-1}} \wedge \nabla_1 \right) \cdot \left(f_1 \wedge f_{\overline{r-1}} \right) \\ &= \frac{1}{r} \left\{ \nabla_{\overline{r-1}} \cdot f_{\overline{r-1}} \quad \nabla \cdot f \quad - \left(\nabla_{\overline{r-2}} \wedge \nabla_1 \right) \cdot \left(f_1^2 \wedge f_{\overline{r-2}} \right) \right\} \\ &= \cdots \\ &= \frac{1}{r} \left\{ \nabla_{\overline{r-1}} \cdot f_{\overline{r-1}} \quad \nabla \cdot f \; - \; \nabla_{\overline{r-2}} \cdot f_{\overline{r-2}} \quad \nabla \cdot f^2 \right. \\ &\qquad \left. + \cdots + (-1)^{r+1} \; \nabla \cdot f^r \right\} \; . \end{aligned}$$

PART II

GEOMETRIC STRUCTURES

5. Rings and Polynomial Rings of Linear Maps

Recall the definitions of the normed linear spaces $\mathcal{L}$ and ${}_P\mathcal{L}_P$ given in section 2b). These spaces can be made into rings by defining the usual operations of composition

$$[F + G](A) \equiv F(A) + G(A) \quad \text{, and} \tag{5.1a}$$

$$FG(A) \equiv F[G(A)] \quad . \tag{b}$$

With these operations $\mathcal{L}$ is a ring with unit Q, the projection operator onto $\mathcal{G}$ (definition (1.42)). Similarly, with these same operations ${}_P\mathcal{L}_P$ is a subring of $\mathcal{L}$ with a unit P, the projection operator onto the finite subalgebra $\mathcal{D}$ of $\mathcal{G}$. In fact both $\mathcal{L}$ and ${}_P\mathcal{L}_P$ are Banach algebras of linear operators on the Banach algebra $\mathcal{G}$. See [3; 221] and [21; 2].

The ring $\mathcal{L}$ can be used to generate a polynomial ring $\mathcal{H}$ consisting of all finite algebraic sums and products of the elements of $\mathcal{L}$. For example if $F,G \in \mathcal{L}$, then

$$H \equiv H[F,G,FG](A,B,C) = F(A)G(B) + F[G(C)] \in \mathcal{H} \ . \tag{5.2}$$

In this example H is the value of the above polynomial at the "point" $[F,G,FG] \in \mathcal{L}\times\mathcal{L}\times\mathcal{L}$, which in turn is evaluated at the "point" $(A,B,C) \in \mathcal{G}\times\mathcal{G}\times\mathcal{G}$. It is essential to note that the operations on the ring $\mathcal{H}$ are the same as the algebraic operations in the geometric algebra $\mathcal{G}$, and are therefore different than the operations of composition (5.1) given on the ring $\mathcal{L}$. In order to lessen the possibility of confusion, we shall generally use the letters E,F,G to represent elements of $\mathcal{L}$, and H,K to represent polynomial elements of $\mathcal{H}$. However, note that as a subset $\mathcal{L} \subset \mathcal{H}$. Similarly, the polynomial subring ${}_P\mathcal{H}_P$ of $\mathcal{H}$ can be constructed from the elements of ${}_P\mathcal{L}_P$.

a) Ring States

By the states of a ring $\mathcal{L}$ we mean the set $\mathcal{S}(\mathcal{L})$ of all bilinear transformations $\mathcal{X}$,

$$\mathcal{X} : \mathcal{G}^1 \times \mathcal{L} \longrightarrow \mathcal{L}. \qquad (5.3a)$$

An element $\mathcal{X} \in \mathcal{S}(\mathcal{L})$ is called a state of $\mathcal{L}$, and we shall write

$$\mathcal{X}[F] = \mathcal{X}[a; F] \qquad b)$$

for its value at $[a; F] \in \mathcal{G}^1 \times \mathcal{L}$. If $\mathcal{X}$ satisfies the additional property that

$$\langle \mathcal{X}[F] \rangle_r = \mathcal{X}[\langle F \rangle_r] \quad \text{for all } r \geq 0, \qquad c)$$

then $\mathcal{X}$ is said to be a homogeneous state.

The linearity of $\mathcal{X}$ in the variable $F \in \mathcal{L}$ can be expressed by the addition formula

$$\mathcal{X}[F + G] = \mathcal{X}[F] + \mathcal{X}[G] \in \mathcal{L}; \qquad (5.4a)$$

Because $\mathcal{X}[F] \in \mathcal{L}$, it can be multiplied by elements $G \in \mathcal{L}$ by the ring operation of composition. Ie.,

$$G\,\mathcal{X}[F] \in \mathcal{L}, \text{ and } \mathcal{X}[F]\,G \in \mathcal{L}. \qquad b)$$

Similarly, by the states of a polynomial ring $\mathcal{H}$ we mean the set $\mathcal{S}(\mathcal{H})$ of all bilinear transformations $\mathcal{X}$,

$$\chi : \mathcal{G}^1 \times \mathcal{H} \rightarrow \mathcal{H} \quad . \tag{5.5a}$$

An element $\chi \in \mathcal{S}(\mathcal{H})$ is called a <u>state</u> of the polynomial ring $\mathcal{H}$, and we shall write

$$\chi[H] = \chi[a; H] \tag{b)}$$

for its value at $[a; H] \in \mathcal{G}^1 \times \mathcal{H}$, just as we wrote (5.3b) for the ring $\mathcal{L}$. The state χ is said to be <u>homogeneous</u> if it satisfies

$$\langle \chi[H] \rangle_r = \chi[\langle H \rangle_r] \text{ for all } r \geq 0 \; . \tag{c)}$$

Since as a subset $\mathcal{L} \subset \mathcal{H}$, it follows that as a subset

$$\mathcal{S}(\mathcal{H}) \subset \mathcal{S}(\mathcal{L}) \tag{5.6}$$

when the domain of the states in $\mathcal{S}(\mathcal{H})$ are restricted to $\mathcal{L}$.

The corresponding formulas to (5.4) for elements of $\mathcal{S}(\mathcal{L})$ can also be written for elements of $\mathcal{S}(\mathcal{H})$, but care must be taked to interpret the operations of addition and multiplication as polynomial addition and multiplication. With this distinction, the corresponding formulas to (5.4) have the same form

$$\chi[H + K] = \chi[H] + \chi[K] \; , \tag{5.7a}$$

and

$$H\,\chi[K] \in \mathcal{H}, \text{ and } \chi[K]H \in \mathcal{H}, \text{ for } H, K \in \mathcal{H}, \tag{b)}$$

but have quite different interpretations.

b) Derivations and Structures

By a <u>derivation</u> on the ring $\mathcal{L}$ we mean a homogeneous state $\mathcal{X} \in \mathcal{S}(\mathcal{L})$ which satisfies for all $F, G \in \mathcal{L}$,

$$\mathcal{X}[FG] = \mathcal{X}[F]G + F\,\mathcal{X}[G] \; . \tag{5.8a}$$

Introducing the symbolism

$$\dot{F}(B) = F_a(B) \equiv \mathcal{X}[a;\, F](B) \quad \text{for all} \quad a, B \in \mathcal{G} \, , \qquad \text{b)}$$

the property a) takes the more workable and familiar form

$$[FG]_a(B) = F_aG(B) + FG_a(B) \quad . \qquad \text{c)}$$

As an application of (5.8a,c), let $F = Q = G$ where Q is the identity map on $\mathcal{G}$. We then find that

$$Q_a(B) = 0 \quad \text{for all} \quad a, B \in \mathcal{G} . \tag{5.9a}$$

More generally, any map $F \in \mathcal{L}$ which satisfies the property

$$F_a(B) = 0 \quad \text{for all} \quad a, B \in \mathcal{G} \, , \qquad \text{b)}$$

is said to be <u>constant</u> on $\mathcal{G}$ with respect to the derivation $\mathcal{X}$. We see from (5.9a) that Q is constant on $\mathcal{G}$ with respect to each derivation on $\mathcal{L}$.

By a <u>derivation</u> on the polynomial ring $\mathcal{H}$ we mean a homogeneous state $\chi \in \mathcal{S}(\mathcal{H})$ which satisfies the corresponding rule to (5.8a) for the product of polynomials in $\mathcal{H}$. Thus we require that for all $H, K \in \mathcal{H}$,

$$\chi[HK] = \chi[H]K + H\,\chi[K] \quad . \tag{5.10a}$$

In the complementary symbolism to (5.8b), (5.10a) takes the suggestive and usable form

$$[HK]_a = H_aK + HK_a \equiv \chi[a;\, HK] \quad , \qquad \text{b)}$$

or even the still more abreviated, but equivalent form

$$[\dot{HK}] = \dot{H}K + H\dot{K} = \chi[HK] \quad . \qquad \text{c)}$$

Again it is essential to remember that the multiplication in (5.10) is polynomial multiplication, and not functional composition as in (5.8a,c).

As an application, let $H = f = K$ in (5.10b) where f is an outermorphism. Then, by taking outer product parts and using the basic property (4.8) of outermorphisms, we find that for all $A, B \in \mathcal{G}$ and $a \in \mathcal{G}^1$ that

$$f_a(A \wedge B) = f_a(A) \wedge f(B) + f(A) \wedge f_a(B) \quad . \tag{5.11}$$

The relation (5.11) expresses the important <u>derivation rule for an outermorphism</u>.

Finally, by a <u>geometric structure</u> on $\mathcal{G}$ we shall mean a homogeneous state $\chi \in \mathcal{S}(\mathcal{H})$ which is both a derivation on $\mathcal{H}$ satisfying (5.10), and (when restricted to $\mathcal{L}$) a derivation on $\mathcal{L}$ satisfying (5.8). Since the concern of the remainder of this work is the study of geometric structures, it is worth while to collect their properties here in terms of the symbolism which will be most often used. For $F, G \in \mathcal{L}$, the geometric structure χ satisfies:

$$[F + G]_a(A) = F_a(A) + G_a(A) \quad , \tag{5.12a}$$

$$[F(A) + G(B)]_a = F_a(A) + G_a(B) \quad , \tag{b}$$

$$[FG]_a(A) = F_aG(A) + FG_a(A) \quad , \tag{c}$$

$$[F(A)G(B)]_a = F_a(A)G(B) + F(A)G_a(B) \quad . \tag{d}$$

A few simple but important properties follow quickly from (5.12). Letting $F = 0 = G$ in a) gives

$$0_a = 0 \text{ for all } a \in \mathcal{G}^1 \quad . \tag{5.13a}$$

Letting $F = Q$ and $G = 0$ in b), and using (5.9a) and (5.13a), gives

$$[A]_a \equiv [Q(A)]_a = Q_a(A) = 0 \quad \text{for} \quad a \in \mathcal{G}^1 , \quad A \in \mathcal{G} . \qquad \text{b)}$$

Finally letting $F = Q$ in (5.12d) gives

$$[A\ G(B)]_a = A\ G_a(B) \quad \text{for all} \quad a,A,B \in \mathcal{G} . \qquad \text{c)}$$

The properties (5.13) should be recognized as the ordinary rules of differential calculus. But note that in this framework, "constants" are always replaced by constant maps, and that only maps are differentiated.

Because the projection operator Q on $\mathcal{G}$ is constant (recall (5.9a)), we will say that $\mathcal{G}$ is flat. It is because $\mathcal{G}$ is flat that we are able to assume that the gradient ∇_Z under the structure χ behaves like a constant. This means that

$$[\nabla_Z]_a \equiv \chi[\nabla_Z] \equiv 0 \quad . \qquad (5.14a)$$

This property makes it possible to "commute" the gradient and structure operations. Thus for $F \in \mathcal{L}$,

$$[\nabla_Z\ F(Z)]_a \equiv \chi[\nabla_Z\ F(Z)] = \nabla_Z \chi[F(Z)] \equiv \nabla_Z\ F_a(Z) . \qquad \text{b)}$$

The following important property is seen to be a simple consequence of (5.12d), (5.14b) and (5.13b):

$$[F^{\dagger}(B)]_a = [\nabla_Z\ F(Z) \odot B]_a = \nabla_Z\ F_a(Z) \odot B \equiv F_a^{\dagger}(B) . \qquad (5.15a)$$

In words (5.15a) says that the derivation of the adjoint of a map is the same as the adjoint of the derivation of the map. It immediately follows from (5.15a) and (4.7), that for a symmetric map F ,

$$F_a(B) = F_a^{\dagger}(B) \quad \text{for all } a,B \in \mathcal{G} , \qquad \text{b)}$$

and in particular for the finite projection operator P ,

$$P_a(B) = P_a^{\dagger}(B) \quad \text{for all} \quad a,B \in \mathcal{G} \ . \qquad c)$$

Thus derivations of symmetric maps are themselves symmetric.

A final simple but important consequence of (5.14a) and (5.13b) is

$$[\nabla_V]_a \equiv [P(\nabla_Z)]_a = P_a(\nabla_Z) \quad , \qquad (5.16a)$$

where $\nabla_V \equiv P(\nabla_Z)$ is the gradient of the finite subalgebra $\mathfrak{D}$. From (5.12d) and a) above, it follows that for $F \in \mathcal{L}$,

$$[\nabla_V F(V)]_a = P_a(\nabla_Z) F(Z) + \nabla_V F_a(V) \quad . \qquad b)$$

If F is a P-map (recall def. (2.4a)), (5.16b) simplifies to

$$[\nabla_V F(V)]_a = P_a(\nabla_V) F(V) + \nabla_V F_a(V) \quad . \qquad c)$$

By comparing (5.16) to (5.14), it is seen that the derivation of the P-gradient of a map is more complicated than the derivation of the full gradient. This complication is further reflected in

$$[F^P(B)]_a = [\nabla_V F(V) \odot B]_a = P_a[F^{\dagger}(B)] + F_a^P(B) \quad , \qquad (5.17a)$$

which is the corresponding expression to (5.15) for the deriva- of the P-adjoint of F . Notice that this expression has a term involving the full adjoint $F^{\dagger}$ of F . If F is a P-map (5.17a) can be written

$$[F^P(B)]_a = P_a F^P(B) + F_a^P(B) \quad , \qquad b)$$

since for P-maps $F^P \equiv F^{\dagger}$. In particular, when $F \equiv P$ in b) above, we get

$$P_a(B) = P_a P(B) + P_a^P(B) \quad \text{for all} \quad a,B \in \mathcal{G} \ , \qquad c)$$

since $P^P = P$. This shows that $P_a(B) = P_a^P(B)$ only when $P(B) = 0$. Contrast this situation with (5.15c).

The extra terms in (5.16b,c) and (5.17) are of course due to the differentiation of the P-gradient ∇_V which, unlike ∇_Z , is not necessarily constant.

c) Powers of a Structure

Let χ be a structure on $\mathcal{G}$, and let H be an arbitrary polynomial in $\mathcal{H}$. The structure χ can be successively applied to H to generate the sequence

$$H\ ,\ \chi[H]\ ,\ \chi^2[H]\ ,\cdots,\ \chi^k[H]\ ,\ \cdots \tag{5.18a}$$

of polynomials in $\mathcal{H}$. Alternatively, the polynomials can be given recursively by

$$\chi^0[H] = H\ ,\quad \chi^1[H] = \chi[a_1;\ H]\ ,$$

and

$$\chi^k[H] = \chi^k[a_1,\ \ldots,a_k;\ H] \equiv \chi\left[a_k;\ \chi^{k-1}[H]\right] \quad \text{for} \quad k>1\ . \tag{b}$$

χ^k is said to be the <u>k^{th}-power</u> of the structure χ . It is clear from (5.18b) that $\chi^k[H]$ will be multilinear in all its variables $a_1,\cdots,a_k \in \mathcal{G}^1$ and $H \in \mathcal{H}$. If, in addition, for each $H \in \mathcal{H}$ and $1 \le r \le k$, $\chi^r[a_1,\cdots,a_r;\ H]$ is symmetric in the variables $[a_1,\ldots,a_r]$, then we will say that the k^{th}-power χ^k of the structure χ is <u>k-regular.</u> We allow the allow the possibility that $k=\infty$. Generalizing the symbolism adopted in (5.8b), we write

$$H_{a_1,\ \ldots,a_k} = \chi^k[H] = \chi\left[\chi^{k-1}[H]\right] = \left[H_{a_1,\ \ldots,a_{k-1}}\right]a_k \tag{c}$$

for the k^{th}-derivation of H. For the remainder of this work, however, only structures which are 2-regular will be considered, even if not always explicidly stated.

As an application of the above notation and the structural rules (5.12) and (5.13), we calculate the 1^{st}- and 2^{nd}-derivations of the example (5.2). We find that

$$H_a = F_a(A)G(B) + F(A)G_a(B) + F_aG(C) + FG_a(C) \quad , \qquad (5.19a)$$

and

$$H_{a,b} = F_{a,b}(A)G(B) + F_a(A)G_b(B) + F_b(A)G_a(B) + F(A)G_{a,b}(B) + F_{a,b}G(C) + F_aG_b(C) + F_bG_a(C) + FG_{a,b}(C) \quad . \qquad b)$$

Note the two Leibnitz product rules at work for the two different kinds of products. Also note that

$$H_{a,b} = H_{b,a} \qquad c)$$

since χ is 2-regular. This is of course analogous to the property (3.19) of chapter 3.

d) Integrable P-Structures

A structure χ_P on $\mathcal{G}$ is said to be a <u>finite P-Structure</u> if for each $F \in \mathcal{L}$ and $a,B \in \mathcal{G}$,

$$F_a(B) \equiv \chi_P[a;\ F](B) = \chi_P[P(a);\ F](B) \equiv F_{P_a}(B) \ . \qquad (5.20a)$$

If in addition the projection operator P satisfies

$$P_aP(b) = P_bP(a) \quad \text{for all} \quad a,b \in \mathcal{G}^1 \quad , \qquad b)$$

then the structure χ_P is said to be an <u>integrable P-structure</u>. (It is possible to generalize (5.20b) to include the notion of <u>torsion</u>, by modifying it to read

$$P_aP(b) - P_bP(a) \equiv T(a \wedge b) \quad , \qquad (5.20b')$$

where the torsion vector $T(a\wedge b)$ would then be subject to certain integrability conditions. However, this generalization will not be pursued here; we will always assume that $T(a\wedge b) \equiv 0$ in this work.) Henceforth only 2-regular integrable P-structures will be studied in this work.

Now let f be a (P',P)-outermorphism. The outermorphism f will be said to be <u>(P',P)-integrable</u> if

$$f_a P(b) = f_b P(a) \quad \text{for all} \quad a,b \in \mathcal{G}^1 \quad . \tag{5.21}$$

It follows from (5.20b) that the projection operator P is (P,P)-integrable.

Finally let us see what (5.20a) implies about 2^{nd}-derivations with respect to a finite P-structure. Using (5.12c) we take the derivation of (5.20a) with respect to a vector b, getting

$$\begin{aligned} F_{a,b}(B) &= [F_a(B)]_b = [F_{Pa}(B)]_b = F_{Pa,b}(B) + F_{P_b a}(B) \\ &= F_{Pa,Pb}(B) + F_{P_a b}(B) \quad . \end{aligned} \tag{5.22a}$$

The last equality follows by applying (5.20a) to the map $F_a \in \mathcal{L}$. The extra term arises from differentiating the P, and of course must vanish when $P(a) = a$, ie.,

$$F_{a,b}(B) = F_{Pa,Pb}(B) \quad \text{whenever} \quad P(a) = a \ . \qquad b)$$

Corresponding rules for higher derivations can also be found, but are not needed in this work.

e) Structural Gradients

Let χ_P be a regular integrable P-structure, and let $H \in {}_P\mathcal{H}_P$, where ${}_P\mathcal{H}_P$ is the set of (P,P)-polynomials generated by ${}_P\mathcal{L}_P$.

By the <u>structural gradient</u> of H we mean

$$\nabla_v H_v = \nabla_v : H_v + \nabla_v \wedge H_v \quad . \tag{5.23}$$

The right side of (5.23) decomposes the structural gradient into what we call the <u>structural divergence</u> and the <u>structural curl</u> of H .

Since $H = PH$, by applying (5.12d) we get

$$H_v = P_v H + P H_v \tag{5.24}$$

which separates H_v into tangent and normal parts. This formula, together with (6.15a) which will be proved in the next chapter, implies that

$$\nabla_v \cdot H_v = P(\nabla_v \cdot H_v) = \nabla_v \cdot P H_v \tag{5.25a}$$

and

$$\nabla_v \wedge H_v = \nabla_v \wedge P_v H + P(\nabla_v \wedge H_v) \quad . \tag{b}$$

The quantity $S(A) = \nabla_v P_v A$ is called the <u>shape operator</u>, and will be studied in the next chapter.

Higher order structural gradients of H can also be taken and are important. For the 2^{nd}-structural gradient of H we have, by using (5.16b),

$$\nabla_{v_2}\left[\nabla_{v_1} H_{v_1}\right]_{v_2} = \nabla_{v_2} P_{v_2}(\nabla_{v_1}) H_{v_1} + \nabla_{v_2}\nabla_{v_1} H_{v_1, v_2} \tag{5.26}$$

which in turn can be further studied —— but this will not be done here. However, in chapters 7 and 8 we will meet and study in greater detail the closely related "intrinsic" structural gradient.

f) Extensions of Structures

We wish now to briefly discuss an alternative approach to the notion of a geometric structure on $\mathcal{G}$. No proofs will be given here, nor will this approach be used further on in this work. The discussion is only included to throw more light on the nature of a geometric structure.

When we defined a structure χ, we defined it with domain all of $\mathcal{L}$ (really $\mathcal{H}$, but only the generating set $\mathcal{L}$ is important in this discussion). Since the range of χ is contained in $\mathcal{L}$, to generate powers of χ it was only necessary to successively reapply χ (to generate the sequence (5.18)).

Suppose now instead that a structure $\tilde{\chi}$ is given on the subring ${}_P\mathcal{L}_P$ of (P,P)-maps. First note that the range $\tilde{\chi}[{}_P\mathcal{L}_P]$ may not be contained in ${}_P\mathcal{L}_P$. Thus extra care must be taken when defining powers of the structure $\tilde{\chi}$. Secondly, it is reasonable to expect that $\tilde{\chi}$ and its powers $\tilde{\chi}^k$ can be extended to a structure χ and its powers χ^k on all of $\mathcal{L}$, and in such a way that the norms $\|\tilde{\chi}\|$ and $\|\tilde{\chi}^k\|$ are preserved (recall that $\mathcal{S}(\mathcal{L})$ is a normed linear space). This is certainly suggestive of but not equivalent to the Hahn Banach theorem, since the extensions must be made preserving the relations (5.8a) and (5.10a).

Still more generally suppose that $\mathcal{R}$ is a subring of ${}_P\mathcal{L}_P$, and that a structure $\tilde{\tilde{\chi}}$ is given on $\mathcal{R}$. It is reasonable to expect that it is always possible to extend $\tilde{\tilde{\chi}}$ to a structure $\tilde{\chi}$ on ${}_P\mathcal{L}_P$ (which can then, in turn, be extended to all of $\mathcal{L}$).

It should further be apparent that when the structure χ in the above considerations is an integral P-structure, then the questions of the existence of substructures and the extensions of these substructures, are closely related to the existence of submanifolds and the Whitney embedding theorem. It is expected that these and other theorems will find new and more general proofs in this language.

6. Derivations of the Projection Operator, Shape and Curvature

Let $\mathcal{X}_P$ be a regular integral P-structure on $\mathcal{G}$. All derivations taken in this chapter will be with respect to this structure. Of course P is the projection operator of the finite subalgebra $\mathcal{D}$.

In this chapter we will learn that the "structure" of an integral P-structure is determined by the projection operator P and its derivations. Particularly important in studying this structure is the shape bivector S(a) of the vector a, defined by

$$S(a) = \nabla_v \wedge P_a(v) \quad \text{for each} \quad a \in \mathcal{D}^1 , \qquad (6.1a)$$

and the Riemann curvature bivector $R(a \wedge b)$ of the bivector $a \wedge b$, defined by

$$R(a \wedge b) = \nabla_v \wedge P_a P_b(v) \quad \text{for each} \quad a \wedge b \in \mathcal{D}^2 . \qquad b)$$

a) Derivations of the Projection Operator

The characterizing property of the finite projection operator is $P^2 = P$ (1.41a), and it is natural that the most important properties of the derivation P_a follow from this property. Taking the derivation of (1.41a) we find, using (5.12c), that

$$P_a(B) = P_a P(B) + P P_a(B) \quad \text{for all} \quad a, B \in \mathcal{G} . \qquad (6.2a)$$

As special cases of this we have

$$PP_a(B) = 0 \quad \text{if} \quad P(B) = B \qquad b)$$

and

$$P_a(B) = PP_a(B) \quad \text{if} \quad P(B) = 0 \quad . \qquad c)$$

In words, P_a takes tangent multivectors (ie. multivectors in $\mathfrak{D}$) into normal ones, and vice-versa. Note that it is the property (6.2b) which makes the "extra" term of (5.22a) vanish in (5.22b), as is necessary.

Taking a 2nd-derivation of the equation (6.2a) with respect to the vector b gives

$$P_{a,b}(B) = P_{a,b}P(B) + P_aP_b(B) + P_bP_a(B) + PP_{a,b}(B) \qquad (6.3a)$$

from which follow the special cases

$$P_aP_b(B) + P_bP_a(B) + PP_{a,b}(B) = 0 \quad \text{if} \quad P(B) = B \qquad b)$$

and

$$P_{a,b}(B) = P_aP_b(B) + P_bP_a(B) + PP_{a,b}(B) \quad \text{if} \quad P(B) = 0 \quad . \qquad c)$$

Note that the special cases b) and c) do not follow from differentiating the special cases (6.2b,c). This is because the validity of (6.2b,c) is subject to a functional condition on B , which would have to be differentiated also.

Because P is an outermorphism, it satisfies the basic rule (5.11). Thus we have for all $A, B \in \mathcal{G}$ and $a \in \mathcal{G}^1$,

$$P_a(A \wedge B) = P_a(A) \wedge P(B) + P(A) \wedge P_a(B) \quad . \qquad (6.4a)$$

It is important to recognize the special cases

$$P_a(A \wedge B) = P_a(A) \wedge B + A \wedge P_a(B) \quad \text{if} \quad A, B \in \mathfrak{D} \, , \qquad b)$$

and

$$P_a(A \wedge B) = 0 \quad \text{if} \quad P(A) = 0 = P(B) \quad . \qquad c)$$

Differentiating (6.4a) a second time gives

$$P_{a,b}(A \wedge B) = P_{a,b}(A) \wedge P(B) + P_a(A) \wedge P_b(B) + P_b(A) \wedge P_a(B) + P(A) \wedge P_{a,b}(B) \quad , \qquad (6.5a)$$

which has the special case

$$P_{a,b}(A \wedge B) = P_a(A) \wedge P_b(B) + P_b(A) \wedge P_a(B) \text{ if } P(A)=0=P(B). \quad b)$$

The relation (6.5b) is interesting in that the left side is a 2^{nd}-derivation, but the right side involves only 1^{st}-derivations.

Return now to equation (6.4a). Applying P_b to both sides of this equation gives, with the help of (6.4a)

$$P_bP_a(A \wedge B) = P_bP_a(A) \wedge P(B) + PP_a(A) \wedge P_bP(B) \\ P_bP(A) \wedge PP_a(B) + P(A) \wedge P_bP_a(B) \quad . \qquad (6.6a)$$

If $A, B \in \mathfrak{D}$, this simplifies to

$$P_bP_a(A \wedge B) = P_bP_a(A) \wedge B + A \wedge P_bP_a(B) \quad . \qquad b)$$

The equation b) will be used when discussing Riemann curvature.

To complete this section we give two important consequences of the basic integrability condition (5.20b). The first is

$$[\nabla_v : A_r] \wedge P_v(B) = (-1)^{r+1} P_v(A_r) \wedge [\nabla_v : B] \text{ for } A_r, B \in \mathfrak{D} \quad (6.7)$$

and can be established by an inductive argument on the degrees of the multivectors. (The case for $A_1 = a$, and $B = b$ is (5.20b).) The second is

$$P_{a,c}P(b) + P_aP_c(b) = P_{b,c}P(a) \text{ for all } a, b, c \in \mathcal{G}^1 \quad (6.8a)$$

and follows directly by taking the 2^{nd}-derivation of (5.20b) with respect to the vector c. Reordering the terms in (6.8a) and using (5.20b) gives

$$P_{a,c}(b) - P_{b,c}(a) = P_b P_a(c) - P_a P_b(c) \quad \text{for} \quad a,b,c \in \mathcal{D}^1, \qquad \text{b)}$$

or equivalently,

$$P_{a,c}(b) - P_{b,c}(a) = [P_b, P_a](c) \quad \text{for all} \quad a,b,c \in \mathcal{D}^1 \quad , \qquad \text{c)}$$

where the bracket $[P_b, P_a]$ of the derivations P_b and P_a is defined by

$$[P_b, P_a] \equiv P_b P_a - P_a P_b \quad . \qquad (6.9)$$

The relation (6.8c) is closely related to the curvature bivector (6.1b) as we will later see.

b) Shape of a Structure

We have already defined in (6.1a) the shape bivector of a vector $a \in \mathcal{D}^1$. It is possible to define a more general <u>shape operator</u> on all of $\mathcal{G}$ by

$$S(A) = \nabla_v P_v(A) \quad \text{for each} \quad A \in \mathcal{G} \; . \qquad (6.10)$$

That (6.10) does indeed reduce to (6.1a) when $A = a \in \mathcal{D}^1$ is shown below, as well as more general properties.

To show the equivalence of (6.10) to (6.1a) for each vector $a \in \mathcal{D}^1$, it is only necessary to use (6.2b) and the integrability condition (5.20b) to get

$$S(a) = \nabla_v P_v(a) = \nabla_v \wedge P_v(a) = \nabla_v \wedge P_a(v) \; . \qquad (6.11a)$$

On the other hand if $b \in \mathcal{G}^1$ and $P(b) = 0$, then

$$\nabla_v \wedge P_v(b) = \nabla_v \wedge P_v^P(b) = \nabla_v \wedge \nabla_{v_1} P_v(v_1) \cdot b = 0 \quad , \qquad \text{b)}$$

from which it follows that

$$S(b) = \nabla_v \cdot P_v(b) + \nabla_v \wedge P_v(b) = \nabla_v \cdot P_v(b) \quad . \qquad \text{c)}$$

The first equality in b) is a consequence of (5.17c), the second of (3.12), and the third of (5.20b).

The integrability condition (5.20b) implies that the shape operator is symmetric, ie.

$$\nabla_v \wedge S(v) = 0 \quad , \tag{6.12a}$$

or equivalently that

$$a \cdot S(b) = P_a(b) = P_b(a) = b \cdot S(a) \quad \text{for all} \quad a, b \in \mathfrak{D}^1. \tag{b}$$

The above equivalence is easily verified by showing that

$$(a \wedge b) \cdot [\nabla_v \wedge S(v)] \equiv a \cdot S(b) - b \cdot S(a) \quad \text{for} \quad a \wedge b \in \mathfrak{D}^2 .$$

Where-as the curl of the shape operator is zero by a) above, in general its divergence is not. This leads us to define the <u>shape normal</u> of the structure χ_P by

$$N(\chi_P) \equiv \nabla_v \cdot S(v) = \nabla_{v_1} \cdot \nabla_v \; P_v(v_1) \quad . \tag{6.13}$$

The shape normal is a fundamental quantity, but will not be studied in this work.

The general identity

$$S(A_r \wedge B) \equiv S(A_r) \wedge P(B) + [\nabla_v : P(A_r)] \wedge P_v(B) \tag{6.14a}$$
$$+ (-1)^r P_v(A_r) \wedge [\nabla_v : P(B)] + (-1)^r P(A_r) \wedge S(B)$$

which is true for all $A_r, B \in \mathcal{G}$, follows from the definition (6.10), (6.4a), and the algebraic identities (1.17a), (1.18a) and (1.20). Using the generalized integrability condition (6.7), (6.14a) simplifies to

$$S(A_r \wedge B) = S(A_r) \wedge B + (-1)^r A_r \wedge S(B) \quad \text{for} \quad A_r, B \in \mathfrak{D}. \tag{b}$$

The identities (6.14a,b) make it possible to generalize (6.11a,c). For $A \in \mathfrak{D}$ we find that

$$S(A) = \nabla_v : P_v(A) + \nabla_v \wedge P_v(A) = \nabla_v \wedge P_v(A) \,, \; P[S(A)] = 0 \,, \quad (6.15a)$$

and for $P(A) = 0$

$$S(A) = \nabla_v : P_v(A) + \nabla_v \wedge P_v(A) = \nabla_v \cdot P_v(A) = P[S(A)] \,. \quad b)$$

For both $P(A) = 0 = P(B)$,

$$S(A \wedge B) = 0 \,. \quad c)$$

These identities should be compared with the identities (6.4). Note that (6.15a) implies that for $A \in \mathcal{D}$, $\nabla_v \cdot P_v(A) = 0$, which is exactly what was required to establish (5.25) of chapter 5. Note also that the shape operator S takes tangent r-vectors into normal (r+1)-vectors, and conversely.

From the last remark it follows that the operator $S^2 \equiv SS$ will preserve both tangent and normal multivectors, and also their degrees. By applying S to both sides of (6.14a), and using (6.14a,b) and (6.15a,b), we calculate the special cases

$$S^2(a \wedge b) = S^2(a) \wedge b + P_b S(a) - P_a S(b) + a \wedge S^2(b) \quad (6.16a)$$

for $a, b \in \mathcal{D}^1$, and

$$S^2(a \wedge b) = S(a)S(b) + S[a \cdot S(b)] \quad \text{if} \quad P(a) = 0,\; P(b) = b \quad b)$$

In the next section we shall show that the operator S^2 is closely related to the Ricci operator.

Several other important relationships between the shape operator S and the derivation P_a are

$$P_a(B) = a \cdot S(B) + S(a:B) \quad \text{for} \quad a, B \in \mathcal{D} \,, \quad (6.17a)$$

$$P_a(B) = \frac{1}{2} [B, S(a)] \quad \text{for} \quad a, B \in \mathcal{D} \,, \quad b)$$

and

$$P_a(B) = \frac{1}{2} P[S(a), B] \quad \text{for} \quad P(a) = a \,, \quad P(B) = 0 \quad c)$$

where the bracket in b) and c) is the commutator bracket defined in (1.24). These relationships can be easily established by applying the principle of multivector decomposition (1.13).

If we let $B = I$ in (6.17b), we can with the help of (1.20) solve for $S(a)$. Thus we find

$$S(a) = I^{\dagger} P_a(I) \quad . \tag{6.18}$$

The above relationship shows that $S(a)$ is completely determined by the a-derivation P_a operating on the unit pseudoscalar I of $\mathcal{D}$, and should be compared with (6.11a).

Using (6.17b,c), and with the help of (6.15a,b), it is easy to derive the composition formulas

$$P_b P_a(A) = \frac{1}{4} P\left\{\left[S(b), [A, S(a)]\right]\right\} \text{ for } A \in \mathcal{D} \, , \tag{6.19a}$$

and

$$P_b P_a(A) = \frac{1}{4} \left[S(b), P\left([a, S(a)]\right)\right] \text{ for } P(A) = 0 \tag{b)}$$

The formula b) above will be used in the discussion of curvature in the next section.

Finally we calculate the derivation of $S(A)$ with respect to the vector b. Using definition (6.10) and (5.16c), we find that

$$S_b(A) = P_b(\nabla_v) \; P_v(A) + \nabla_v \, P_{v,b}(A) \text{ for } b, A \in \mathcal{G} . \tag{6.20}$$

This identity will also be used in the next section.

c) Curvature of a Structure

In this section we will examine the curvature bivector and its close relationship to the shape bivector. A more general curvature operator will be defined, and several of its properties studied.

We start by giving several equivalent expressions for the Riemann curvature bivector. For all $a,b \in \mathfrak{D}^1$

$$R(a\wedge b) = \nabla_v \wedge P_a P_b(v) = P_a S(b) = \frac{1}{2} P\left[S(a), S(b)\right] = \nabla_{v_2} P_{v_1}(a) P_{v_2}(b) \quad . \tag{6.21a}$$

The equivalence is quickly established: The first equality is definition (6.1b), the second follows using (6.1a), (6.2b) and (6.4a), the third using (6.15a) and (6.17c), and the last using (1.27). Note that we have implicidly assumed that $R(a\wedge b)$ is a function of the bivector $a\wedge b$, but that all onr expressions for $R(a\wedge b)$ are given in terms of the vectors a and b. The following identity gives R explicidly as a function of the bivector $a\wedge b$, and can be easily verified.

$$R(a\wedge b) = \frac{1}{4}(a\wedge b)\cdot\nabla_{v_2} P\left[S(v_1), S(v_2)\right] \quad . \tag{b}$$

There is a very close relationship between the curvature bivector $R(a\wedge b)$ and the bracket of the derivations P_a and P_b defined by (6.9). Specifically we have the general identity

$$\left[P_b, P_a\right](C) = \frac{1}{2}\left[R(a\wedge b), C\right] \quad \text{for all } C \in \mathfrak{D} , \tag{6.22}$$

which is an easy consequence of (6.19a) and the Jacobi identity (1.26). This identity suggests that the bracket $[P_b, P_a]$ be called the <u>general curvature operator</u> on $\mathfrak{D}$. Since the right side of (6.22) is the commutator product of the bivector $R(a\wedge b)$ with C, some of its properties have already been determined in section 1c).

Several equivalent ways are now given for writing (6.22) when $C = c \in \mathfrak{D}^1$.

$$R(a\wedge b)\cdot c = \frac{1}{2}\left[R(a\wedge b), c\right] = \left[P_b, P_a\right](c) = P_{a,c}(b) - P_{b,c}(a) \tag{6.23}$$

The first two equalities are a consequence of (1.23) and (6.22), and the last of (6.8c).

Putting $C = I$ in (6.22) and using (1.20) gives the integrability condition

$$\left[P_b, P_a\right](I) = 0 \tag{6.24a}$$

or equivalently using (6.18),

$$P_b\left[I\, S(a)\right] - P_a\left[I\, S(b)\right] = 0 \tag{b}$$

where I is the unit pseudoscalar of $\mathcal{D}$.

Finally we carry out a calculation to show that

$$S_b(a) - S_a(b) = \left[S(a), S(b)\right] \tag{6.25}$$

the tangent part of which, by (6.21a), is exactly $2R(a\wedge b)$.

$$\begin{aligned} S_b(a) - S_a(b) &= \nabla_v\left[P_{v,b}(a) - P_{v,a}(b)\right] + P_b(\nabla_v)\,P_v(a) - P_a(\nabla_v)\,P_v(b) \\ &= \nabla_v\; v\cdot R(a\wedge b) - 2\,P_a(\nabla_v)\wedge P_b(v) \\ &= 2\,\nabla_{v_2}\,P_{v_1}(a)\cdot P_{v_2}(b) - 2\,\nabla_{v_2}\cdot\nabla_{v_1}\,P_a(v_1)\wedge P_b(v_2) \\ &= \left[S(a), S(b)\right] . \end{aligned}$$

(6.20) is used in the first step, (6.23) in the second, (A.2b) and (6.21a) in the third, and (1.27) in the last step.

d) Riemann, Ricci and Scalar Curvatures

In this section we will see how the various classical notions of curvature are dressed in this language and closely interrelated. Particularly simple expressions for and proofs of the so called Bianchi identities are given, and it is shown that the shape operator is a "square root" of the Ricci operator.

The Riemann curvature operator $R(a \wedge b)$ is completely symmetric since it satisfies the Bianchi condition (4.4). This is easily established in the following identity

$$\nabla_v \wedge R(v \wedge b) = -\nabla_v \wedge P_b S(v) = -P_b\left[\nabla_v \wedge S(v)\right] = 0 \ . \qquad (6.26a)$$

The first equality is a consequence of (6.21a), the second of (6.4a), and the last of (6.12a). (6.26a) implies that $R(a \wedge b)$ is symmetric (4.7a) and therefore satisfies

$$R(a \wedge b)\cdot(c \wedge d) = (a \wedge b)\cdot R(c \wedge d) \quad \text{for all} \quad a \wedge b, c \wedge d \in \mathfrak{D}^2 . \qquad b)$$

Another consequence of (6.26a) is that

$$R(a \wedge b)\cdot c + R(c \wedge a)\cdot b + R(b \wedge c)\cdot a = 0 \quad \text{for} \quad a,b,c \in \mathfrak{D}^1 \qquad c)$$

and is known as the 1st Bianchi identity. The identity c) follows from a) and b) with the help of the algebraic identity

$$(a \wedge b \wedge c)\cdot\left[\nabla_v \wedge R(v \wedge d)\right] \equiv (a \wedge b)\cdot R(c \wedge d) + (b \wedge c)\cdot R(a \wedge d) + (c \wedge a)\cdot R(b \wedge d) \ .$$

From this identity it should also be apparent that b) and c) toghather are equivalent to a).

The 2nd Bianchi identity is established directly by showing

$$P\left[\nabla_v \wedge R_v(a \wedge b)\right] = \nabla_v \wedge \nabla_{v_2} P_{v_1,v}(a)\cdot P_{v_2}(b) + \nabla_v \wedge \nabla_{v_2} P_{v_1}(a)\cdot P_{v_2,v}(b) = 0 \ . \qquad (6.27a)$$

The first equality is a consequence of (5.16c), (5.12d) and (6.2a), and the second of the symmetry of the variables v_1 and v, v_2 and v in the above respective terms. A more recognizable (but less usable!) form of this identity is

$$P\left[R_a(b\wedge c) + R_b(c\wedge a) + R_c(a\wedge b)\right] = 0 \qquad b)$$

and can be established in the same way that (6.26c) was established from (6.26a), but this time using the symmetry of R_a which follows from (5.15b).

In terms of the curvature bivector, the classical Ricci and scalar curvatures are best defined by

$$R(a) = \nabla_v \cdot R(v\wedge a) \qquad (6.28a)$$

and

$$R = \nabla_v \cdot R(v) = 2\,\nabla_{V_2}\, R(V_2) \quad . \qquad b)$$

No confusion can result from using the same symbol R for the Riemann, Ricci, and Scalar curvatures, since they are respectively bivector, vector, and scalar operators, and doing so draws attention to their close relationship.

The Ricci curvature vector trivially satisfies

$$\nabla_v \wedge R(v) = 0 \qquad (6.29a)$$

or equivalently

$$R(a)\cdot b = a\cdot R(b) \quad \text{for all} \quad a,b \in \mathcal{D}^1 \,, \qquad b)$$

since by (4.2) and (6.26a)

$$\nabla_{v_1}\wedge R(v_1) = \nabla_{v_1}\cdot\left[\nabla_v \wedge R(v\wedge v_1)\right] = 0 \quad .$$

The following simple calculation

$$\begin{aligned} S^2(a) &= S\left[S(a)\right] = \nabla_{v_1}\cdot P_{v_1}\left[\nabla_v \wedge P_v(a)\right] \\ &= \nabla_{v_1}\cdot\left[\nabla_v \wedge P_{v_1} P_a(v)\right] = \nabla_{v_1}\cdot R(v_1\wedge a) = R(a) \end{aligned} \qquad (6.30)$$

shows that the shape operator is a "square root" of the Ricci operator. Note also that (6.16a) shows that

$$S^2(a \wedge b) = R(a) \wedge b + a \wedge R(b) - 2R(a \wedge b) \qquad (6.31)$$

by using (6.30) and (6.21a).

Finally we give several general curvature relations which will be needed later.

$$\left[\nabla_{v_2}, R(v_2)\right] = 0 \quad . \qquad (6.32)$$

$$\frac{1}{2} \nabla_{v_2}\left[R(v_2), C\right] = \frac{1}{4}\left[\nabla_{v_2}, \left[R(v_2), C\right]\right] \quad . \qquad (6.33a)$$

Identity (6.32) is a consequence of (1.23), (6.28b) and (1.32b). Identity (6.33a) follows from (1.23), (6.32), and (1.32d). With the help of (1.21) and (1.23), (6.33a) implies that

$$\nabla_{v_2} \cdot \left[R(v_2), C\right] = 0 \quad \text{if} \quad \langle C \rangle_1 = 0 \qquad b)$$

and

$$\nabla_{v_2} \wedge \left[R(v_2), C\right] = 0 \quad \text{for all } C \in \mathfrak{D} \ . \qquad c)$$

For the case when $C = a \wedge b \wedge c$, (6.32b) is equivalent to (6.26a,c).

Note also the identities

$$\nabla_{v_2} R(v_2) a = R(a) \qquad (6.34a)$$

and

$$\left[a \cdot \nabla_{v_2}\right] \wedge \left[R(v_2) b\right] = R(a \wedge b) \quad , \qquad b)$$

which can be easily established.

Suppose now that in (6.33a), $C = A \wedge B$. Then using (1.25c), the right side can be broken down into

$$\begin{aligned} \frac{1}{4}\left[\nabla_{v_2}, \left[R(v_2), A \wedge B\right]\right] &= \frac{1}{4}\left[\nabla_{v_2}, \left[R(v_2), A\right]\right] \wedge B \\ &+ \frac{1}{4}\left[R(v_2), A\right] \wedge \left[\nabla_{v_2}, B\right] + \frac{1}{4}\left[\nabla_{v_2}, A\right] \wedge \left[R(v_2), B\right] \\ &+ \frac{1}{4} A \wedge \left[\nabla_{v_2}, \left[R(v_2), B\right]\right] \ , \end{aligned} \qquad (6.35a)$$

which has the special case

$$\frac{1}{2}\nabla_{v_{\bar{2}}}\left[R(v_{\bar{2}}), a\wedge b\right] = R(a)\wedge b + a\wedge R(b) - 2\,R(a\wedge b) \qquad \text{b)}$$
$$= S^2(a\wedge b)$$

by using (6.34), (6.33a) and (6.31).

More generally it can be shown that

$$\frac{1}{2}\nabla_{v_{\bar{2}}}\left[R(v_{\bar{2}}), A\right] = S^2(A) \quad \text{for all } A\in\mathfrak{D}. \qquad (6.36)$$

Thus there is a very close relationship between the general shape operator (6.10) and the general curvature operator (6.22).

e) Equivalent Structures

We see from the previous sections that the shape and curvature of a structure are completely determined by derivations of the projection operator. This leads us to make the following definition:

Two regular integrable P-structures χ_P and $\widetilde{\chi}_P$ will be said to be equivalent if

$$\chi_P[a;\ P] = \widetilde{\chi}_P[a;\ P] \quad \text{for all } a\in\mathfrak{D}^1. \qquad (6.37)$$

Now let $F\in{}_P\mathcal{L}_P$, and let

$$F_a \equiv \chi_P[a;\ F] \quad \text{and} \quad \widetilde{F}_a \equiv \widetilde{\chi}_P[a;\ F] .$$

Then the following simple calculation

$$\begin{aligned}\widetilde{F}_a - F_a &= \widetilde{[PFP]}_a - [PFP]_a \qquad (6.38)\\ &= \widetilde{P}_aFP + P\widetilde{F}_aP + PF\widetilde{P}_a - P_aFP - PF_aP - PFP_a\\ &= P\widetilde{F}_aP - PF_aP\end{aligned}$$

shows that on (P,P)-maps $\tilde{\chi}_P$ and χ_P are identical up to their tangent parts.

If we now define a third structure,

$$\mathcal{X}_P = \tilde{\chi}_P - \chi_P \quad , \tag{6.39a}$$

it is easy to see that $\mathcal{X}_P$ will be a <u>flat</u> regular integrable P-structure, ie. that

$$\mathcal{X}_P[a;\ P] \equiv 0 \quad \text{for all} \quad a \in \mathcal{G}^1 \ . \tag{b}$$

Finally we will say that a P-structure χ_P is a <u>minimum P-structure</u> if it satisfies the condition that for each

$F \in {}_P\mathcal{L}_P$,

$$PF_aP = 0 \quad \text{for all} \quad a \in \mathcal{G}^1 \ . \tag{6.40}$$

7. Intrinsic Structure

In this chapter we sometimes refer to a geometric P-structure χ_P as an extrinsic structure, and derivations with respect to χ_P as extrinsic derivations. This is done to emphasize the difference between the notion of a structure, until now discussed, and the notion of an intrinsic structure to be defined in this chapter.

a) Intrinsic Structure of a Map

Let $F \in \mathcal{L}$, and let P' and P be finite projection operators. We define a (P',P)-map $\bar{F}$, called the intrinsic structure of F with respect to (P',P), by

$$\bar{F}(A) = P'FP(A) \quad \text{for all} \quad A \in \mathcal{G} . \qquad (7.1a)$$

From this definition it is clear that a (P',P)-map F has an equivalent (P',P)-structure. Ie. if $F \in_{P'} \mathcal{L}_P$, then

$$\bar{F}(A) \equiv F(A) \quad \text{for all} \quad A \in \mathcal{G} . \qquad (7.1b)$$

By the adjoint of the $\bar{F}$ we shall always mean the map $\bar{F}^{\dagger}$ given by

$$\begin{aligned} \bar{F}^{\dagger}(A) &\equiv \nabla_Z \bar{F}(Z) \odot A = \nabla_Z P'FP(Z) \odot A \\ &= \nabla_Z Z \odot PF^{\dagger}P'(A) = PF^{\dagger}P'(A) \quad . \end{aligned} \qquad (7.2)$$

We see from (7.2) that the adjoint of a (P',P)-map is a (P,P')-map, as might be expected.

b) Intrinsic Structure of a Structure

Let P' and P be finite projection operators.

In terms of a regular integral P-structure χ_P, we now define a new transformation $\bar{\chi}_P$ called the <u>intrinsic structure of χ_P with respect to (P',P)</u>. It is given by

$$\bar{\chi}_P[F] \equiv P'\chi_P[F]P \quad \text{for each} \quad F \in \mathcal{L} \quad . \tag{7.3a}$$

For the case when $P' = P$, we will also write

$$\bar{\chi}_P \equiv {}_P\bar{\chi}_P \quad , \tag{b)}$$

and say that ${}_P\bar{\chi}_P$ is the <u>intrinsic structure of χ_P</u>. Unlike χ_P, $\bar{\chi}_P$ will always take (P',P)-maps into (P',P)-maps. Powers of $\bar{\chi}_P$ are defined in the natural way. Thus the second power of $\bar{\chi}_P$ is given by

$$\bar{\chi}_P^2[F] \equiv \bar{\chi}_P\left[\bar{\chi}_P[F]\right] = P'\chi_P\left[P'\chi_P[F]P\right]P \quad . \tag{c)}$$

Actually, $\bar{\chi}_P$ can be extended to $\mathcal{H}$, the set of polynomials generated by $\mathcal{L}$, by defining for all $H,K \in \mathcal{H}$:

$$\bar{\chi}_P[HK] = \bar{\chi}_P[H]K + H\,\bar{\chi}_P[K] \quad . \tag{7.4}$$

Since $\bar{\chi}_P$ is defined in terms of χ_P, its properties are completely determined by those of χ_P. In order to more clearly see what these properties are, we introduce the following shortened notation: For each $F \in \mathcal{L}$ write

$$\bar{F}_a \equiv \bar{\chi}_P[a;\, F] = P'\chi_P[a;\, F]P = P'F_aP \tag{7.5a}$$

and

$$\bar{F}_{a,b} \equiv \bar{\chi}_P^2[a,b;\, F] = P'\left[\bar{F}_a\right]_b P = P'\left[P'F_aP\right]_b P \quad . \tag{b)}$$

By the adjoint $\bar{F}_a^\dagger$ of F_a we shall always mean

$$\bar{F}_a^\dagger \equiv PF_a^\dagger P' = [\bar{F}_a]^\dagger \quad , \tag{7.6a}$$

and for the adjoint $\bar{F}_{a,b}^\dagger$ of $\bar{F}_{a,b}$

$$\bar{F}_{a,b}^\dagger \equiv P[PF_a^\dagger P']_b P' = [\bar{F}_{a,b}]^\dagger \quad , \qquad b)$$

as might be expected. However, a certain amount of care must be taken in the use of the bar to indicate intrinsic operations, or confusion can arise. Immediate consequences of definition (7.6) are that

$$\bar{F}_a(A) \odot B = A \odot \bar{F}_a^\dagger(B) \tag{7.7a}$$

and

$$\bar{F}_{a,b}(A) \odot B = A \odot \bar{F}_{a,b}^\dagger(B) \quad \text{for } A,B \in \mathcal{G} . \qquad b)$$

We now need several simple properties of derivations of (P',P)-maps (with respect to $\mathcal{X}_P$). Let $F \in {}_{P'}\mathcal{L}_P$, then

$$F_a \equiv [P'FP]_a = P_a'F + P'F_aP + FP_a \quad , \tag{7.8a}$$

and has the special cases

$$F_a(A) = P_a'F(A) + P'F_a(A) \quad \text{for all } A \in \mathcal{D} \qquad b)$$

and

$$F_a(A) = FP_a(A) \quad \text{if } P(A) = 0 \quad . \qquad c)$$

Using the properties (7.8b,c) and (6.2b,c) we now calculate $\bar{F}_{a,b}$ from (7.5b). For $F \in {}_{P'}\mathcal{L}_P$ we have

$$\begin{aligned} \bar{F}_{a,b} &= P'[P'F_aP]_bP = P'P_b'F_aP + P'F_{a,b}P + P'F_aP_bP \\ &= P_b'P_a'F + P'F_{a,b}P + FP_aP_bP \quad , \end{aligned} \tag{7.9a}$$

from which it follows that for tangent $A \in \mathcal{D}$

$$\bar{F}_{a,b}(A) = P_b'P_a'F(A) + P'F_{a,b}(A) + FP_aP_b(A) \quad . \qquad b)$$

The special cases of (7.5a) and (7.9b) when $F = P' = P$ are of interest. For these cases we have

$$\bar{P}_a(A) = 0 = \bar{P}_{a,b}(A) \quad \text{for all } b, A, a \in \mathfrak{D}. \qquad (7.10)$$

Equation (7.10) shows that it would not be possibie to use (6.1) with intrinsic derivations to define the shape and curvature bivectors. In fact in such an approach the shape bivector would have to be given up entirely, and curvature would then make its appearence only in an indirect way as we shall see below. The integrability condition (5.20b) would also have to be re-expressed, as will be done in the next chapter. Of course there is no reason why the advantages of both intrinsic and extrinsic derivations can't be exploited, depending only on the problem at hand which or whether both are used.

The intrinsic derivation (7.5a) preserves (P',P)-maps by projecting away extrinsic or orthogonal parts. This process of "projecting away", in effect, replaces lost information about the extrinsic part with intrinsic or "curvature" information. This new information then finds its expression in the non-commutativity of extrinsic derivations. We find using (7.9b) and the commutativity of extrinsic derivations, that for $F \in P' \mathcal{L} P$

$$\bar{F}_{a,b}(C) - \bar{F}_{b,a}(C) = [P'_b, P'_a]F(C) - F[P_b, P_a](C) \quad . \qquad (7.11a)$$

Note that the right side of (7.11a) involves only commutator brackets of derivations of the projection operators P' and P. In the case that $P' = P$ we find more explicitly that

$$\bar{F}_{a,b}(C) - \bar{F}_{b,a}(C) = \frac{1}{2}[R(a\wedge b), F(C)] - \frac{1}{2} F\left([R(a\wedge b), C]\right) \qquad b)$$

by using (6.22).

If in addition F is scalar valued, then the first term on the right vanishes and we are left with

$$\bar{F}_{a,b}(C) - \bar{F}_{b,a}(C) = -\frac{1}{2} F\left(\left[R(a \wedge b), C\right]\right) \quad \text{for} \quad F \in {}_P\mathcal{L}^{0}_{P} . \qquad c)$$

The relationships b) and c) suggest that we define

$$\frac{1}{2} \bar{F}_{v_1 \wedge v_2} C \equiv \frac{1}{2}\left[\bar{F}_{v_1, v_2}(C) - \bar{F}_{v_2, v_1}(C)\right] \equiv \bar{F}_{v_{\bar{2}}}(C) \quad , \qquad d)$$

since this quantity is dependent only on the bivector $v_{\bar{2}} \equiv \frac{1}{2} v_1 \wedge v_2$ and not the vectors v_1 and v_2 independently. In the next chapter (7.11) will be used when discussing linear forms on $\mathcal{G}$.

As an application of intrinsic derivations, we can equivalently write

$$\nabla_v \wedge \bar{R}_v (a \wedge b) = 0 \qquad (7.12a)$$

for the 2^{nd}-Bianchi identity (6.27a), and

$$\bar{S}_b(a) - \bar{S}_a(b) = P\left[S(a), S(b)\right] = 2R(a \wedge b) \qquad b)$$

for the tangent part of the relationship (6.25). (7.12b) is a good example of how both intrinsic and extrinsic derivations can make their appearance in the same problem; the extrinsic derivation is involved in the definition of shape, but we are taking intrinsic derivations of it.

c) Intrinsic Structural Gradients

Intrinsic Structural gradients are better behaved than their extrinsic counterparts of section 5e).

Let $F \in \mathcal{L}$. By the intrinsic structural gradient of F we mean

$$\nabla_v \bar{F}_v = \nabla_v : \bar{F}_v + \nabla_v \wedge \bar{F}_v \quad , \qquad (7.13)$$

where $\bar{F}_v$ is of course the intrinsic v-derivation of F with respect to ${}_P\chi_P$. The right side of (7.13) uses (1.17a) to algebraically decompose the intrinsic gradient into the intrinsic divergence and intrinsic curl of F. It is immediately clear from (5.25a) that if $F \in {}_P\mathcal{L}_P$, then the extrinsic and intrinsic structural divergences of F are equivalent.

For the remainder of this section let $F \in {}_P\mathcal{L}_P$.

The analogous expression to (5.26) for the 2nd-intrinsic gradient of F is

$$\nabla_{v_2} P \left[\nabla_{v_1} \bar{F}_{v_1}\right]_{v_2} P = \nabla_{v_2} \nabla_{v_1} \bar{F}_{v_1,v_2} \quad , \qquad (7.14a)$$

but is considerably easier to work with. The right side of (7.14a) can be algebraically decomposed into

$$\nabla_{v_2} \nabla_{v_1} \bar{F}_{v_1,v_2} = \nabla_{v_2} \cdot \nabla_{v_1} \bar{F}_{v_1,v_2} + \nabla_{v_2} \wedge \nabla_{v_1} \bar{F}_{v_1,v_2} \; . \qquad b)$$

The second term on the right side of this equation can be further expressed in terms of curvature. We find by using (7.11)

$$\begin{aligned} \nabla_{v_2} \bar{F}_{v_1,v_2}(C) &= \frac{1}{2} \nabla_{v_{\bar{2}}} \left[\bar{F}_{v_1,v_2} - \bar{F}_{v_2,v_1}\right](C) \\ &= \frac{1}{2} \nabla_{v_{\bar{2}}} \left[R(v_{\bar{2}}), F(C)\right] - \frac{1}{2} \nabla_{v_{\bar{2}}} F\left(\left[R(v_{\bar{2}}); C\right]\right) \qquad c) \\ &= \nabla_{v_{\bar{2}}} \bar{F}_{v_{\bar{2}}}(C) \quad . \end{aligned}$$

The first term of the middle equality can be further analized with the help of (6.33); The second is more complicated and depends on F. We shall better understand the significance of these properties when we meet them again in the next chapter.

8. Forms, Fields, and the Bracket Operation

Let χ_P be a regular integrable P-structure with intrinsic structure ${}_P\bar{\chi}_P$, and let $\mathcal{L}^o_P$ be the subset of $\mathcal{L}^o$ consisting of P-forms on $\mathcal{G}$.

a) Forms and Fields

It is well-known that each bounded linear functional $F \in \mathcal{L}^o$ on a Hilbert space $\mathcal{G}$ can be uniquely represented in the form

$$F(Z) = Z \odot A^F \tag{8.1a}$$

for some unique element $A^F \in \mathcal{G}$, and conversely. (See [3; 178] for example.) We can explicitly solve for the element A^F by differentiating both sides of (8.1a) by ∇_Z, getting

$$A^F = \nabla_Z F(Z) \quad . \qquad \text{b)}$$

The element A^F is called the field of the form F. If in addition F is a P-form, then

$$A^F = \nabla_Z F(Z) \equiv \nabla_V F(V) = P(A^F) \quad , \qquad \text{c)}$$

and A^F is then called the P-field of the P-form F. We shall denote by $\mathcal{G}^F$ the set of field representations of the multivectors of $\mathcal{G}$ given by (8.1b), and by $\mathcal{B}^F$ the set of P-field representations of the multivectors of $\mathcal{B}$ given by (8.1c).

As a simple excercise we use (8.1b) to calculate the adjoints of a form F. Using definitions (3.11a), (3.12) and

(1.13), and (8.1b) we find that for each $B \in \mathcal{G}$

$$F^{\dagger}(B) = \nabla_Z F(Z) \circledcirc B = \nabla_Z \langle F(Z)B \rangle_0 = \beta A^F \tag{8.2a}$$

and

$$F^P(B) = \nabla_V F(V) \circledcirc B = \beta P(A^F) \qquad \text{b)}$$

where $\beta \equiv \langle B \rangle_0$. If in addition F is a P-form, then (8.1c) gives

$$F^{\dagger}(B) = F^P(B) = \beta A^F \quad . \qquad \text{c)}$$

Equations (8.1) and (8.2) show that the adjoint of a form or P-form is a field or P-field, and also conversely.

We now calculate derivations of fields in terms of derivations of their corresponding forms. By directly applying (5.14b) to (8.1b), we find that

$$A^F_a \equiv [\nabla_Z F(Z)]_a = \nabla_Z F_a(Z) = A^{F_a} \tag{8.3a}$$

and

$$A^F{}_{a,b} \equiv \left[\nabla_Z F_a(Z)\right]_b = \nabla_Z F_{a,b}(Z) = A^{F_{a,b}} \quad . \qquad \text{b)}$$

In the case that F is a P-form (8.3a) can be written

$$A^F_a = [\nabla_V F(V)]_a = P_a(A^F) + P(A^F_a) \tag{8.4}$$

by using (5.16c) and (8.1c), or alternatively (7.8b).

From definition (7.5) we now calculate intrinsic derivations of A^F, getting

$$\bar{A}^F_a \equiv \nabla_Z \bar{F}_a(Z) = A^{\bar{F}_a} \equiv \nabla_V F_a(V) = P(A^F_a) \tag{8.5a}$$

and

$$\bar{A}^F_{a,b} \equiv \nabla_Z \bar{F}_{a,b}(Z) = \nabla_V \bar{F}_{a,b}(V) = A^{\bar{F}_{a,b}} \quad . \qquad \text{b)}$$

Finally using (8.5b), (7.11c), (8.1a) and (1.25b) we calculate the commutator

$$\bar{A}^F_{a,b} - \bar{A}^F_{b,a} = \nabla_V \left[\bar{F}_{a,b}(V) - \bar{F}_{b,a}(V)\right] = \tag{8.6a}$$

$$= -\frac{1}{2}\nabla_V F\Big(\big[R(a\wedge b),V\big]\Big) = -\frac{1}{2}\nabla_V \big[R(a\wedge b),V\big]\odot A^F$$

$$= \frac{1}{2}\nabla_V V\odot\big[R(a\wedge b),A^F\big] = \frac{1}{2} P\big[R(a\wedge b),A^F\big]$$

of intrinsic derivations of the field $A^F \in \mathcal{G}^F$. The right side of (8.6a) shows that this commutator is dependent only on the curvature of the structure χ_P and the field A^F. For the case when F is a 1-form, (8.6a) simplifies to

$$\tilde{c}^F_{a,b} - \tilde{c}^F_{b,a} = \frac{1}{2}\big[R(a\wedge b),c^F\big] = R(a\wedge b)\cdot c^F \qquad \text{b)}$$

where $c^F \equiv \nabla_Z F(Z) \in \mathcal{G}^F$.

b) Brackets of P-fields

In this section we study an important P-bracket operation defined by

$$[H,K]_P \equiv (H:\nabla_v)\wedge K_v - H_v\wedge(\nabla_v:K) \qquad (8.7)$$

for all $H,K \in \mathcal{H}$. We shall be particularly interested in determining the properties of this P-bracket operation as applied to P-fields, even though the properties are more generally true for all (P,P)-polynomials $H,K \in {}_P\mathcal{H}_P$. The P-bracket is a generalization of the familiar Lie bracket of vector fields, and was first defined and studied in [25].

The P-bracket makes it possible to give a still more general form of the integrability condition (5.20b). Noting the identity

$$\big[P(A),P(B)\big]_P \equiv \big[P(A):\nabla_v\big]\wedge P_v B - P_v(A)\wedge\big[\nabla_v:P(B)\big] \qquad (8.8a)$$

for all $A,B \in \mathcal{G}$, the general integrability condition is the special case of this when A and B are tangent multivectors.

Thus for $A, B \in \mathcal{B}$ we have

$$[A,B]_P \equiv (A: \nabla_v)\wedge P_v(B) - P_v(A)\wedge(\nabla_v:B) = 0 \qquad \text{b)}$$

as a direct consequence of (6.7) and (1.20a).

The most important property of the P-bracket is that it preserves P-fields, ie. if A^F and B^G are P-fields, then so is $[A^F,B^G]_P$. This follows easily from the steps below using definition (8.7), (8.4), (8.8b), and (8.5a).

$$\begin{aligned}[A^F,B^G]_P &= (A^F: \nabla_v)\wedge P(B^G_v) + (A^F: \nabla_v)\wedge P_v(B^G) \\ &\quad - P_v(A^F)\wedge(\nabla_v:B^G) - P(A^F_v)\wedge(\nabla_v:B^G) \qquad (8.9a) \\ &= (A^F: \nabla_v)\wedge \bar{B}^G_v - \bar{A}^F_v\wedge(\nabla_v:B^G) \equiv [A^F,B^G]_{\bar{P}}\end{aligned}$$

The last line can be taken as the definition of the P-bracket with respect to intrinsic derivations, and shows that the intrinsic and extrinsic P-bracket operations of P-fields are identical. For vector P-fields a^F and b^G , (8.9a) takes the simpler form

$$[a^F,b^G]_P = a^F\cdot\nabla_v\, b^G_v - b^G\cdot\nabla_v\, a^F_v = [a^F,b^G]_{\bar{P}} \qquad \text{b)}$$

and should be recognized as being closely related to the Lie bracket of vector fields. The fact that $[a^F,b^G]_P$ is a vector P-field is another expression of the integrability condition (5.20b), and is the one refered to in the discussion after (7.10).

If in (8.6b) we replace the vectors $a,b \in \mathcal{B}^1$ with P-field representations $a^E, b^G \in \mathcal{B}^F$, we get the more complicated but perhaps more familiar expression

$$\left[\bar{c}^F{}_{a^E}\right]_{b^G} - \left[\bar{c}^F{}_{b^G}\right]_{a^E} - \bar{c}^F{}_{[b^G,a^E]_P} = R(a^E\wedge b^G)\cdot c^F \quad . \qquad (8.10)$$

Equation (8.10) is often called a "structure equation" in the literature (see for example [26; 7.40]). The extra term involving the P-bracket arises from the necessity of subtracting off the contributions from taking derivations of the fields a^E and b^G, and is unnecessary in (8.6b).

We now list and briefly discuss the most important properties of the P-bracket operation.

Let $A^E, B^F, C^G \in \mathfrak{D}^F$. Then

$$[A^E_r, B^F_s]_P = -(-1)^{(r-1)(s-1)}[B^F_s, A^E_r]_P \tag{8.11a}$$

$$[A^E + B^F, C^G]_P = [A^E, C^G]_P + [B^F, C^G]_P \qquad \text{b)}$$

$$[A^E_r, B^F_s \wedge C^G]_P = [A^E_r, B^F_s]_P \wedge C^G + (-1)^{(r-1)s} B^F_s \wedge [A^E_r, C^G]_P \qquad \text{c)}$$

$$\nabla_v : [A^E_r \wedge B^F]_v = [\nabla_v : A^E_{rv}] \wedge B^F + (-1)^r A^E_r \wedge [\nabla_v : B_v] + (-1)^{r+1}[A^E_r, B^F]_P \qquad \text{d)}$$

$$\nabla_v : ([A^E_r, B^F]_P)_v = [\nabla_v : A^E_{rv}, B^F]_P + (-1)^{r-1}[A^E_r, \nabla_v : B^F_v]_P \;. \qquad \text{e)}$$

Identity (8.11a) shows how the order of the terms of a P-bracket may be reversed, paying attention only to the changes of signs of its various multivector parts, and is a simple consequence of (1.32). Identities (8.11b,c) express distributive type rules of the P-bracket. Identity (8.11d) relates the divergencesof the outer product of fields to the P-bracket of fields and is easily established by using (1.18a) and (8.9b). Finally identity (8.11e) is a kind of Leibnitz product rule for the divergence of a P-bracket, and follows from (8.11d) with the help of (8.14b) from the next section.

c) Structural Gradients of Fields

This section should be compared with the closely related sections 5e) and 7c).

Let A^F be a P-field. Then

$$\nabla_v A_v^F = \nabla_v : A_v^F + \nabla_v \wedge A_v^F \qquad (8.12a)$$

is called the structural gradient of A^F, and is the algebraic sum of the structural divergence and structural curl of A^F. By using the decomposition formula (8.4), it is evident from (6.15a) and (8.5a) that

$$\nabla_v \cdot A_v^F = \nabla_v \cdot P_v(A^F) + \nabla_v \cdot P(A_v^F) = \nabla_v \cdot \bar{A}_v^F \qquad b)$$

and

$$\nabla_v \wedge A_v^F = \nabla_v \wedge P_v(A^F) + \nabla_v \wedge P(A_v^F) = S(A^F) + \nabla_v \wedge \bar{A}_v^F . \qquad c)$$

Identity (8.12b) shows that the extrinsic and intrinsic divergences of P-fields are equivalent. Identity (8.12c) separates the extrinsic curl of A^F into tangent and normal parts; the tangent part being the intrinsic curl of A^F, and the normal part the shape operator of A^F.

Taking a 2nd-intrinsic gradient of A^F gives

$$\nabla_{v_2} \nabla_{v_1} \bar{A}^F_{v_1, v_2} = \nabla_{v_2} \cdot \nabla_{v_1} \bar{A}^F_{v_1, v_2} + \nabla_{v_{\bar{2}}} \bar{A}^F_{v_{\bar{2}}} . \qquad (8.13)$$

We will call the first term on the right side the Laplacian of A^F. For the second term on the right we find, using (8.6a) and (6.33a), that

$$\begin{aligned}\nabla_{v_{\bar{2}}} \bar{A}^F_{v_{\bar{2}}} &= \tfrac{1}{2} \nabla_{v_{\bar{2}}} \left[\bar{A}^F_{v_1, v_2} - \bar{A}^F_{v_2, v_1}\right] \\ &= \tfrac{1}{2} \nabla_{v_{\bar{2}}} \left[R(v_{\bar{2}}), A^F\right] = \tfrac{1}{4} \left[\nabla_{v_{\bar{2}}}, R(v_{\bar{2}}), A^F\right] .\end{aligned} \qquad (8.14a)$$

This implies, using (6.33b,c), that

$$\nabla_{v_2} \cdot \bar{A}^F_{v_2} = 0 \quad \text{when} \quad \langle \bar{A}^F_{v_2} \rangle_1 = 0 \quad , \tag{b}$$

and

$$\nabla_{v_2} \wedge \bar{A}^F_{v_2} = 0 \quad \text{for all} \quad A^F \in \mathfrak{D}^F \quad . \tag{c}$$

In the language of differential forms, c) is equivalent to $d^2F = 0$. But note that in this language it would be superfluous to construct an exterior calculus of differential forms. Rather, forms are directly represented as fields, and are then subject to analysis in the framework of geometric algebra and the theory of geometric structures.

Finally we consider two special cases of (8.14a). For $A^F = a^F \in \mathfrak{D}^F$ we find, with the help of (6.34a), that

$$\nabla_{v_2} \bar{a}^F_{v_2} = \frac{1}{4}\left[\nabla_{v_2}, \left[R(v_2), a^F\right]\right] = R(a^F) = S^2(a^F) \; , \tag{8.15a}$$

and for $A^E = a^F \wedge b^G$, (6.35b) gives

$$\begin{aligned} \nabla_{v_2} \overline{\left[a^F \wedge b^G\right]}_{v_2} &= R(a^F) \wedge b^G + a^F \wedge R(b^G) - 2R(a^F \wedge b^G) \\ &= S^2(a^F \wedge b^G) \quad . \end{aligned} \tag{b}$$

More generally, from (6.36), it is seen that

$$\nabla_{v_2} \bar{A}^F_{v_2} = S^2(A^F) \quad \text{for all} \quad A^F \in \mathfrak{D}^F \quad . \tag{c}$$

9. Related Structures

Let χ_P be a regular integrable P-structure and $\chi'_{P'}$ a regular integrable P'-structure, where P and P' are the projection operators of the finite geometric subalgebras $\mathfrak{D}(I)$ and $\mathfrak{D}'(I')$.

a) Basic Definitions and Properties

We shall say that the structures χ_P and $\chi'_{P'}$ are f-related if the following two conditions are satisfied:

i) f is a (P',P)-integrable outermorphism which relates $\mathfrak{D}(I)$ and $\mathfrak{D}'(I')$. (Recall definitions (5.21) and (4.17).)

$$\text{ii) } \chi_P[a;\, F] = \chi'_{P'}[f(a);\, F] \quad \text{for each } a \in \mathfrak{D}^1 \text{ and } F \in \mathcal{L}. \tag{9.1}$$

We begin our study of f-related structures by establishing conventions that will be used. For each multivector $A \in \mathfrak{D}$, by the related multivector to A we shall always mean the unique $A' \in \mathfrak{D}'$ which satisfies

$$A' = f(A) \quad , \tag{9.2}$$

and the use of primes will always imply this relatedness, unless otherwise specified.

Let $G \in {}_{P'}\mathcal{L}_P$. Then from (9.1ii) we can write

$$\chi_P[a;\, G] \equiv G_a = G_{a'} \equiv \chi'_{P'}[a';\, G] \quad , \tag{9.3a}$$

and for intrinsic (P', P)-derivations

$$\bar{\chi}_P[a;\, G] = P'G_aP = \bar{G}_a = \bar{G}_{a'} = P'G_{a'}P \equiv \bar{\chi}'_{P'}[a';\, G]. \qquad b)$$

Note that writing G_a or $G_{a'}$ means that derivations are being taken with respect to χ_P or $\chi'_{P'}$, and represents a "change of variables" formula in this language. The relationship between the 2^{nd}-derivations $G_{a,b}$ and $G_{a';b'}$ is slightly more complicated. By taking b-derivations of the identities (9.3a,b) we get, using (5.12c) and (9.3),

$$G_{a,b} = \left[G_{f(a)}\right]_b = G_{a';b'} + G_{f_b(a)} \qquad (9.4a)$$

and

$$\bar{G}_{a,b} = P'\left[G_{f(a)}\right]_b P = \bar{G}_{a';b'} + \bar{G}_{f_b(a)} \qquad b$$

for the respective extrinsic and intrinsic 2^{nd}-derivations.

Applying (9.3) and (9.4) to the case when $G = fF$, gives

$$[fF]_{a'} = f_{a'}F + fF_a \qquad (9.5a)$$

and

$$\overline{[fF]}_{a'} = \bar{f}_{a'}F + f\bar{F}_a \qquad b)$$

by using (5.12c) and (9.1 ii), and also

$$[fF]_{a';b'} = [fF]_{a,b} - [fF]_{f_b(a)}$$

and

$$\overline{[fF]}_{a';b'} = \overline{[fF]}_{a,b} - \overline{[fF]}_{f_b(a)} \;.$$

From these formulas we calculate, by again using (5.12c) and (9.4), that

$$[fF]_{a';b'} = f_{a';b'}F + f_{a'}F_b + f_{b'}F_a + fF_{a,b} - fF_{f^{-1}f_b(a)} \qquad a)$$

and

$$\overline{[fF]}_{a';b'} = \bar{f}_{a';b'}F + \bar{f}_{a'}\bar{F}_b + \bar{f}_{b'}\bar{F}_a + f\bar{F}_{a,b} - f\bar{F}_{f^{-1}f_b(a)} \qquad (9.6)\ b)$$

Of course by f^{-1} we mean the inverse of the outermorphism f. More exactly, f^{-1} means the unique outermorphism in ${}_{P}\mathcal{L}_{P'}$ which satisfies

$$ff^{-1} = P' \quad \text{and} \quad f^{-1}f = P \ . \tag{9.7a}$$

By taking derivations of (9.7a) we get

$$f_{b'}f^{-1} + ff_b^{-1} = P'_{b'} \quad \text{and} \quad f_b^{-1}f + f^{-1}f_{b'} = P_b \ , \tag{b}$$

which imply that

$$ff_b^{-1}(a') = -\bar{f}_{b'}(a) \quad \text{and} \quad f^{-1}f_{b'}(a) = -\bar{f}_b^{-1}(a') \ . \tag{c}$$

Identity (9.7c) can be used to slightly improve the form of the last terms in (9.6a,b). This last identity also shows that f^{-1} is (P,P')-integrable whenever f is (P',P)-integrable, as might be expected.

For the remainder of this section we shall restrict our attention to learning properties of intrinsic derivations of f-related structures.

By applying (7.11) to (9.4b) with $G = fF$, and using (6.22) and the integrability condition (9.1a) of f, we find that

$$\begin{aligned}\overline{[fF]}_{v'_{\bar{2}}}(C) = \overline{[fF]}_{v_{\bar{2}}}(C) &= \tfrac{1}{2}\left[\overline{[fF]}_{v_1,v_2} - \overline{[fF]}_{v_2,v_1}\right](C) \\ &= \tfrac{1}{2}\left[R'(v'_{\bar{2}}), fF(C)\right] - \tfrac{1}{2}\, fF\left(\left[R(v_{\bar{2}}), C\right]\right)\end{aligned} \tag{9.8a}$$

where $v'_{\bar{2}} \equiv f(v_{\bar{2}})$. For the case when $F \equiv P$, and $C = c \in \mathfrak{D}^1$, (9.8a) simplifies to the important expression

$$\bar{f}_{v'_{\bar{2}}}(c) \equiv \tfrac{1}{2}\left[\bar{f}_{v'_1,v'_2}(c) - \bar{f}_{v'_2,v'_1}(c)\right] = R'(v'_{\bar{2}})\cdot c' - f\left[R(v_{\bar{2}})\cdot c\right] \tag{b}$$

which relates the curvature of the structure $\chi'_{P'}$ to the curvature of the structure χ_P through the map f. By using the identity (6.34a), and the basic relationships (4.10a) and (4.17b), we further find that

$$\nabla'_{v'_2} \cdot f_{v'_2}(c) = R'(c') - f[R(c)] \quad , \tag{9.9}$$

which relates the Ricci curvatures of the structures $\chi'_{P'}$ and χ_P through the map f. In the next chapter we shall use (9.8b) and (9.9) in characterizing projectively and conformally related structures.

Finally we give several equivalent ways of writing the integrability condition (9.1i) of f. They are

$$f_{a'}(b) = f_{b'}(a) \quad \text{for all} \quad a, b \in \mathcal{B}^1 \tag{9.10a}$$

$$f_{a'}(B) = f_{v'}(a) \wedge (\nabla'_{v'} : B') \quad \text{for} \quad a, B \in \mathcal{B} \ , \tag{b}$$

and

$$[\nabla'_{v'} : A'_r] \wedge f_{v'}(B) = (-1)^{r+1} f_{v'}(A_r) \wedge [\nabla'_{v'} : B'] \tag{c}$$

for all $A_r, B \in \mathcal{B}$. These conditions should be compared with the integrability conditions (5.20b) and (5.7) of P, and can be established in exactly the same way.

b) Auxiliary Functions

Many properties of f-related structures are best studied in terms of auxiliary functions defined by the relation

$$\phi'(B) = \frac{1}{\mu} \nabla'_{v'} \cdot f_{v'}(B) = f[\phi(B)] \quad , \tag{9.11}$$

where μ is a normalizing factor chosen for convenience. The function $\phi'(B)$ is the structural divergence of f, and $\phi(B)$ is its inverse image.

By applying the decomposition formula

$$f_{v'} = [fP]_{v'} = f_{v'}P + fP_v \quad ,$$

we find, with the help of (4.10a) and (4.17b), that

$$\nabla'_{v'}\cdot f_{v'}(B) = \nabla'_{v'}\cdot f_{v'}P(B) + \nabla'_{v'}\cdot fP_v(B)$$

$$= \nabla'_{v'}\cdot f_{v'}[P(B)] + f[S(B)] \quad ,$$

which implies that

$$\phi'(B) = \phi'[P(B)] + f[S(B)] = f[\phi(B)] \text{ for } B \in \mathcal{G}, \quad (9.12a)$$

and

$$\phi'(B) = f[S(B)] = f[\phi(B)] \quad \text{when } P(B) = 0 \ . \qquad b)$$

Thus the auxiliary functions are essentially new quantities only when evaluated at tangent multivectors $B \in \mathfrak{D}$; and henceforth we shall restrict the domain of them to $\mathfrak{D}$. On the other hand, the range of ϕ' lies in $\mathfrak{D}'$ (because of (5.25a)), whereas the range of ϕ lies in $\mathfrak{D}$.

The most important properties of the auxiliary functions are

$$\phi'(A_r \wedge B) = \phi'(A_r) \wedge f(B) + (-1)^r f(A_r) \wedge \phi'(B) \qquad (9.13a)$$

for ϕ' , and

$$\phi(A_r \wedge B) = \phi(A_r) \wedge B + (-1)^r A_r \wedge \phi(B) \qquad b)$$

for ϕ . These properties should be compared with the similar property (6.14b) of the shape operator; and they can be established in the same way, but this time using the integrability condition (9.10c) of the outermorphism f .

As an important application of (9.13a), let $A_r = a \in \mathfrak{D}^1$ and $B = I$ in it. We then find that

$$\phi'(a) f(I) = a' \wedge \phi'(I) \quad ,$$

or equivalently, by using (1.19a), (9.11) and (5.12d),

$$\phi'(a) = [a'\wedge\phi'(I)][f(I)]^{-1} = \frac{1}{\mu}\, a'\cdot\nabla'_{v'}\, f_{v'}(I)\cdot f(I^{\dagger})$$
$$= \frac{1}{\mu}\, a'\cdot\nabla'_{v'}\left[\ln|f(I)|\right]_{v'} = \frac{1}{\mu}\left[\ln|J_f|\right]_{a'} \quad , \qquad (9.14a)$$

where J_f is the generalized Jacobian of f given in (4.15a) For $\phi(a)$ we have

$$\phi(a) = \frac{1}{\mu}\, a\cdot\nabla_v\left[\ln|J_f|\right]_v = \frac{1}{\mu}\left[\ln|J_f|\right]_a \quad , \qquad b)$$

where in this case the derivation is being taken with respect to χ_p .

More generally, for each $A \in \mathfrak{D}$

$$\phi'(A) = \frac{1}{\mu}\left\{\nabla'_{v'}\left[\ln|J_f|\right]_{v'}\right\}:A' = \left[\nabla'_{v'}\,\phi'(v)\right]:A' \qquad (9.15a)$$

and

$$\phi(A) = \frac{1}{\mu}\left\{\nabla_v\left[\ln|J_f|\right]_v\right\}:A = \left[\nabla_v\,\phi(v)\right]:A \quad . \qquad b)$$

These properties are established in the steps below:

$$\phi'(A) = \frac{1}{\mu}\,\nabla'_{v'_1}\cdot f_{v'_1}(A) = \frac{1}{\mu}\,\nabla'_{v'_1}\cdot\left[f_{v'_1}(v)\wedge\left(\nabla'_{v'}:A'\right)\right]$$
$$= \phi'(v)\nabla'_{v'}:A' - \frac{1}{\mu}\, f_{v'_1}(v)\wedge\left[\left(\nabla'_{v'_1}\wedge\nabla'_{v'}\right):A'\right]$$
$$= \left[\nabla'_{v'}\,\phi'(v)\right]:A' = f\left\{\left[\nabla_v\,\phi(v)\right]:A\right\} \quad .$$

The successive steps above are justified by (9.11), (9.10b), (1.18a) and (1.19a), (9.10a), (4.10a) and (4.17b), respectively. The relationships (9.15) are important consequences of the integrability of f .

It is usually expedient to work as much as possible in terms of the auxiliary function $\phi(B)$ because both its domain and range lie in $\mathfrak{D}$. As a consequence of this, $\phi(B)$

satisfies several additional nice properties which can not be so easily stated for $\phi'(B)$. They are

$$\phi(a:B) = a:\phi(B) \quad \text{for} \quad a, B \in \mathfrak{D} \tag{9.16a}$$

and

$$\phi^2(B) = 0 \quad \text{for} \quad B \in \mathfrak{D}, \tag{b}$$

and can be easily established using (9.13b) and the Principle of multivector decomposition (1.13).

c) Derivations of Auxiliary Functions

We wish first to show the properties

$$\phi'_{a'}(b) = \phi'_{b'}(a) \quad \text{for all} \quad a,b \in \mathfrak{D}^1 \tag{9.17a}$$

and

$$\phi_a(b) = \phi_b(a) \quad \text{for all} \quad a,b \in \mathfrak{D}^1 \ . \tag{b}$$

These properties are easily established by using (9.14) and the regularity of the structures. Thus for (9.17b),

$$\phi_b(a) = \frac{1}{\mu}\left[\ln|J_f|\right]_{a,b} = \frac{1}{\mu}\left[\ln|J_f|\right]_{b,a} = \phi_a(b) \ ,$$

and (9.17a) can be established similarly. Note that

$$\phi'_{a'}(b) \equiv \bar{\phi}'_{a'}(b) \quad \text{and} \quad \phi_a(b) \equiv \bar{\phi}_a(b) \quad , \tag{9.18}$$

ie., it is immaterial whether intrinsic or extrinsic derivations are taken. Actually, we have already established (9.17) for intrinsic derivations in (8.14) with $F(V) \equiv V_o \ln|J_f|$ and $A^F \equiv \nabla_V F(V) = \ln|J_f|$.

More generally, using (9.15) and (A.2b), we find that

$$\frac{1}{r}\nabla'_{V_r'}\phi'(V_r) = \nabla'_{v'}\phi'(v) \quad \text{and} \quad \frac{1}{r}\nabla_{V_r}\phi(V_r) = \nabla_v\phi(v), \tag{9.19}$$

from which it follows, using (9.17), that

$$\frac{1}{r}\nabla'_{v'}\wedge\nabla'_{v'_r}\bar{\Phi}'_{v'}(v_r) = \nabla'_{v'_{\bar{2}}}\phi'_{v'_2}(v_1) = 0 \qquad (9.20a)$$

and

$$\frac{1}{r}\nabla_v\wedge\nabla_{v_r}\bar{\Phi}_v(v_r) = \nabla_{v_{\bar{2}}}\phi_{v_2}(v_1) = 0 \quad . \qquad b)$$

d) Brackets of f-Related Fields

Let $F\in\mathcal{L}^0_P$ and $F'\in\mathcal{L}^0_{P'}$ be P- and P'- forms respectively, and suppose their respective P- and P'-fields are given by

$$A^F = \nabla_v F(v) \text{ and } A^{F'} = \nabla'_{v'}F'(v') \ .$$

We shall say that the fields A^F and $A^{F'}$ are f-related if

$$A^{F'} = f(A^F) \quad . \qquad (9.21)$$

Taking a v'-derivation of (9.21) gives, using (9.5a),

$$A^{F'}_{v'} = f_{v'}(A^F) + f(A^F_v) \ , \qquad (9.22a)$$

from which it follows, using (4.10a) and (4.17b), and (9.11), that

$$\nabla'_{v'}\cdot A^{F'}_{v'} = f(\nabla_v\cdot A^F_v) + \mu\phi'(A^F) \quad . \qquad b)$$

The familiar transformation formula

$$\nabla'\cdot a^{F'} = \nabla\cdot a^F + a\cdot\nabla\ \ln|J_f| \qquad c)$$

for the divergence of a vector field is a special case of b).

To finish this section let A^E and $A^{E'}$, and B^F and $B^{F'}$, be f-related fields. By an argument almost identical to the argument (8.9a) showing that the Lie Bracket preserves P-fields, we now show that

$$\left[A^{E'},B^{F'}\right]_{P'} = f\left(\left[A^{E},B^{F}\right]_{P}\right) \quad , \tag{9.23}$$

ie., the Lie brackets of f-related fields are themselves f-related. Thus with the help of (9.22a),

$$\begin{aligned}\left[A^{E'},B^{F'}\right]_{P'} &= \left(A^{E'}:\nabla'_{V'}\right)\wedge B^{F'}_{V'} - A^{E'}_{V'}\wedge\left(\nabla'_{V'}:B^{F'}\right)\\ &= \left(A^{E'}:\nabla'_{V'}\right)\wedge f_{V'}\left(B^{F}\right) + f\left[\left(A^{E}:\nabla_{V}\right)\wedge B^{F}_{V}\right]\\ &\quad - f_{V'}\left(A^{E}\right)\wedge\left[\nabla'_{V'}:B^{F'}\right] - f\left[A^{E}_{V}\wedge\left(\nabla_{V}:B^{F}\right)\right]\\ &= f\left(\left[A^{E},B^{F}\right]_{P}\right) \quad .\end{aligned}$$

Note that we are using the general integrability condition

$$\begin{aligned}\left[f(A),f(B)\right]_{P'} &= \left(A':\nabla'_{V'}\right)\wedge f_{V'}(B) - f_{V'}(A)\wedge\left[\nabla'_{V'}:B'\right]\\ &\equiv 0 \qquad \text{for all } A,B \in \mathfrak{V}\end{aligned} \tag{9.24}$$

of f, which is a consequence of (9.10c); and is analogous to the condition (8.8b) of P. The general relationship (9.23) was first established in [25].

10. Projective and Conformal Structures

Many of the computations in this chapter involve simple formulas for differentiation which can be found in appendix A.

a) Projective Structures

Two f-related geometric structures are said to be projective structures if for each $a \in \mathfrak{D}^1$

$$\bar{f}_{a'}(a) \wedge a' = 0 \quad \text{where} \quad a' \equiv f(a) \quad . \tag{10.1a}$$

In studying projective structures it is convenient to choose the normalization factor of (9.11) to be $n+1$ where $n = \dim(\mathfrak{D}^1)$. Thus for all $A \in \mathfrak{D}$

$$\phi'(A) \equiv \frac{1}{n+1} \nabla'_{v'} \cdot f_{v'}(A) = f\,\phi(A) \quad . \tag{b}$$

Taking the divergence of (10.1a) by $\nabla'_{a'}$, and simplifying, gives

$$\bar{f}_{a'}(a) = 2\,\phi(a)\,a' = 2f\left[\phi(a)a\right] \quad . \tag{10.2a}$$

Further differentiating this new expression by $c' \cdot \nabla'_{a'}$, and using (9.10a), gives

$$\bar{f}_{a'}(c) = f\left[\phi(c)\,a + \phi(a)\,c\right] \quad , \tag{b}$$

which is a special case of the more general relationship

$$\bar{f}_{a'}(C_r) = f\left[r\,\phi(a)C_r + a\,\phi(C_r)\right] . \tag{c}$$

This last relationship can be established by induction on r=1. The relationship (10.1a) is trivially equivalent to each of the relationships in (10.2).

Taking now a b′-derivation of (10.2b) gives, with the help of (9.5b) and (10.2b),

$$\bar{f}_{a',b'}(c) = \bar{f}_{b'}\left[\phi(c)\ a + \phi(a)\ c\right] + f\left[\phi_b(c)a + \phi_b(a)c\right] \quad (10.3)$$

$$= f\left\{\left[\phi_b(c) + \phi(b)\,\phi(c)\right]a + 2\phi(a)\phi(b)\ b + \left[\phi_a(b) + \phi(a)\phi(b)\right]c\right\}$$

from which it follows, using (9.17b) and (7.11), that

$$\bar{f}_{v'_{\bar{2}}}(c) = \frac{1}{2}\left[\bar{f}_{v'_1,v'_2}(c) - \bar{f}_{v'_2,v'_1}(c)\right] = f\left[v_{\bar{2}}\cdot\nabla_v\Psi(v,c)\right] \quad (10.4a)$$

where $\Psi(v,c) \equiv \phi_v(c) - \phi(v)\,\phi(c)$. Taking the outer product of both sides of (10.4a) with $v'_{\bar{2}}$ gives the simpler but equivalent expression,

$$\bar{f}_{v'_{\bar{2}}}(c)\wedge v'_{\bar{2}} \equiv 0 \qquad , \qquad b)$$

which can, in turn, by (9.8b), be equivalently written as

$$\left[R'(v'_{\bar{2}})\cdot c'\right]\wedge v'_{\bar{2}} - f\left\{\left[R(v_{\bar{2}})\cdot c\right]\wedge v_{\bar{2}}\right\} = 0 \qquad . \qquad c)$$

(Note that for curvature, $f\left[R(v_{\bar{2}})\right] \neq R'(v'_{\bar{2}})$.)

The quantity

$$\mathcal{W}_P \equiv \left[R(v_{\bar{2}})\cdot c\right]\wedge v_{\bar{2}} \quad (10.5)$$

is classically known as the projective Weyl tensor; and (10.4c) shows that it's form is preserved under f , for projectively related structures. A more recognizable form of (10.4c) can be obtained by taking the simplicial divergence of both sides of it with $\nabla'_{v'_{\bar{2}}}$. Doing this, we find, with the help of (A.7), (4.10a) and (4.17b), that

$$R'(v'_{\bar{2}})\cdot c' - \frac{1}{n-1}\,v'_{\bar{2}}\cdot R'(c') = f\left[R(v_{\bar{2}})\cdot c - \frac{1}{n-1}\,v_{\bar{2}}\cdot R(c)\right] \quad . \quad (10.6)$$

Although (10.6) is fully equivalent to (10.4b,c), (10.4b,c) should be recognized as being more fundamental; it being trivial, for example, that $\mathcal{W}_p \equiv 0$ for $n = 2$ (because there are only degenerate 3-vectors in 2-dimensions).

b) Conformal Structures

Two f-related geometric structures are said to be conformal structures if for each $a \in \mathfrak{D}^1$,

$$\bar{f}_{a'}(a) \wedge a' = a' \cdot a' \; a' \wedge \nabla' \phi' \tag{10.7a}$$

where

$$\phi'(A) = \frac{1}{n} \nabla_{v'} \cdot f_{v'}(A) = \frac{1}{n} f[\phi(A)] \qquad \text{b)}$$

and

$$\nabla' \phi' \equiv \nabla'_{v'} \phi'(v) \quad . \qquad \text{c)}$$

Note that in b) the normalizing factor of the auxiliary function has been specified to be $n = \dim(\mathfrak{D}^1)$. Comparing this definition with (10.1a) shows immediately that conformal structures are projective only when $\nabla' \phi' = 0$, as is well-known. For a "standard" statement and proof of this fact see [6; 117].

We now carry out for conformal structures the analogous but more involved calculations that we did for projective structures. Taking the divergence of (10.7a) with $\nabla'_{a'}$ gives

$$\begin{aligned} \bar{f}_{a'}(a) &= f[2\phi(a)a - a^2 \nabla \phi] = f[\phi(a)a + a \cdot (\nabla \phi \wedge a)] \\ &= f[a \nabla \phi a] = a' \nabla' \phi' a' \end{aligned} \tag{10.8a}$$

Further differentiating by $c' \cdot \nabla'_{a'}$ gives, using (9.10a),

$$\bar{f}_{a'}(c) = f[\phi(c)a + a \cdot (\nabla \phi \wedge c)] = \frac{1}{2} f[c \nabla \phi a + a \nabla \phi c] \qquad \text{b)}$$

The relationship (10.7a) is trivially equivalent to each of the relationships in (10.8). Note the geometric significance of the last equality in (10.8a): It shows that the a'-derivation of a conformal map f , evaluated at the vector a , is the reflection of the vector a' through the vector $\nabla'\phi'$.

Our purpose now is to calculate what is classically known as the conformal Weyl tensor. Taking a b'-derivation of (10.8b) gives, using (9.5b),

$$\bar{f}_{a;b'}(c) = \bar{f}_{b'}\left[\phi(c)a + a\cdot(\nabla\phi\wedge c)\right] + f\left[\phi_b(c)a + a\cdot(\nabla\phi_b\wedge c)\right] \quad (10.9a)$$

from which we further calculate, by again using (10.8b), that

$$\bar{f}_{a;b'}(c)\wedge a'\wedge b' = f\Big[\Big\{\left[\phi_b(a) + \phi(a)\phi(b)\right]c - \left[a\cdot c\nabla\phi_b + \left(\phi(a)b\cdot c + a\cdot b\phi(c)\right)\nabla\phi\right]\Big\}\wedge a\wedge b\Big] \quad b)$$

Using this last expression, we find that

$$\begin{aligned}\bar{f}_{v_{\bar 2}'}(c)\wedge v_{\bar 2}' &= \frac{1}{2}\left[f_{v_1',v_2'}(c) - f_{v_2',v_1'}(c)\right]\wedge v_{\bar 2}' \\ &= f\left\{\left[\nabla_v\Psi(v,\bar c)\right]\wedge v_{\bar 2}\right\}\end{aligned} \quad (10.10a)$$

where

$$\Psi(v,\bar c) \equiv \left[\phi_{\bar c}(v) - \phi(\bar c)\phi(v)\right] \text{ and } \bar c \equiv v_{\bar 2}\cdot c \quad . \qquad b)$$

Now let

$$A = \left[R'(v_{\bar 2}')\cdot c'\right]\wedge v_{\bar 2}' - f\left\{\left[R(v_{\bar 2})\cdot c\right]\wedge v_{\bar 2}\right\}$$

and

$$B = R'(\bar c)\wedge v_{\bar 2}' - f\left[R(\bar c)\wedge v_{\bar 2}\right] \quad . \qquad (10.11a)$$

Then using the differentiation formulas (A.7), (A.8), (A.9), and (4.10a), (4.17b), we find that

$$\left\{\left[\nabla_c'\wedge(\nabla_{v_{\bar 2}}'\cdot A)\right]\cdot c'\right\}\wedge v_{\bar 2}' = -n(n-1)A + \frac{n}{2}B \quad . \qquad b)$$

But equation (10.10a) together with (9.8b) shows also that

$$A = f\left\{\left[\nabla_v\Psi(v,\bar c)\right]\wedge v_{\bar 2}\right\} \quad . \qquad (10.12a)$$

and by a similar calculation to (10.11b) we find that

$$\left\{\left[\nabla'_c\cdot\wedge\left(\nabla'_{v_{\bar{2}}}\cdot\cdot A\right)\right]\cdot c'\right\}\wedge v'_{\bar{2}} = -\frac{n^2}{2}A \quad . \qquad \text{b)}$$

Equating (10.11b) to (10.12b) gives the relationship

$$A - \frac{1}{n-2}B = 0 \quad . \qquad (10.13a)$$

Finally, by defining the quantity

$$\mathcal{W}_C = \left[R(v_{\bar{2}})\cdot c - \frac{1}{n-2}R(v_{\bar{2}}\cdot c)\right]\wedge v_{\bar{2}} \quad , \qquad \text{b)}$$

the relationship (10.13a) takes the new form

$$\mathcal{W}'_C = \left[R'(v'_{\bar{2}})c' - \frac{1}{n-2}R'(v'_{\bar{2}}\cdot c')\right]\wedge v'_{\bar{2}} \qquad \text{c)}$$

$$= f\left\{\left[R(v_{\bar{2}})\cdot c - \frac{1}{n-2}R(v_{\bar{2}}\cdot c)\right]\wedge v_{\bar{2}}\right\} = f\left(\mathcal{W}_C\right)$$

which expresses the conformal Weyl relationship between conformally related structures. The quantity $\mathcal{W}_C$ is classically known as the conformal Weyl tensor.

A more recognizable, but equivalent form of the conformal Weyl tensor can be obtained by taking the simplicial divergence of (10.13b) with $\nabla_{v_{\bar{2}}}$. Doing this we find that

$$\frac{2}{n(n-1)}\nabla_{v_{\bar{2}}}\cdot\mathcal{W}_C = R(v_{\bar{2}})\cdot c - \frac{1}{n-2}\left[v_{\bar{2}}\cdot R(c) + R(v_{\bar{2}}\cdot c)\right] + \frac{1}{(n-1)(n-2)}R\; v_{\bar{2}}\cdot c \quad . \qquad (10.14)$$

Although (10.14) and (10.13b) are fully equivalent, (10.13b) is more fundamental. As an indication of this, it is trivial to argue that (10.13b) is identically zero for $n=3$; this is not the case with (10.14),

APPENDICES

A. Differentiation Formulas

Let $\mathfrak{D} = \mathfrak{D}(I_n)$ be a finite subalgebra of $\mathcal{G}$. In this appendix we consider differentiation formulas involving the finite gradient operators of $\mathfrak{D}$.

a) Basic Identities

Let $A_s \in \mathfrak{D}^s$ where $o \le s \le n$, and consider the combinatorial identity

$$\binom{n}{r}A_s = \binom{s}{o}\binom{n-s}{r}A_s + \binom{s}{1}\binom{n-s}{r-1}A_s + \cdots + \binom{s}{r-1}\binom{n-s}{1}A_s + \binom{s}{r}\binom{n-s}{o}A_s \tag{A.1a}$$

where $\binom{l}{k}$ means the combination of l, k at a time for $k \le l$ and $\binom{l}{k} \equiv 0$ for $k > l$. Consider now the algebraic identity

$$\nabla_{V_r} V_r A_s = \nabla_{V_r} V_r \wedge A_s + \nabla_{V_r} \langle V_r A_s \rangle_{s+r-2} + \cdots + \nabla_{V_r} \langle V_r A_s \rangle_{|s-r|+2} + \nabla_{V_r} V_r : A_s \tag{b)}$$

which is an application of (1.21). The identities a) and b) above are <u>termwise</u> equivalent, as can be verified by an inductive argument on r and s. Proofs of special cases of this can be found in [25].

We now list several immediate consequences of the above equivalence for easy reference:

$$\nabla_{V_r} V_r = \binom{n}{r} \tag{A.2a}$$

$$\nabla_{V_r} V_r \cdot A_s = \left\{ \begin{array}{ll} \binom{s}{r} A_s & \text{for } r \leq s \\ \binom{n-s}{r-s} A_s & \text{for } r \geq s \end{array} \right\} = A_s \cdot \nabla_{V_r} V_r \tag{b}$$

$$\nabla_{V_r} V_r \wedge A_s = \left\{ \begin{array}{ll} \binom{n-s}{r} A_s & \text{for } r+s \leq n \\ 0 & \text{for } r+s > n \end{array} \right\} = A_s \wedge \nabla_{V_r} V_r \tag{c}$$

Using (A.2a,b) and the Leibnitz product rule (3.18b), it is easy to further calculate

$$\nabla_{V_r} \frac{V_r}{|V_r|^k} = \left[\binom{n}{r} - k\right] \frac{1}{|V_r|^k} \qquad \text{for } k \geq 1 \tag{A.3a}$$

$$\nabla_{V_r} |V_r|^{2-k} = (2-k) \frac{V_r^{\dagger}}{|V_r|^k} \tag{b}$$

$$\nabla_{V_r} \ln |V_r| = \frac{V_r^{\dagger}}{|V_r|^2} \quad . \tag{c}$$

b) Signatures of a Multivector

We now consider a special class of identities which can also be directly established from (A.1) with the help of the rules (1.20) for commuting multivectors. Define the k^{th}-signature of a multivector $A \in \mathfrak{D}$ to be

$$\text{sig}_k(A) = \nabla_{V_k} A \, V_k \quad . \tag{A.4}$$

Since $\text{sig}_k(A+B) = \text{sig}_k(A) + \text{sig}_k(B)$, it is sufficient to determine the signatures of homogeneous multivectors. These are given recursively by

$$\mathrm{sig}_1(A_r) = (-1)^r(n-2r)A_r \tag{A.5a}$$

$$\mathrm{sig}_2(A_r) = \frac{1}{2}\left[(n-2r)^2 - n\right] A_r \qquad \text{b)}$$

and

$$\mathrm{sig}_k(A_r) = \frac{1}{k}\left[(-1)^r(n-2r)\,\mathrm{sig}_{k-1}(A_r) - (n-k+2)\,\mathrm{sig}_{k-2}(A_r)\right] \qquad \text{c)}$$

for $k > 2$.

Tables of Signatures

Let $A = \alpha + a + A_2 + A_3 + A_4$.

n=2	α	a	A_2
sig_0	1	1	1
sig_1	2	0	-2
sig_2	1	-1	1

n=3	α	a	A_2	A_3
sig_0	1	1	1	1
sig_1	3	-1	-1	3
sig_2	3	-1	-1	3
sig_3	1	1	1	1

n=4	α	a	A_2	A_3	A_4
sig_0	1	1	1	1	1
sig_1	4	-2	0	2	-4
sig_2	6	0	-2	0	6
sig_3	4	2	0	-2	-4
sig_4	1	-1	1	-1	1

c) Completely Symmetric Bivector Maps

We now summarize the differential identities used in chapter 10. Let $R(v_2)$ be a linear, completely symmetric, bivector-valued map. Recall that this just means $R(v_2)$

satisfies the Bianchi condition (4.4). Just as in section 6d), let

$$R(v_2) = 2\ \nabla_{v_1} \cdot R(v_{\bar{2}}) = \nabla_{v_1} \cdot R(v_1 \wedge v_2)$$

and

$$R = \nabla_{v_2} \cdot R(v_2) = 2\ \nabla_{\bar{2}} \cdot R(v_{\bar{2}}) \quad .$$

Then $R(v_{\bar{2}})$ will satisfy the following differential identities:

$$\nabla_{\bar{2}} \cdot \left[\left[R(v_{\bar{2}}) \cdot c\right] \wedge v_{\bar{2}}\right] = \frac{n(n-1)}{2} R(v_{\bar{2}}) \cdot c - \frac{n}{2} v_{\bar{2}} \cdot R(c) \quad . \tag{A.7}$$

$$v_{\bar{2}} \cdot \nabla_v R(v) = \frac{1}{2}\left[v_1 \wedge R(v_2) + R(v_1) \wedge v_2\right] = \nabla_v R(v) \cdot v_{\bar{2}} \tag{A.8a}$$

$$\left[v_{\bar{2}} \cdot \nabla_v R(v)\right] \cdot c = v_{\bar{2}} \cdot R(c) + R(v_{\bar{2}} \cdot c) \quad . \tag{b}$$

$$\nabla_{\bar{2}} \cdot \left[R(v_{\bar{2}} \cdot c) \wedge v_{\bar{2}}\right] = \frac{n(n-1)}{2} R(v_{\bar{2}} \cdot c) + \frac{n}{2}\left[v_{\bar{2}} \cdot R(c) - v_{\bar{2}} \cdot c\ R\right] . \tag{A.9}$$

These identities are consequences of (A.2), (A.5), the Leibnitz product rule (3.18), and the Bianchi identities (6.26).

It is important to note that differentiation in the above identities is taken with respect to the simplicial variable $v_{\bar{2}}$. For comparison we give the corresponding formulas to (A.7) and (A.9) when differentiation is instead taken with respect to the bivector variable V_2. Thus

$$\nabla_{V_2} \cdot \left\{\left[R(V_2) \cdot c\right] \wedge V_2\right\} = \frac{n^2-3n+4}{2} R(V_2) \cdot c - V_2 \cdot R(c) \tag{A.10}$$

and

$$\nabla_{V_2} \cdot \left\{\left[R(V_2 \cdot c) \wedge V_2\right]\right\} = \frac{n^2-3n+4}{2} R(V_2 \cdot c) + V_2 \cdot R(c) - V_2 \cdot c\ R \tag{A.11}$$

More insight can be gained into these two different kinds of differentiation by considering the following two easier calculations:

$$\nabla_{V_2} R(V_2) \cdot V_2 = R^{\dagger}(V_2) + R(V_2) = 2\ R(V_2) \quad , \tag{A.12a}$$

whereas

$$\nabla_{v_{\bar{2}}} R(v_{\bar{2}})\cdot v_{\bar{2}} \equiv \frac{1}{4}\, \nabla_2 \wedge \nabla_1 \; R(v_1 \wedge v_2)\cdot(v_1 \wedge v_2)$$

$$= \frac{1}{2}\, \nabla_2 \wedge \dot{\nabla}_1 \; \dot{v}_1 \cdot [v_2 \cdot R(v_1 \wedge v_2)] = \frac{1}{2}\, \nabla_2 \wedge [v_2 \cdot R(v_1 \wedge v_2)]$$

$$= R(v_1 \wedge v_2) + \frac{1}{2}\, \dot{\nabla}_2 \wedge [v_2 \cdot R(v_1 \wedge \dot{v}_2)] \qquad \text{b)}$$

$$= \frac{3}{2}\, R(v_1 \wedge v_2) - \frac{1}{2} v_2 \cdot [\dot{\nabla}_2 \wedge R(v_1 \wedge \dot{v}_2)] = 3\, R(v_{\bar{2}}) \quad .$$

The calculation in a) only makes use of the fact that $R(V_2)$ is symmetric (ie., $R(V_2) = R^{\dagger}(V_{\bar{2}})$), whereas the calculation in b) makes full use of the Bianchi condition in the last step.

A simple but important consequence of (A.12b) is that

$$R(v_{\bar{2}})\cdot v_{\bar{2}} \equiv 0 \quad \text{for all simple} \quad v_{\bar{2}} \in \mathfrak{D}$$

is equivalent to (A.13)

$$R(v_{\bar{2}}) \equiv 0 \quad \text{for all simple} \quad v_{\bar{2}} \in \mathfrak{D} \; .$$

This result is equivalent to proposition 8 of [26; II 4D-17], and a closer examination further shows that our Bianchi condition (4.4) for $R(v_{\bar{2}})$ is equivalent to the hypothesis of proposition 7 of the same reference.

Finally we note that all the identities of this section can be easily generalized to completely symmetric linear maps of an r-vector, or an r-simplicial variable.

B. Manifolds

In this appendix we give definitions which suggest how the usual notion of a differentiable manifold is related to the notion of a geometric structure. We conclude with a brief discussion of integration on a manifold.

a) Definition of a Manifold

Let M be a connected subset of $\mathcal{G}^1$. Vectors $x \in M$ will be called points. Let x_o be a point of M. By a curve on M at x_o we shall mean a continuous mapping $r(t)$ with the properties

i) $r : -\varepsilon < t < \varepsilon \longrightarrow M$, and $r(o) = x_o$.

ii) $\dot{r}(o) \equiv \frac{dr}{dt}\Big|_{t=o}$ exists. (B.1)

The vector $\dot{r}_o \equiv \dot{r}(o)$ will be said to be the velocity of the curve r at x_o.

Let $\mathfrak{D} = \mathfrak{D}(I)$ be a finite subalgebra of $\mathcal{G}$. We will say that $\mathfrak{D}$ is tangent to M at x_o provided

i) For each $a \in \mathfrak{D}^1$ there is a curve r on M at x_o with velocity $\dot{r}_o = a$.

ii) Conversely, if r is a curve on M at x_o, then $\dot{r}_o \in \mathfrak{D}^1$. (B.2)

Now let χ_P be a regular integrable P-structure of a finite subalgebra $\mathfrak{D}$ which is tangent to M at x_o. The structure χ_P will be said to <u>fit M at x_o</u> if there is a neighborhood N_o (with respect to the relative topology from $\mathcal{G}$) of x_o in M, and a transformation $\mathcal{J}_o$ with the following properties:

$$\text{i)}\quad \mathcal{J}_o : \mathcal{L} \longrightarrow \mathcal{L}_{N_o} \tag{B.3}$$

where $\mathcal{L}_{N_o}$ is the set of all $\mathcal{L}$-valued continuous mappings with domain N_o. For $F \in \mathcal{L}$, we will write ${}_oF_x \equiv \mathcal{J}_o[F](x) \in \mathcal{L}_{N_o}$ for the value of the transformation $\mathcal{J}_o$ at F. Note that for $x \in N_o$, ${}_oF_x \in \mathcal{L}$.

$$\text{ii)}\quad {}_oF_{x_o} = \mathcal{J}_o[F](x_o) = F\quad , \tag{B.3}$$

ie., the value of ${}_oF_x$ at $x=x_o$ is F.

$$\text{iii)}\quad \frac{d}{dt}\left[{}_oF_r\right]_{t=o} = \chi_P\left[\dot{r}_o;\ F\right] = \frac{d}{dt}\mathcal{J}_o[F](r(t))\Big|_{t=o} \tag{B.3}$$

for each curve $r(t)$ at x_o. The transformation $\mathcal{J}_o$ will be called an <u>integrating transformation</u> of the structure χ_P.

Suppose now that we are given a second transformation $\mathcal{J}'_{o'} : \mathcal{L} \longrightarrow \mathcal{L}_{N'_o}$, where N'_o is the neighborhood of a second a second point $x'_o \in M$. We shall say that the transformations $\mathcal{J}_o$ and $\mathcal{J}'_{o'}$ are <u>consistent</u> if whenever

$$ {}_oF_x = {}_{o'}F'_x \quad \text{for some} \quad x \in N_o \cap N'_o\ ,$$

then (B.4)

$$ {}_oF_x = {}_{o'}F'_x \quad \text{for all} \quad x \in N_o \cap N'_o\ ,$$

where of course ${}_oF_x \equiv \mathcal{J}_o[F](x)$, and ${}_{o'}F'_{x'} = \mathcal{J}'_{o'}[F'](x')$.

Finally by a manifold $\mathcal{M}$ with structure χ we shall mean a triple $\mathcal{M} = (M, \chi, \mathcal{J})$, where

i) M is a connected point set in $\mathcal{G}^1$.

ii) For each $x_o \in M$, $\chi(x_o) = \chi_P$ where χ_P is a regular integrable P-structure which fits M at the point x_o with the integrating transformation $\mathcal{J}_o = \mathcal{J}(x_o)$. (B.5)

iii) The integrating transformations $\mathcal{J}_o = \mathcal{J}(x_o)$ are consistent in overlapping neighborhoods for all $x_o \in M$.

b) Forms and Fields on a Manifold

Let $\mathcal{M} = (M, \chi, \mathcal{J})$ be a manifold. By a structural form on $\mathcal{M}$ we mean a function

$$F : M \longrightarrow \mathcal{L}^o \tag{B.6a}$$

which we will denote by F_x , with the property that for each $x_1 \in M$, there is an $x_o \in M$ and a neighborhood N_o containing both x_1 and x_o such that for some $F' \in \mathcal{L}$

$$F_x \equiv {}_oF'_x = \mathcal{J}_o[F'](x) \quad \text{for all} \quad x \in N_o \quad . \tag{b}$$

Of course $\mathcal{J}_o$ is the integrating transformation of the manifold at x_o . If F_x satisfies the additional property that

$$F_x \equiv F_x P_x \quad \text{for all} \quad x \in M \quad , \tag{c}$$

where P_x is the finite projection operator onto the tangent algebra $\mathcal{D}_x$ to the manifold at the point x, then F_x will be called a <u>structural P_x-form</u>.

As might be expected, by the <u>structural field</u> of a structural form F_x we mean the multivector-valued function defined on M by

$$A^F(x) \equiv \nabla_Z F_x(Z) \quad . \qquad \text{(B.7a)}$$

If F_x is a structural P_x-form, then $A^F(x)$ will satisfy the additional property that

$$A^F(x) = \nabla_Z F_x(Z) = \nabla_V F_x(V) = P_x\left[A^F(x)\right] \qquad \text{b)}$$

where $V \equiv P_x(Z)$, and will be called a <u>structural P_x-field</u>.

c) Integration on Manifolds

Let $\mathcal{M}$ be an oriented n-manifold in $\mathcal{G}^1$, and let $\mathcal{R}$ be a region of $\mathcal{M}$ with a well defined boundary $\partial\mathcal{R}$. Let $A(x)$ be a multivector field defined in $\mathcal{R}$. Following [13; 317], we define the <u>directed Riemann integral</u> of $A(x)$ over $\mathcal{R}$ by

$$\int_{\mathcal{R}} dV_x\, A(x) \equiv \lim_{\substack{k \to \infty \\ \Delta V_{x_i} \to 0}} \sum_{i=1}^{k} \Delta V_{x_i} A(x_i) \qquad \text{(B.8)}$$

where ΔV_{x_i} is the directed (n-vector) measure of volume of the region $\mathcal{R}$ at the point x_i.

One of the most important theorems about integrals of multivector fields is the so-called fundamental theorem of calculus. It is given by

$$\int_{\mathcal{R}} dV_x \, \nabla_x \, A(x) = \int_{\partial\mathcal{R}} ds_x \, A(x) \quad . \tag{B.9a}$$

This theorem relates the integral of the (structural) gradient of a field over an n-region to the integral of the field over the (n-1)-boundary. The special case of this theorem

$$\int_{\mathcal{R}} dV_x \cdot \left[\nabla_x \wedge A_{n-1}(x)\right] = \int_{\partial\mathcal{R}} dS_x \cdot A_{n-1}(x) \tag{b}$$

when $A_{n-1}(x)$ is an (n-1)-field, is equivalent to the Stokes theorem for differential forms. For a discussion of the fundamental theorem of calculus and other general integration formulas in the language of geometric algebra see [13]. For an account of the close parallel between operations on differential forms and the corresponding algebraic operations on fields, see [25; 93].

Suppose now that a second n-manifold $\mathcal{M}'$ in $\mathcal{G}^1$ is given which is related pointwise to $\mathcal{M}$ by the map $x' = f(x)$. Suppose further that the tangent map $\dot{f}_x$ of f at the point x, $\dot{f}_x$-relates the tangent algebras $\mathcal{G}_x$ and $\mathcal{G}'_{x'}$ of the manifolds $\mathcal{M}$ and $\mathcal{M}'$. Then we have the following basic formula for transforming integrals on $\mathcal{R}'$ to integrals on $\mathcal{R}$:

$$\int_{\mathcal{R}'} dV'_{x'} \cdot A'(x') = \int_{\mathcal{R}} dV_x \, J_{\dot{f}_x} \, A'\left[f(x)\right] \tag{B.10}$$

where $J_{\dot{f}}$ is the generalized Jacobian (4.15a) of the tangent map $\dot{f}$ defined in chapter 4.

We will say that the transformation $x' = f(x)$ represents a <u>change of coordinates</u> with positive orientation in $\mathcal{R}$ if

i) $f : \mathcal{R} \longrightarrow \mathcal{R}'$ is one-to-one, onto and regular.

(B.11)

ii) $J_{\dot{f}_x} = \langle J_{\dot{f}_x} \rangle_0 > 0$ for each $x \in \mathcal{R}$.

If f satisfies (B.11), and $A'(x')$ is a pseudoscalar-valued field, then (B.10) is equivalent to the usual transformation of coordinates formula for integrals.

If $A'(x')$ and $A(x)$ are pseudoscalar fields on $\mathcal{R}'$ and $\mathcal{R}$ respectively, and

$$A(x) = \dot{f}_x^{\dagger}\left[A'(x')\right] \qquad \text{(B.12a)}$$

then (B.10) reduces to

$$\int_{\mathcal{R}'} dV'_{x'} \cdot A'(x') = \int_{\mathcal{R}} dV_x \cdot A(x) \quad , \qquad \text{b)}$$

which is equivalent to the rule for "pulling back" differential forms.

Transformation formulas for integrals of the type (B.10-12) were first considered in [25].

LIST OF SYMBOLS

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