Geometric Matrices of Dual Null Vectors

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Abstract

A (Clifford) geometric algebra is usually defined in terms of a quadratic form. A null vector v is an algebraic quantity with the property that $v^2 = 0$. Null vectors over the real or complex numbers are taken as fundamental and are added and multiplied together according to the familiar rules of real or complex square matrices. In a series of ten definitions, the concepts of a Grassmann algebra, its dual Grassmann algebra, the associated real and complex geometric algebras, and their isomorphic real or complex coordinate matrix algebras are laid down.

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1 Introduction

The development of the concept of duality in mathematics has a robust history dating back more than 100 years, and involving 20th Century mathematicians of the first rank such as F. Reisz and S. Banach, but also encompassing first rank 19th Century mathematicians such as Gauss, Lobachevsky and Bolyai. A fascinating history of the seminal Hahn-Banach Theorem, and all its ramifications regarding the issue of duality in finite and infinite dimensional Hilbert spaces is given in [3].

Considering infinite dimensional vector spaces broadens and immensely deepens the mathematical issues involved in the concept of duality [1, 5]. In this paper we consider duality only in regard to a finite dimensional vector space, where it is well known that duality is equivalent to defining a Euclidean inner product. The purpose of this paper is to show how defining an *inner product* on a vector space of *null vectors*, over the real or complex numbers, not only captures the notion of duality but also nails down the corresponding isomorphic real or complex matrix algebra of a geometric algebra. In a series of 10 definitions, we define a Grassmann algebra, its dual Grassmann algebra and their associated Clifford geometric algebra. The isomorphic real or complex coordinate matrices of a geometric algebra, called geometric matrices, are directly constructed alongside and serve as a powerful computational and pedagogical tool.

2 Ten Definitions

1. Null vectors are algebraic quantities $x \neq 0$ with the property that $x^2 = 0$. They are added and multiplied together using the same rules as the addition and multiplication of real or complex square matrices. The trivial null vector is denoted by 0.

2. Two null vectors a_1 and a_2 are said to be *anticommutative* if
$$a_1 a_2 + a_2 a_1 = 0.$$

3. A set of mutually anticommuting null vectors $\{a_1, \ldots, a_n\}_{\mathcal{F}}$ is said to be *linearly independent* over $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, if $a_1 \cdots a_n \neq 0$.[1] In this case they generate the 2^n-dimensional Grassmann algebra
$$G_n(\mathcal{F}) := gen\{a_1, \ldots, a_n\}_{\mathcal{F}}.$$

4. Let $A_i := \begin{pmatrix} 1 \\ a_i \end{pmatrix}$ for $i = 1, \ldots, n$. By the *right directed Kronecker* product, $G_2(F) := A_1 \overrightarrow{\otimes} A_2 := \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_{12} \end{pmatrix}$. More generally, the right directed Kronecker product
$$G_n(\mathcal{F}) := A_1 \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} A_n$$
gives a 2^n-*column* matrix of the basis elements
$$\{1; a_1, \ldots, a_n; \ldots; a_{\lambda_1} \cdots a_{\lambda_k}; \ldots; a_1 \cdots a_n\}_{\mathcal{F}}$$
of the 2^n-dimensional Grassmann algebra $\mathbb{G}_n(\mathcal{F})$. The $\begin{pmatrix} n \\ k \end{pmatrix}$ elements
$$a_{\lambda_1 \cdots \lambda_k} := a_{\lambda_1} \cdots a_{\lambda_k}$$
for $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$ are called k-vectors. Similarly, the *left directed Kronecker* product
$$G_2(\mathcal{F}) := A_2^T \overleftarrow{\otimes} A_1^T := (A_1 \overrightarrow{\otimes} A_2)^T = \begin{pmatrix} 1 & a_1 & a_2 & a_{21} \end{pmatrix},$$

[1] More general fields $\mathcal{F}$ can be considered as long as *characteristic* $\mathcal{F} \neq 2$. For an interesting discussion of this issue see [2].

and more generally,

$$G_n(\mathcal{F}) := (A_n^T \overleftarrow{\otimes} \cdots \overleftarrow{\otimes} A_1^T) := (A_1 \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} A_n)^T,$$

gives a 2^n-*row* matrix of the 2^n basis of the Grassmann algebra $G_n(\mathbb{R})$, [7, p.82].

5. A pair of null vectors a and b are said to be *algebraically dual* if

$$ab + ba = 1.$$

In this case, we define $a^* := b$ and $b^* := a$, from which it follows that

$$(a^*)^* = b^* = a.$$

6. Let $G_n(\mathbb{R}) := A_1 \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} A_n$ and $G_n^{\#}(\mathbb{R}) = B_n^T \overleftarrow{\otimes} \cdots \overleftarrow{\otimes} B_1^T$ be two 2^n-dimensional real Grassmann algebras. The Grassmann algebras $G_n(\mathbb{R})$ and $G_n^{\#}(\mathbb{R})$ are said to be *dual Grassmann algebras* if there exists generating bases

$$G_n(\mathbb{R}) := gen\{a_1, \ldots, a_n\}_{\mathbb{R}} \quad \text{and} \quad G_n^{\#}(\mathbb{R}) := gen\{b_1, \ldots, b_n\}_{\mathbb{R}}$$

such that

$$2a_i \cdot b_j := a_i b_j + b_j a_i = \delta_{ij}.$$

In this case, $G_n^*(\mathbb{R}) := G_n^{\#}(\mathbb{R})$. Of course, nothing is surprising because it is well known that any standard vector space V, and it dual space V^* can be represented in terms of an equivalent *inner product*.

7. The *real geometric algebra*

$$\mathbb{G}_{n,n}(\mathbb{R}) := G_n(\mathbb{R}) \otimes G_n^*(\mathbb{R}) = gen\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$$

where for $i, j = 1, \ldots, n$,

$$2a_i \cdot b_j = a_i b_j + b_j a_i = \delta_{ij}.$$

8. Defining the idempotents $u_i := b_i a_i$, the quantity

$$u_{1\cdots n} := u_1 \cdots u_n = \prod_{i=1}^{n} b_i a_i = b_1 a_1 \cdots b_n a_n$$

is a *primitive idempotent* in the geometric algebra $\mathbb{G}_{n,n}$. The *nilpotent matrix basis* of the geometric algebra $\mathbb{G}_{n,n}$ is specified by

$$\mathbb{G}_{n,n} := A_1 \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} A_n u_{1\cdots n} B_n^T \overleftarrow{\otimes} \cdots \overleftarrow{\otimes} B_1^T = \left(\overrightarrow{\otimes}_{i=1}^n A_i\right) u_{1\cdots n} \left(\overrightarrow{\otimes}_{i=1}^n A_i\right)^*,$$

where $A_i^* := B_i^T$ for $i = 1, \ldots, n$. In the nilpotent matrix basis, any $g \in \mathbb{G}_{n,n}$ is explicitly expressed in terms of its *coordinate geometric matrix* $[g] := [g_{ij}]$ for $g_{ij} \in \mathbb{R}$, by

$$g = \left(\overrightarrow{\otimes}_{i=1}^n A_i^T\right) u_{1\cdots n} [g] \left(\overleftarrow{\otimes}_{i=n}^1 B_i\right),$$

[7, Chapter 5].

9 . The *standard basis* of $\mathbb{G}_n(\mathbb{R}) = \mathbb{G}_{n,n}(\mathbb{R})$, is specified by

$$\mathbb{G}_{n,n} := \mathbb{R}(e_1, \ldots, e_n, f_1, \ldots, f_n) = gen\{e_1, \ldots, e_n, f_1, \ldots, f_n\}_{\mathbb{R}}$$

where $e_i := a_i + b_i$ and $f_i := a_i - b_i$ for $i = 1, \ldots, n$. The basis vectors are mutually anticommutative and satisfy the basic property

$$e_i^2 = 1, \quad \text{and} \quad f_i^2 = -1$$

for $i = 1, \ldots, n$, as can be easily verified.

10. The real geometric algebra

$$\mathbb{G}_{n,n+1} := \mathbb{R}(e_1, \ldots, e_n, f_1, \ldots, f_{n+1}) = \mathbb{G}_n(\mathbb{C})$$

where the imaginary unit

$$i = \sqrt{-1} := (e_1 f_1) \cdots (e_n f_n) f_{n+1} \quad \Longleftrightarrow \quad f_{n+1} := i(e_1 f_1) \cdots (e_n f_n).$$

How geometric matrices arise as algebraically isomorphic coordinate matrices, for the simplest case of 2×2 real or complex matrices, and their practical application to the classical Plücker relations is given in [6]. A general introduction to geometric algebras and their coordinate geometric matrices is given in [7]. A periodic table of all of the classical Clifford geometric algebras is derived from three *Fundamental Structure Theorems* in [8]. Other links to relevant research can be found on my website $<$ https : //www.garretstar.com/ $>$.

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References

[1] P. De La Harpe, *The Clifford algebra and the Spinor group of a Hilbert space*, Compositio Mathematica, tome 25 No. 3, p. 245-261 (1972).

[2] P. Lounesto, *Crumeyrolle's Bivectors and Spinors*, in R. Ablamowicz, P. Lounesto (eds.), *Clifford Algebras and Spinor Structures*, pp. 137-166, Kluwer (1995).

[3] L. Narici, *On the Hahn-Banach Theorem*, p. 87 - 123, in Proceedings of the Second International Course of Mathematical Analysis in Andalucia, Sept. 20-24, Granada, Editors: M.V. Velasco & Al., World Scientific (2004).

[4] Z. Oziewicz, *From Grassmann to Clifford*, p.245-256 in *Clifford Algebras and Their Applications in Mathematical Physics*, eds. J.S.R. Chisholm, A.K. Common, NATO ASI Series C: Mathematical and Physical Sciences Vol. 183 (1986).

[5] D. Shale, W. Stinespring, *Spinor Representations of Infinite Orthogonal Groups*, Journal of Mathematics and Mechanics, Vol. 14, No. 2, pp. 315-322 (1965).

[6] G. Sobczyk, *Notes on Plückers relations in geometric algebra*, Advances in Mathematics, 363 (2020) 106959. An early version of this article is https://arxiv.org/pdf/1809.09706.pdf

[7] G. Sobczyk, *Matrix Gateway to Geometric Algebra, Spacetime and Spinors*, Independent Publisher, Nov. 7, 2019.

[8] G. Sobczyk, *Periodic Table of Geometric Numbers*, 12 March 2020. https://arxiv.org/pdf/2003.07159.pdf