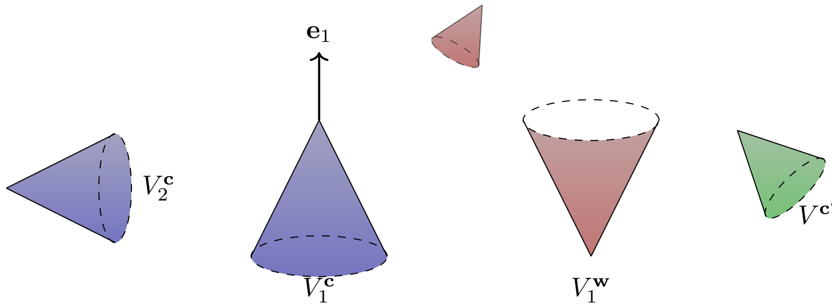
Grassmann-Clifford Geometric Algebras from a Universal Null Substrate

Garret Sobczyk[1], Jesus Cruz Guzman[2] Bill Page[3]



Depiction of the Universal Null Substrate (UNS)

"Da quel punto depende il cielo e tutta la natura."
"From that point depend the heavens and all of nature."
— **Dante Alighieri**, *Paradiso* XXVIII, 41–42

Abstract

Clifford Geometric Algebras $\mathbb{G}_{p,q}$ are traditionally studied in terms of their defining quadratic forms. Here, we introduce a complementary Quadratic Space of Null Vectors (QSNV), where geometric algebras emerge from a Universal Null Substrate (UNS). The resulting hybrid Grassmann-Clifford Geometric Algebra is defined by a recursive orthogonalization process, offering powerful new tools for studying classical Lie groups and their Lie algebras. At the core is a Zero Residue Factorization (ZRF), defining the geometric stability of the algebra. This QSNV classification offers great computational advantages by replacing matrix operations with an index arithmetic on null vectors. It is conjectured that the famous Valley of Stability in nuclear physics closely mirrors the algebraic structure of the QSNV. An Appendix further speculates that QSNV is a natural language to investigate that spacetime geometry is just a physical manifestation of triangular inequalities of quantum information on emergent null cones, and considers its philosophical precursors in Dante's Divine Comedy.

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[1]Corresponding Author: garretudla@gmail.com

Contents

1 Introduction

The classification of Clifford geometric algebras has historically relied upon the rigid definition of metric signatures (p, q), dictating the number of basis vectors squaring to $+1$ or -1 [11]. While this approach, solidified by the work of Cartan, Bott, and others, successfully categorizes the periodicity of these algebras, it obscures the computational elegance found in the underlying null geometry [15, 16].

This work proposes a shift in perspective. Rather than viewing the metric signature as a primary constraint, specific geometric algebras arise from a *Universal Null Substrate* (UNS) defining a *Quadratic Space of Null Vectors* (QSNV). In this framework, the fundamental objects are the null vectors themselves, arranged in specific configurations (simplexes). The standard orthonormal bases of $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ are derived via a recursive orthogonalization of these null simplexes.

By establishing the isomorphism between these null bases and the standard bases, we unlock a powerful computational tool. The breakdown of high-dimensional algebras into *Zero Residue* factors reduces geometric multiplications to simple arithmetic rules on a null substrate. This paper details the construction of $V_{r,s,k}$ spaces, provides an inductive proof of recursive orthogonalization, and catalogs the resulting geometric algebras for dimensions $p + q \leq 8$. The defining properties of the classical Lie groups and their Lie algebras are greatly simplified, taking advantage of the trivial multiplication rules of the emergent null vectors in $V_{r,s,k}$.

Particularly noteworthy is the striking similarity of the algebraic structure of QSNV to the structure of nuclear decay in the famous *Valley of Stability* [17]. The QSNV and UNS framework provides a natural algebraic language for the Holographic Principle and the Covariant Entropy Bound, and Discrete Max-Focusing [2, 3]. The Holographic Principle has many famous historic precursors, such as in the Greek's Famous Cave Allegory, and Dante Alighieri's *Divine Comedy*, *Paradiso* XXVIII.

2 Core Definitions

Definition 1. Standard Basis of a Geometric Algebra.

A geometric algebra over the real numbers $\mathbb{R}$ is generated by a *Standard Orthonormal Basis* of a quadratic form:

$$\mathbb{G}_{p,q,k} := \mathbb{R}(\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{f}_1, \dots, \mathbf{f}_q, \mathbf{n}_1, \dots, \mathbf{n}_k), \tag{1}$$

where all of the basis vectors $\mathbf{e}_i, \mathbf{f}_j, \mathbf{n}_h$ are *anti-commutative*, and

$$\mathbf{e}_i^2 = 1 = -\mathbf{f}_j^2, \text{ and } \mathbf{n}_h^2 = 0. \tag{2}$$

[13, 14]. When $k = 0$ in the standard basis, $\mathbb{G}_{p,q}$ is said to be *indefinite*, and when $q = k = 0$, the geometric algebra $\mathbb{G}_p$ is said to be *positive definite*, or *Euclidean*. In the case that $p = k = 0$ the geometric algebra $\mathbb{G}_{0,q}$ is *negative definite*, or *pseudo-Euclidean*.

The most famous identity for the geometric product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{G}^1_{p,q,k}$ is

$$\mathbf{ab} := \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \frac{1}{2}(\mathbf{ab} - \mathbf{ba}), \tag{3}$$

defining the *symmetric* inner product and the *anti-symmetric* wedge product of the vectors $\mathbf{a}$ and $\mathbf{b}$. For a comprehensive treatment of geometric algebra and calculus, see [9].

Definition 2. Null Vector: A null vector $\mathbf{n}$ is defined by two properties:

1. It possesses an *oriented direction*.
2. Since $\mathbf{n}^2 = 0$, it lacks a metric defined length.

Definition 3. Quadratic Null Vector Space $V_{r,s,k}$:

The $V_{r,s,k}$ is an $(r + s + k)$-dimensional vector space spanned by the union of three subsets of null vectors:

$$V_{r,s,k} := \{\mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{w}_1, \dots, \mathbf{w}_s, \mathbf{n}_1, \dots, \mathbf{n}_k\}_{\mathbb{R}} = V_r^{\mathbf{c}} \cup V_s^{\mathbf{w}} \cup V_k^{\mathbf{n}}.$$

Inner Product Rules

Using the Kronecker delta δ_{ij}, inner products of null vectors define a non-orthogonal basis where every distinct pair in the same subset share a constant inner product:

1. **Positive-like c-vectors:** $\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(1 - \delta_{ij})$

2. **Negative-like w-vectors:** $\mathbf{w}_i \cdot \mathbf{w}_j := -\frac{1}{2}(1-\delta_{ij})$

3. **Zero-like n-vectors**: $\mathbf{n}_i \cdot \mathbf{n}_j := 0$.

Definition 4. Quadratic Grassmann-Clifford Algebra $\mathcal{G}^{\perp}_{r,s,k}$

The 2^{r+s+k}-dimensional Grassmann algebra $\mathcal{G}^{\perp}_{r,s,k}$, under the anti-commutative wedge product, is generated by the QSVN basis null vectors $V_{r,s,k}$,

$$\mathcal{G}^{\perp}_{r,s,k} := \mathcal{G}(V_{r,s,k}). \tag{4}$$

We impose the additional rule that basis vectors from different signature sets anti-commute under the *geometric product*. Because they are mutually orthogonal,

$$\mathbf{c}_i \cdot \mathbf{w}_j = \mathbf{c}_i \cdot \mathbf{n}_j = \mathbf{w}_i \cdot \mathbf{n}_j = 0,$$

so that

$$\mathbf{c}_i \wedge \mathbf{w}_j = \mathbf{c}_i \mathbf{w}_j = -\mathbf{w}_j \mathbf{c}_i = -\mathbf{w}_j \wedge \mathbf{c}_i, \quad \mathbf{c}_i \wedge \mathbf{n}_j = \mathbf{c}_i \mathbf{n}_j = -\mathbf{n}_j \mathbf{c}_i = -\mathbf{n}_j \wedge \mathbf{c}_i,$$
$$\text{and} \quad \mathbf{n}_i \wedge \mathbf{w}_j = \mathbf{n}_i \mathbf{w}_j = -\mathbf{w}_j \mathbf{n}_i = -\mathbf{w}_j \wedge \mathbf{n}_i. \tag{5}$$

The quadratic Grassmann-Clifford algebra $\mathcal{G}^{\perp}_{r,s,k}$ is a hybrid structure combining the formal Grassmann antisymmetric wedge product with the independent symmetric structure that defines the quadratic form of a Clifford geometric algebra.

Multiplication Tables

The following Tables identify the geometric algebras defined by each of the subspaces $V_r^{\mathbf{c}}, V_s^{\mathbf{w}}, V_k^{\mathbf{n}}$ of null vectors in the Quadratic Grassmann-Clifford Algebra (QGA) $\mathcal{G}^{\perp}_{r,s,k}$. This is proved in Theorem 1 of the next Section.

Table 1: Multiplication Table for $V_r^{\mathbf{c}}$ ($\mathbf{c}$-basis)

	$\mathbf{c}_i$	$\mathbf{c}_j$	$\mathbf{c}_i\mathbf{c}_j$	$\mathbf{c}_j\mathbf{c}_i$
$\mathbf{c}_i$	0	$\mathbf{c}_i\mathbf{c}_j$	0	$\mathbf{c}_i$
$\mathbf{c}_j$	$\mathbf{c}_j\mathbf{c}_i$	0	$\mathbf{c}_j$	0
$\mathbf{c}_i\mathbf{c}_j$	$\mathbf{c}_i$	0	$\mathbf{c}_i\mathbf{c}_j$	0
$\mathbf{c}_j\mathbf{c}_i$	0	$\mathbf{c}_j$	0	$\mathbf{c}_j\mathbf{c}_i$

Table 2: Multiplication Table for $V_s^{\mathbf{w}}$ ($\mathbf{w}$-basis)

	$\mathbf{w}_i$	$\mathbf{w}_j$	$\mathbf{w}_i\mathbf{w}_j$	$\mathbf{w}_j\mathbf{w}_i$
$\mathbf{w}_i$	0	$\mathbf{w}_i\mathbf{w}_j$	0	$-\mathbf{w}_i$
$\mathbf{w}_j$	$\mathbf{w}_j\mathbf{w}_i$	0	$-\mathbf{w}_j$	0
$\mathbf{w}_i\mathbf{w}_j$	$-\mathbf{w}_i$	0	$-\mathbf{w}_i\mathbf{w}_j$	0
$\mathbf{w}_j\mathbf{w}_i$	0	$-\mathbf{w}_j$	0	$-\mathbf{w}_j\mathbf{w}_i$

Tables 1 and 2 demonstrate how multiplication of all multivectors in the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$, is reduced to two simple rules of multiplication for distinct pairs of null vectors in the Quadratic Grassmann-Clifford Algebras $\mathcal{G}(V_r^{\mathbf{c}}), \mathcal{G}(V_s^{\mathbf{w}})$, and in $\mathcal{G}(V_k^{\mathbf{n}})$ for $n \geq 1$.

$$\mathbb{G}_{1,n} := \mathcal{G}^{\mathbf{c}}_{n+1} = \mathcal{G}(V^{\mathbf{c}}_{n+1}) \quad \text{and} \quad \mathbb{G}_{n,1} := \mathcal{G}^{\mathbf{w}}_{n+1} = \mathcal{G}(V^{\mathbf{w}}_{n+1}). \tag{6}$$

Table 3 is the standard multiplication table for the multiplication for any two null vectors $\mathbf{n}_i, \mathbf{n}_j$ in the 2^k-dimensional graded Grassmann algebra $\mathcal{G}^{\mathbf{n}}_k$. This Grassmann algebra also defines the *degenerate geometric algebra*

$$\mathbb{G}_{0,0,k} := \mathcal{G}^{\mathbf{n}}_k = \mathcal{G}(V^{\mathbf{n}}_k). \tag{7}$$

Examples are given in the next section.

Table 3: Multiplication Table for $V_k^{\mathbf{n}}$ ($\mathbf{n}$-basis)

	$\mathbf{n}_i$	$\mathbf{n}_j$	$\mathbf{n}_i\mathbf{n}_j$	$\mathbf{n}_j\mathbf{n}_i$
$\mathbf{n}_i$	0	$\mathbf{n}_i\mathbf{n}_j$	0	0
$\mathbf{n}_j$	$\mathbf{n}_j\mathbf{n}_i$	0	0	0
$\mathbf{n}_i\mathbf{n}_j$	0	0	0	0
$\mathbf{n}_j\mathbf{n}_i$	0	0	0	0

2.1 Recursive Orthogonalization Theorem

We introduce the partial sums (centroids) $C_k := \sum_{i=1}^k \mathbf{c}_i$ and $W_k := \sum_{i=1}^k \mathbf{w}_i$.

Theorem 1. The Quadratic Grassmann Algebras $\mathcal{G}(V_{n+1}^{\mathbf{c}})$, and $\mathcal{G}(V_{n+1}^{\mathbf{w}})$ define the Standard Basis of the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ as follows:

Case 1) Construction of $\mathbb{G}_{1,n}$: The algebra $\mathcal{G}(V_r^{\mathbf{c}})$ generates the geometric algebra

$$\mathbb{G}_{1,n} := \mathcal{G}(V_{n+1}^{\mathbf{c}}) = \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n),$$

where $\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2$, $\mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2$, and for $2 \le k \le n$:

$$\mathbf{f}_k := \alpha_k \Big(C_k - (k-1)\mathbf{c}_{k+1} \Big), \quad \text{with} \quad \alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}. \tag{8}$$

Case 2) Construction of $\mathbb{G}_{n,1}$: The algebra $\mathcal{G}(V_{n+1}^{\mathbf{w}})$ generates the geometric algebra

$$\mathbb{G}_{n,1} := \mathcal{G}(V_{n+1}^{\mathbf{w}}) = \mathbb{R}(\mathbf{f}_1, \mathbf{e}_1, \dots, \mathbf{e}_n),$$

where $\mathbf{f}_1 := \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{e}_1 := \mathbf{w}_1 - \mathbf{w}_2$, and for $2 \le k \le n$:

$$\mathbf{e}_k := \alpha_k \Big(W_k - (k-1)\mathbf{w}_{k+1} \Big), \quad \text{with } \alpha_k \text{ defined as above.} \tag{9}$$

Proof. We prove Case 1 by induction on n. The proof for Case 2 follows identically by replacing $\mathbf{c}$ with $\mathbf{w}$, but redefining $\mathbf{f}_1$ and $\mathbf{e}_1$ appropriately in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$.
Base Case ($n = 1$): Consider $\mathbb{G}_{1,1} = \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1)$ derived from $\{\mathbf{c}_1, \mathbf{c}_2\}$.

$$\begin{aligned} \mathbf{e}_1^2 &= (\mathbf{c}_1 + \mathbf{c}_2)^2 = \mathbf{c}_1^2 + \mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2\mathbf{c}_1 + \mathbf{c}_2^2 = 0 + 2(\mathbf{c}_1 \cdot \mathbf{c}_2) + 0 = 2(1/2) = 1. \\ \mathbf{f}_1^2 &= (\mathbf{c}_1 - \mathbf{c}_2)^2 = -(\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2\mathbf{c}_1) = -2(1/2) = -1. \\ \mathbf{e}_1 \cdot \mathbf{f}_1 &= (\mathbf{c}_1 + \mathbf{c}_2) \cdot (\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_1^2 - \mathbf{c}_2^2 = 0. \end{aligned}$$

Thus, $\{\mathbf{e}_1, \mathbf{f}_1\}$ forms a standard orthonormal basis for $\mathbb{G}_{1,1}$.
Inductive Step: Assume that for $k < n$, the set $\{\mathbf{f}_2, \dots, \mathbf{f}_k\}$ consists of mutually orthogonal vectors squaring to -1, and that all are orthogonal to $\mathbf{e}_1$. We must show that $\mathbf{f}_{k+1}$ defined by (8) satisfies $\mathbf{f}_{k+1}^2 = -1$ and is orthogonal to the previous basis vectors.

First, we calculate the square of the unnormalized vector $\mathbf{v}_{k+1} = C_{k+1} - k\mathbf{c}_{k+2}$. Note that $C_m \cdot C_m = (\sum \mathbf{c}_i)^2 = \sum_{i \ne j} \mathbf{c}_i \cdot \mathbf{c}_j = m(m-1)(1/2)$. Also, $C_m \cdot \mathbf{c}_{ext} = m(1/2)$.

Let $m = k + 1$. Then $\mathbf{f}_m = \alpha_m(C_m - (m-1)\mathbf{c}_{m+1})$.

$$\begin{aligned} (C_m - (m-1)\mathbf{c}_{m+1})^2 &= C_m^2 - 2(m-1)(C_m \cdot \mathbf{c}_{m+1}) + (m-1)^2\mathbf{c}_{m+1}^2 \\ &= \frac{m(m-1)}{2} - 2(m-1)\frac{m}{2} + 0 \\ &= \frac{m(m-1)}{2} - m(m-1) = -\frac{m(m-1)}{2}. \end{aligned}$$

To normalize this to -1, we require a scalar α_m such that $\alpha_m^2[-\frac{m(m-1)}{2}] = -1$.

$$\alpha_m^2 = \frac{2}{m(m-1)} \implies \alpha_m = \frac{-\sqrt{2}}{\sqrt{m(m-1)}}.$$

This matches the definition in the Theorem.

Finally, observe that by construction, $\mathbf{f}_m$ lies in the subspace $V^{\mathbf{c}}_{m+1}$. Due to the symmetry of the simplex basis where $\mathbf{c}_i \cdot \mathbf{c}_j = 1/2$, the vector $C_m - (m-1)\mathbf{c}_{m+1}$ represents the direction from the vertex $\mathbf{c}_{m+1}$ to the centroid of the face defined by C_m. This direction is orthogonal to the subspace spanned by the differences of the components of C_m, which contains all previous $\mathbf{f}_j$ $(j < m)$.

Thus, $\mathbf{f}_n$ is orthogonal to all prior basis vectors and squares to -1, completing the induction. □

Remark on Geometry: The construction in Case 1) corresponds to the standard *Hyperbolic Plane* definition found in Artin [1] for $k = 1$. For $k \geq 2$, the construction differs from standard Witt decompositions. Since $\mathbf{c}_i \cdot \mathbf{c}_j = 1/2$ for all $i \neq j$, the basis vectors $\{\mathbf{c}_1, \dots, \mathbf{c}_{n+1}\}$ form the vertices of a regular simplex embedded in the null cone (or light cone) [12]. The recursive formula (8) represents a specialized Gram-Schmidt orthogonalization exploiting this simplex symmetry.

Examples

The following Examples should help familiarize readers with notation being used, and newly introduced.

1. $\mathbb{G}_{1,1} = \mathcal{G}(V_2^{\mathbf{c}}) = \mathcal{G}(V_2^{\mathbf{w}}) = \mathbb{R}(\mathbf{c}_1, \mathbf{c}_2) = \mathbb{R}(\mathbf{w}_1, \mathbf{w}_2)$

2. $\mathbb{G}_{2,2} = \mathbb{G}^{\mathbf{c}}_{1,1} \oplus^{\perp} \mathbb{G}^{\mathbf{w}}_{1,1} = \mathbb{R}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{w}_1, \mathbf{w}_2) \equiv \mathcal{G}(V_2^{\mathbf{c}}) \oplus^{\perp} \mathcal{G}(V_2^{\mathbf{c}'}) = \mathbb{R}(\mathbf{c}_1, \mathbf{c}_2; \mathbf{c}'_1, \mathbf{c}'_2)$,
   where
   $$\mathbf{e}_1 = \mathbf{c}_1 + \mathbf{c}_2,\ \mathbf{f}_1 = \mathbf{c}_1 - \mathbf{c}_2,\ \mathbf{e}_2 = \mathbf{w}_1 - \mathbf{w}_2,\ \mathbf{f}_2 = \mathbf{w}_1 + \mathbf{w}_2. \tag{10}$$
   Alternatively, $\mathbf{e}_2$ and $\mathbf{f}_2$ can be defined by an orthogonal $\mathbf{c}'$ basis $\{\mathbf{c}'_1, \mathbf{c}'_2\}$,
   $$\mathbf{e}_2 := \mathbf{c}'_1 + \mathbf{c}'_2,\ \mathbf{f}_2 := \mathbf{c}'_1 - \mathbf{c}'_2, \tag{11}$$
   where $\mathbf{c}_i \cdot \mathbf{c}'_j = 0$ for $i, j \in \{1, 2\}$.

3. $\mathbb{G}_{r,r} = \oplus^{\perp^r} \mathcal{G}^{\mathbf{c}}_2 := \mathcal{G}(V_2^{\mathbf{c}}) \oplus^{\perp_2} \cdots \oplus^{\perp_r} \mathcal{G}(V_2^{\mathbf{c}''\cdots'})$, for r orthogonal copies of $\mathcal{G}(V_2^{\mathbf{c}})$ as defined in Example 2.

3 Universal Null Substrate

In standard Clifford algebra classifications, the metric signature (p, q) is often viewed as a fundamental rigidity of the space. In the QSNV framework, we invert this view. The fundamental objects are the **null vectors**, themselves existing in an infinite "reservoir" of orthogonal potentialities:

$$\mathcal{U} = \{\mathbf{c}_1, \mathbf{c}_2, \dots\} \cup \{\mathbf{c}'_1, \mathbf{c}'_2, \dots\} \cup \{\mathbf{w}_1, \mathbf{w}_2, \dots\} \cup \{\mathbf{n}_1, \mathbf{n}_2, \dots\} \cup \dots \tag{12}$$

Any specific geometric algebra $\mathbb{G}_{p,q}$ is a finite selection of null vectors from the various positive-like, negative-like and other orthogonal sets. The rules of interaction (the inner products) are unchanging constants. This unchanging foundation emphasizes that the geometric properties are emergent from the combinatorial selection of null vectors. In Section 7, it is shown that crucial Lie Algebra and Lie Group structures of all the Classical Groups directly emerge from the universal substrate. Because of the complementary interplay between symmetric and antisymmetric structures in $\mathcal{G}^{\perp}(V_{r,s,k})$, many exotic algebraic structures can emerge from the universal null substrate.

As depicted in Figure 1, the substrate is not a rigid lattice but a reservoir of potentiality. The random orientations of the null cones signify that before a metric signature is imposed, the null vectors exist in mutually orthogonal subspaces with no preferred *time* or *space* direction until selected.

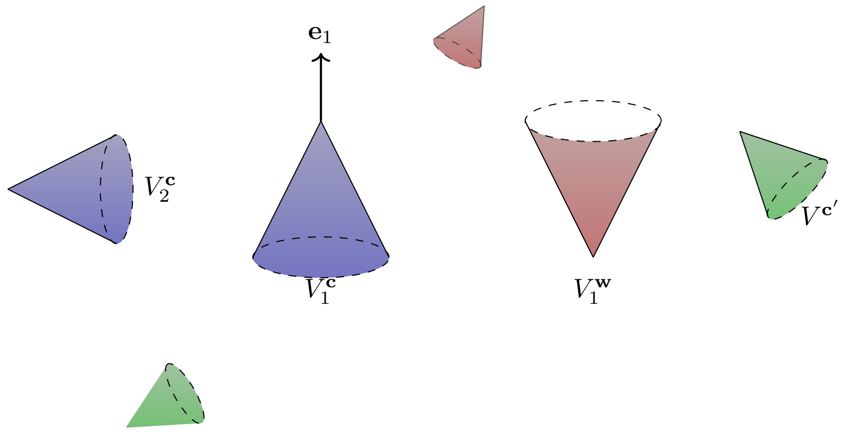

Figure 1: Depiction of the Universal Null Substrate. Geometric algebras are constructed by selecting null vectors from these locally independent null cones. The variety of sizes and orientations recognizes the independent orthogonal subspaces available, existing in potentiality before a basis is chosen. The Substrate is not a rigid lattice, but a reservoir of independent null forms.

We review the matrix representation of geometric algebras in the next section, emphasizing the relationship between the new approach expounded here and the historic approach used over the last 150 years.

4 Coordinate Matrices of Geometric Numbers

All geometric algebras $\mathbb{G}_{p,q}$ are known to be Bott 8-periodic. Let $[g_{ij}]_{m,n}$ be a real $(m \times n)$-matrix. The *basis elements* $M_{mn}(r,s)$ of an $(m \times n)$-matrix M_{mn} are specified by its mn basis elements:

$$M_{mn}(r,s) := [1_{rs}]_{m,n} = [\delta_{ir}\delta_{js}], \text{ where } 1 \le r \le m, \text{ and } 1 \le s \le n. \tag{13}$$

All of the $r(r-1)$ basis elements $M_{rr}(i,j)$ off of the main diagonal of a square $r \times r$ matrix have square 0. However, the set of $k = \frac{r(r-1)}{2}$ upper triangular basis matrices $M_{rr}(i,j)$ is **not** isomorphic to the Grassmann algebra $\mathcal{G}_r^{\mathbf{n}} = gen\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$.

4.1 Coordinate matrices of Grassmann algebras

Grassmann algebras are isomorphic to subalgebras of matrix algebras of the appropriate rank and order. The (2×2)-matrices

$$[\mathbf{a}] := [1_{21}]_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad [\mathbf{b}] := [1_{12}]_2 = [\mathbf{a}]^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{14}$$

are a *double matrix covering* of the isomorphic 1-dimensional Grassmann algebra,

$$\mathbb{G}_{0,0,1} = \mathcal{G}_1 := gen\{\mathbf{a}\} = \overline{\mathcal{G}}_1 := gen\{\mathbf{b}\} \cong gen\{[\mathbf{a}]\} \cong gen\{[\mathbf{b}]\}. \tag{15}$$

For the Grassmann algebra $\mathcal{G}_2$,

$$\mathbb{G}_{0,0,2} = \mathcal{G}_2 := gen\{\mathbf{a}_1, \mathbf{a}_2\} = \overline{\mathcal{G}_2} := gen\{\mathbf{b}_1, \mathbf{b}_2\} \cong gen\{[\mathbf{a}_1], [\mathbf{a}_2]\} \cong gen\{[\mathbf{b}_1], [\mathbf{b}_2]\}. \tag{16}$$

where

$$[\mathbf{a}_1] = \begin{pmatrix} 1_{21} \\ 1_{43} \end{pmatrix} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ [\mathbf{a}_2] = \begin{pmatrix} 1_{31} \\ -1_{42} \end{pmatrix} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{b}_1] = [\mathbf{a}_1]^T = \begin{pmatrix} 1_{12} \\ 1_{34} \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ [\mathbf{b}_2] = [\mathbf{a}_2]^T = \begin{pmatrix} 1_{13} \\ -1_{24} \end{pmatrix} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

For $\mathcal{G}_3$,

$$\mathbb{G}_{0,0,3} = \mathcal{G}_3^{\mathbf{a}} = \overline{\mathcal{G}}_3^{\mathbf{b}} \cong gen\{[\mathbf{a}_1], [\mathbf{a}_2], [\mathbf{a}_3]\} \cong gen\{[\mathbf{b}_1], [\mathbf{b}_2], [\mathbf{b}_3]\}, \tag{18}$$

and $[\mathbf{b}_i] = [\mathbf{a}_i]^T$, where the $(2^3 \times 2^3)$-coordinate matrices are given by

$$[\mathbf{a}_1] = \begin{pmatrix} 1_{21} & 1_{43} & 1_{65} & 1_{87} \end{pmatrix}$$

$$[\mathbf{a}_2] = \begin{pmatrix} 1_{31} & -1_{42} & 1_{75} & -1_{86} \end{pmatrix} \quad \text{and} \quad [\mathbf{a}_3] = \begin{pmatrix} 1_{51} & -1_{62} & -1_{73} & 1_{84} \end{pmatrix}. \tag{19}$$

4.2 Coordinate matrices of geometric algebras $\mathbb{G}_{r,r}$

A geometric algebra $\mathbb{G}_{p,q}$ with $p+q>0$ is said to be *positive dominant* or *negative dominant* if $p>q$, or $p<q$, respectively. In the case that $p=q$, the geometric algebra $\mathbb{G}_{r,r}$ is said to have *neutral signature*. The double covering of the Grassmann algebras $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3$ by the matrix algebras $\{gen\{[\mathbf{a}_k]\},gen\{[\mathbf{b}_k]\}\}$ for $k\in\{1,2,3\}$, are used to find sets of isomorphic coordinate matrix algebras of the geometric algebras $\mathbb{G}_{1,1},\mathbb{G}_{2,2},\mathbb{G}_{3,3}$. Letting $\mathbf{e}:=\mathbf{a}+\mathbf{b}$ and $\mathbf{f}:=\mathbf{a}-\mathbf{b}$, it is easy to show that $\mathbf{e}^2=1=-\mathbf{f}^2$ and $\mathbf{ef}=-\mathbf{fe}$. Taken with the coordinate matrices $[\mathbf{a}]$ and $[\mathbf{b}]$ given in (14),

$$[\mathbb{G}_{1,1}]\cong\mathbb{G}_{1,1}=gen\{\mathbf{e},\mathbf{f}\}=gen\{\mathbf{a},\mathbf{b}\}. \tag{20}$$

Similarly, letting $\mathbf{e}_i:=\mathbf{a}_i+\mathbf{b}_i$ and $\mathbf{f}_i:=\mathbf{a}_i-\mathbf{b}_i$, then $\mathbf{e}_i^2=1=-\mathbf{f}_i^2$ and $\mathbf{e}_i\mathbf{f}_i=-\mathbf{f}_i\mathbf{e}_i$, and

$$\mathbb{G}_{r,r}:=\mathbb{R}(\mathbf{e}_1,\ldots,\mathbf{e}_r,\mathbf{f}_1,\ldots\mathbf{f}_r)=\mathcal{G}_r\oplus^{\perp}\overline{\mathcal{G}_r}=\mathbb{G}_{1,1}^{\perp^r} \tag{21}$$

where $\mathcal{G}_r:=\mathbb{R}(\mathbf{a}_1,\ldots,\mathbf{a}_r)$ and $\overline{\mathcal{G}_r}:=\mathbb{R}(\mathbf{b}_1,\ldots,\mathbf{b}_r)$. The Grassmann algebras $\mathcal{G}_r$ and $\overline{\mathcal{G}}_r$, defined by the null vector bases $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_j\}$, are said to be *globally dual* because

$$\mathbf{a}_i\cdot\mathbf{b}_j=\frac{1}{2}(\mathbf{a}_i\mathbf{b}_j+\mathbf{b}_j\mathbf{a}_i)=\frac{1}{2}\delta_{ij}, \tag{22}$$

[16]. Taken together (14), (17), and (19), define the geometric algebras (21) and the isomorphic coordinate matrix algebras $[\mathbb{G}_{r,r}]\cong\mathbb{G}_{r,r}$ for $r=1,2,3$. Compare this with the construction given in (6), (10) and (11).

4.3 Building blocks of geometric algebras

In contrast to definition (22), the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ defined in (15), are said to be *locally dual* because their null vector bases satisfy

$$\mathbf{c}_i\cdot\mathbf{c}_j=\frac{1}{2}(1-\delta_{ij})\quad\text{or}\quad\mathbf{w}_i\cdot\mathbf{w}_j=-\frac{1}{2}(1-\delta_{ij}), \tag{23}$$

respectively [16]. The basic building blocks of the geometric algebras are $\mathbb{G}_{1,1}$, $\mathbb{G}_{1,2}$ and $\mathbb{G}_{2,1}$, generated by the locally dual null vectors $\mathbf{a}\equiv\mathbf{a}_1=\mathbf{c}_1$, $\mathbf{b}\equiv\mathbf{b}_1=\mathbf{c}_2$, and $\mathbf{c}_3$, and $\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3$ in the case of $\mathbb{G}_{2,1}$. The geometric algebra $\mathbb{G}_{1,2}$ and its coordinate matrix $[\mathbb{G}_{1,2}]$ is given below.

$$\mathbb{G}_{1,2}:=\mathcal{G}(V_3^{\mathbf{c}})=\mathbb{R}(\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3)\cong[\mathbb{G}_{3,0}].$$

where

$$[\mathbb{G}_{1,2}]=gen\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}j&1\\1&-j\end{pmatrix}\} \tag{24}$$

for $j:=\sqrt{-1}$.

For $\mathbb{G}_{2,1}:=\mathcal{G}(V_3^{\mathbf{w}})$,

$$\mathbb{G}_{2,1}:=\mathbb{R}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{f}_1)=\text{span}\{u_+,\mathbf{f}_1,\mathbf{f}_1\mathbf{e}_2,\mathbf{e}_2\}\cup\text{span}\{u_-,\mathbf{e}_1,\mathbf{f}_1\mathbf{e}_1,\mathbf{e}_1\mathbf{e}_2$$
$$=u_+\mathbb{R}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{f}_1)+u_-\mathbb{R}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{f}_1), \tag{25}$$

where the mutually annihilating idempotents $u_\pm:=\frac{1}{2}(1\pm\mathbf{e}_1\mathbf{e}_2\mathbf{f}_1)$ play the role of the different identity elements in the two 4-dimensional projective subspaces.

5 Zero Residue Factorization

Positive and negative dominant geometric algebras can be factored into the products of geometric algebras of the same kind. If $\mathbb{G}_{p,q}$ is negative dominant, with $p\le q$, Zero Residue Factorization (ZRF) of $\mathbb{G}_{p,q}$ occurs in two cases: 1) Negative dominance ZRF when $p\le q$, and 2) Positive dominance ZRF when $p\ge q$. For the negative dominant case $p\le q$, we seek integers $k,l\ge 0$ such that the orthogonal direct sum $\mathbb{G}_{p,q}:=\mathbb{G}_{1,1}^{\wedge k}\oplus^{\perp}\mathbb{G}_{1,2}^{\wedge l}$ is metric preserving, giving the system of linear equations

$$\begin{pmatrix}p=k+l\\q=k+2l\end{pmatrix}.$$

Solving for k and l gives $l=q-p\ge 0$ and $k=2p-q\ge 0$. The zero residue, metric-preserving **c**-decomposition is only possible under the condition that both k and l are non-negative:

$$\mathbf{p\le q\le 2p}$$

Table 4: Zero-Residue: Negative Dominance ($p \le q \le 2p$, $p+q<8$)

$\mathbb{G}_{p,q}$	$p+q$	$k=2p-q$	$l=q-p$	Lounesto (2001) Notation
$\mathbb{G}_{1,1}$	2	1	0	$\mathbb{R}(2)$
$\mathbb{G}_{1,2}$	3	0	1	$\mathbb{C}(2)$
$\mathbb{G}_{2,2}$	4	2	0	$\mathbb{R}(4)$
$\mathbb{G}_{2,3}$	5	1	1	$\mathbb{C}(4)$
$\mathbb{G}_{2,4}$	6	0	2	$\mathbb{H}(4)$
$\mathbb{G}_{3,3}$	6	3	0	$\mathbb{R}(8)$
$\mathbb{G}_{3,4}$	7	2	1	$\mathbb{C}(8)$

Classification Table 4: Negative Dominance ($p \le q \le 2p$)

These algebras are perfectly tiled by the Hyperbolic Factor ($\mathbb{G}_{1,1} \cong \mathbb{R}(2)$) and the Complexifier Factor ($\mathbb{G}_{1,2} \cong \mathbb{C}(2)$).

Classification Table 5: Positive Dominance ($q \le p \le 2q$)

By swapping the $\mathbb{G}_{1,2}$ factor for $\mathbb{G}_{2,1}$, we obtain zero-residue $\wedge$-decomposition when the positive signature is dominant. We seek integers $k, l \ge 0$ such that $\mathbb{G}_{p,q} = \mathbb{G}_{1,1}^{\perp^k} \oplus^\perp \mathbb{G}_{2,1}^{\perp^l}$.

$$\begin{aligned} p = k(1) + l(2) &\implies p = k + 2l \\ q = k(1) + l(1) &\implies q = k + l \end{aligned}$$

Solving for k and l yields the required factors:

$$\begin{aligned} l &= p - q \\ k &= 2q - p \end{aligned}$$

The zero-residue, metric-preserving decomposition is only possible under the condition:

$$\mathbf{q \le p \le 2q}$$

Table 5: Zero-Residue: Positive Dominance ($q \le p \le 2q$, $p+q<8$)

$\mathbb{G}_{p,q}$	$p+q$	$k=2q-p$	$l=p-q$	Lounesto (2001) Notation
$\mathbb{G}_{1,1}$	2	1	0	$\mathbb{R}(2)$
$\mathbb{G}_{2,1}$	3	0	1	${}^2\mathbb{R}(2) = \mathbb{R}(2) \oplus \mathbb{R}(2)$
$\mathbb{G}_{2,2}$	4	2	0	$\mathbb{R}(4)$
$\mathbb{G}_{3,2}$	5	1	1	$\mathbb{R}(4) \oplus \mathbb{R}(4)$
$\mathbb{G}_{4,2}$	6	0	2	$\mathbb{R}(8)$
$\mathbb{G}_{3,3}$	6	3	0	$\mathbb{R}(8)$
$\mathbb{G}_{4,3}$	7	2	1	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

5.1 The Anti-Zero Residue Boundary Cases

An **Anti-Zero Residue** algebra is one that cannot be decomposed entirely into null simplexes $\mathbb{G}_{1,1}$, $\mathbb{G}_{1,2}$, or $\mathbb{G}_{2,1}$. These algebras, falling outside the zero-residue bands ($q > 2p$ or $p > 2q$), possess an irreducible **Euclidean Core** (isomorphic to $\mathbb{G}_{2,0}$ or $\mathbb{G}_{0,2}$) or a **Quaternion Core** (isomorphic to $\mathbb{H} := \mathbb{G}_{0,3}$) that prevents the total collapse into a null substrate.

The primary boundary cases for low dimensions are the Spacetime algebra $\mathbb{G}_{1,3}$ and the Majorana algebra $\mathbb{G}_{3,1}$. However, as dimension increases, a specific "Anti-Zero" boundary layer emerges, including $\mathbb{G}_{1,4}$, $\mathbb{G}_{1,5}$, and $\mathbb{G}_{2,5}$, as detailed in Tables 6 and 7.

Table 6: Anti-Zero Residue: Negative Dominance Boundary ($q > 2p$)

$\mathbb{G}_{p,q}$	$p+q$	Isomorphism (Lounesto p.217)
$\mathbb{G}_{1,3}$	4	$\mathbb{H}(2)$ (Spacetime Algebra)
$\mathbb{G}_{1,4}$	5	${}^2\mathbb{H}(2) = \mathbb{H}(2) \oplus \mathbb{H}(2)$
$\mathbb{G}_{1,5}$	6	$\mathbb{H}(4)$
$\mathbb{G}_{1,6}$	7	$\mathbb{C}(8)$
$\mathbb{G}_{2,5}$	7	${}^2\mathbb{H}(4) = \mathbb{H}(4) \oplus \mathbb{H}(4)$

Table 7: Anti-Zero Residue: Positive Dominance Boundary ($p > 2q$)

$\mathbb{G}_{p,q}$	$p+q$	Isomorphism (Lounesto p.217)
$\mathbb{G}_{3,1}$	4	$\mathbb{R}(4)$ (Majorana Algebra)
$\mathbb{G}_{4,1}$	5	$\mathbb{C}(4)$
$\mathbb{G}_{5,1}$	6	$\mathbb{H}(4)$
$\mathbb{G}_{6,1}$	7	$\mathbb{C}(8)$
$\mathbb{G}_{5,2}$	7	$\mathbb{C}(8)$

6 Analysis of Stability

The QSNV classification scheme, based on the Zero Residue condition, defines a bounded region of maximal algebraic stability within the collection of $\mathbb{G}_{p,q}$ algebras. This structural organization presents a compelling analogy to the well-known **Nuclear Valley of Stability** in nuclear physics which classifies atomic nuclei based on the ratio of protons and neutrons [17, 18]. The three regimes of stability common to both systems are summarized in Table 8.

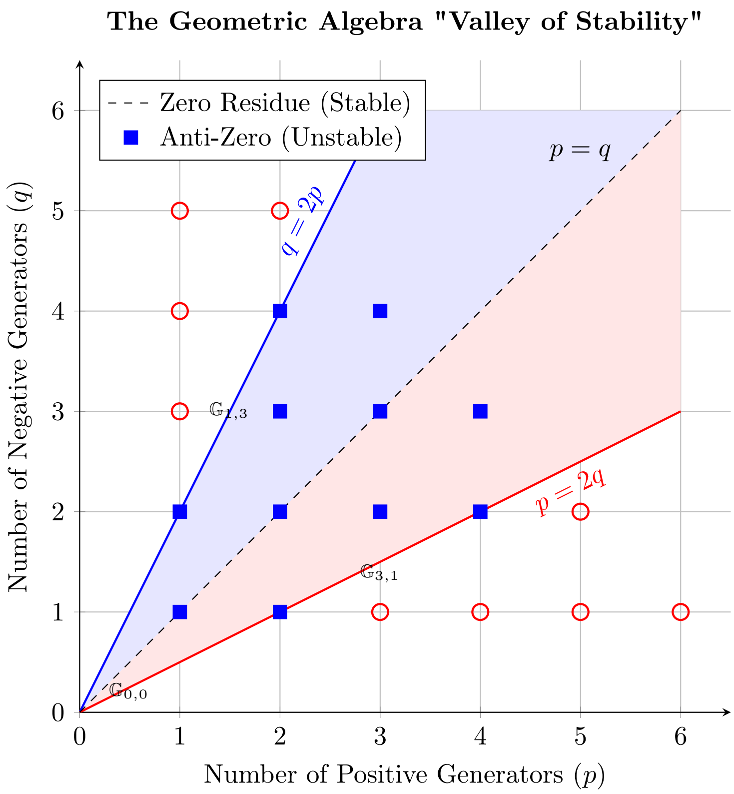

Figure 2: **The Geometric Algebra Valley of Stability.** The chart plots the metric signature components p versus q. The shaded regions represent the "Zero Residue" algebras, which factor completely into null simplexes ($\mathbb{G}_{1,1}, \mathbb{G}_{1,2}, \mathbb{G}_{2,1}$). The red circles indicate "Anti-Zero" boundary cases, such as Spacetime Algebra ($\mathbb{G}_{1,3}$), which require an irreducible Euclidean or Quaternionic core.

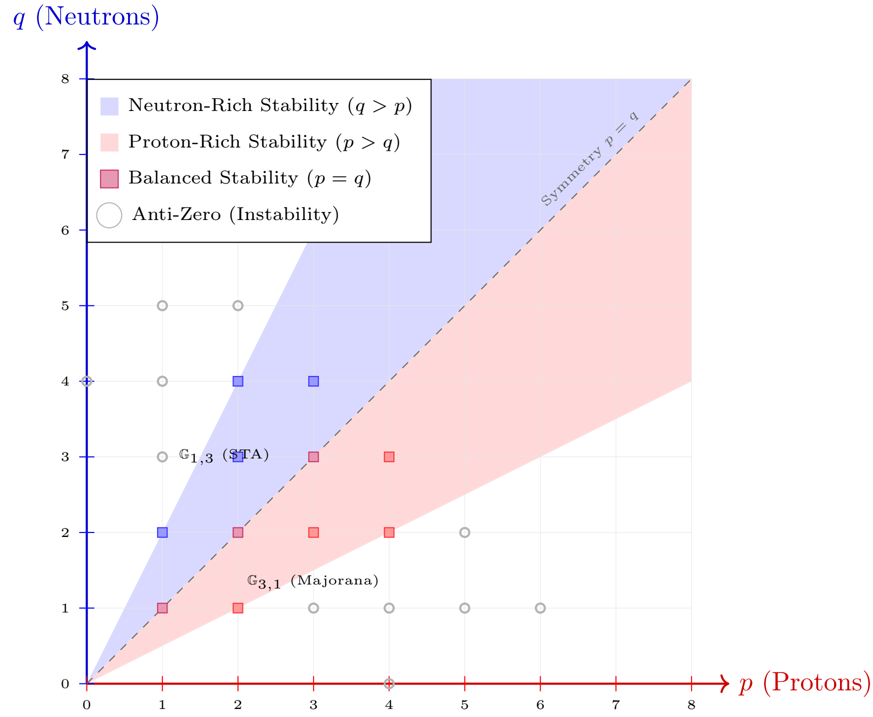

Figure 3: **The Algebraic Segrè Chart with Bipartite Stability.** The chart distinguishes between neutron-dominant (blue) and proton-dominant (red) regions of stability. The "Zero Residue" algebras within these shaded regions mirror the stable isotopes of the nuclear valley, while algebras falling outside these bounds (Anti-Zero) represent the unstable regimes where algebraic factorization is inhibited.

Table 8: **Three Regimes of Stability: Structural Analogy between Geometric Algebras and Atomic Nuclei**

Regime	QSNV (Geometric Algebra)	Nuclear Stability (Physics)	Principle of Stability
I. Perfect Symmetry	$\mathbf{p} = \mathbf{q}$	$\mathbf{Z} = \mathbf{N}$ (Light Nuclei)	Stability achieved through **equality** of competing factors.
Example	$\mathbb{G}_{1,1}$ or $\mathbb{G}_{2,2}$ (Perfect Factorization)	Carbon-12 (C^{12}) or Oxygen-16 (O^{16})	
II. Bounded Asymmetry	$p \leq q \leq 2p$ or $q \leq p \leq 2q$	$\mathbf{N} > \mathbf{Z}$ (Heavy Nuclei)	Stability achieved through precisely **bounded asymmetry**.
Example	$\mathbb{G}_{3,4}$ or $\mathbb{G}_{4,3}$ (Zero Residue)	Gold-197 (Au^{197}) or Lead-208 (Pb^{208})	
III. Critical Instability	$q > 2p$ or $p > 2q$	$\mathbf{N} \gg \mathbf{Z}$ or $\mathbf{Z} \gg \mathbf{N}$ (Extremes)	Instability results when asymmetry **exceeds the critical bound**.
Example	$\mathbb{G}_{1,5}$ or $\mathbb{G}_{5,1}$ (Anti-Zero Residue)	Uranium-238 (U^{238}) or highly unstable isotopes	

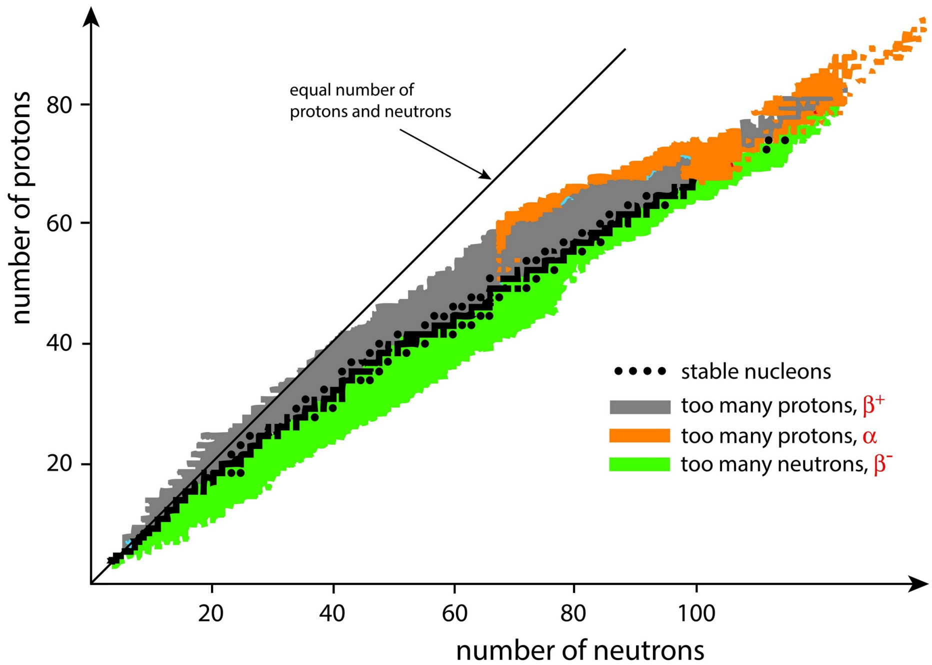

Figure 4: This plot, called the table of isotopes, or Segrè chart, is a graph of all known nuclides (isotopes) as a function of their number of protons and neutrons. The nuclides are color-coded by the type of radioactive decay they undergo. This chart is crucial in nuclear physics for understanding nuclear stability and decay processes. The third dimension, energy, is not shown. *Isospin* $T \cong 0$ for equal numbers of protons and neutrons, occurs at the bottom of the valley. The $SU(2)$ energy states, eigenvalues of the Hamiltonian operator, are constructed from its Lie algebra generators. $SO(3)$ and $SU(3)$ dictate the *shell structure* of the *magic numbers*, and the *unitary group* $U(6)$ helps model heavy complex isotopes.

7 Lie Algebras in Quadratic Grassmann-Clifford Algebra

The basis for the classification of Classical Lie Groups of a quadratic form and their Lie Algebras, is explored in the Quadratic Grassmann-Clifford Algebra $\mathcal{G}^{\perp}(V_{r,s,k})$. The recursive orthogonalization and Zero Residue factorization properties simplify and directly reveal their intrinsic geometric structure. The Zero Residue condition is equivalent to defining the algebra's structure as an $N \times N$ matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, mirroring the structural isomorphisms that define the classical Lie groups $O(N)$, $U(N)$, and $\mathrm{Sp}(N)$ and their Lie Algebras, [13, Chp.18].

From the perspective of the Universal Null Substrate (UNS), the emergent structure of Lie algebras offer new tools in the study of the fundamental nature of *magic numbers* defining the Valley of Stability. Highlighting the central role of null vectors should not be surprising in light of prominent role they play in the development of quantum mechanics, relativity [14, 15, 16], and in the ground breaking recent paper by David Hestenes [8].

7.1 Lie Algebra calculations

Let $A, B \in \mathbb{G}_{p,q}$ be two multivectors. The *symmetric* and *antisymmetric bracket* parts of their geometric product is important.

Definition 5.

$$AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA) := A \circ B + A \otimes B \tag{26}$$

where $A \circ B$ is the *symmetric part* and $A \otimes B$ is the *antisymmetric part*.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{G}^1_{p,q}$ be vectors, and $\mathbf{B} \in \mathbb{G}^2_{p,q}$ be a bivector in $\mathbb{G}_{p,q}$. The important *Jacobi identity* is respected for bivectors in geometric algebra. Noting that for vectors $\mathbf{B} \otimes \mathbf{x} \equiv \mathbf{B} \cdot \mathbf{x}$,

$$\mathbf{B} \otimes (\mathbf{x} \wedge \mathbf{y}) = (\mathbf{B} \cdot \mathbf{x}) \wedge \mathbf{y} + \mathbf{x} \wedge (\mathbf{B} \cdot \mathbf{y}). \tag{27}$$

Example

For the null vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4 \in V_r^{\mathbf{c}}$, consider the Lie algebras generated by the 6 bivectors $\mathbf{B} := \mathbf{c}_i \wedge \mathbf{c}_j$. The following identities are easily established:

- $(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot (\mathbf{c}_1 \wedge \mathbf{c}_3) = \frac{\mathbf{c}_1}{2} \cdot (\mathbf{c}_3 - \mathbf{c}_1) = \frac{1}{4} = (\mathbf{c}_1 \wedge \mathbf{c}_2)^2$.
- $(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot (\mathbf{c}_2 \wedge \mathbf{c}_3) = \frac{\mathbf{c}_1}{2} \cdot (-\mathbf{c}_2) = -\frac{1}{4}$.
- $(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot (\mathbf{c}_3 \wedge \mathbf{c}_4) = \frac{\mathbf{c}_1}{2} \cdot (\mathbf{c}_4 - \mathbf{c}_3) = 0$.
- $(\mathbf{c}_1 \wedge \mathbf{c}_2) \otimes (\mathbf{c}_1 \wedge \mathbf{c}_3) = [(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot \mathbf{c}_1] \wedge \mathbf{c}_3 + \mathbf{c}_1 \wedge [(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot \mathbf{c}_3] = \frac{1}{2}[\mathbf{c}_1 \wedge (\mathbf{c}_3 - \mathbf{c}_2)]$.
- $(\mathbf{c}_1 \wedge \mathbf{c}_2) \otimes (\mathbf{c}_3 \wedge \mathbf{c}_4) = [(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot \mathbf{c}_3] \wedge \mathbf{c}_4 + \mathbf{c}_3 \wedge [(\mathbf{c}_1 \wedge \mathbf{c}_2) \cdot \mathbf{c}_4]$
  $= \frac{1}{2}[(\mathbf{c}_1 - \mathbf{c}_2) \wedge \mathbf{c}_4 + \mathbf{c}_3 \wedge (\mathbf{c}_1 - \mathbf{c}_2)] = \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2) \wedge (\mathbf{c}_4 - \mathbf{c}_3)$.

The Lie algebra of the Zero Residue algebra $\mathbb{G}_{1,1}$, the smallest null simplex, is isomorphic to the linear algebra of 2×2 matrices with trace zero:

$$\mathrm{Lie}(\mathbb{G}_{1,1}) \cong sl(2, \mathbb{R}) \cong so(1, 2).$$

This confirms that the Universal Null Substrate contains the seeds of the Lorentz transformations within its simplest interacting null pair. The bivectors of the Quadratic Grassmann Algebra naturally generate the Lie algebras of the classical groups: **Rotation Generators**: For $\mathbf{c}_i, \mathbf{c}_j \in V_r^{\mathbf{c}}$, the bivector $\mathbf{L}_{ij} = \mathbf{c}_i \wedge \mathbf{c}_j$ generates rotations preserving the null cone structure. **Boost Generators**: The bivector $u = \mathbf{e}_1\mathbf{f}_1 = 2\mathbf{c}_2 \wedge \mathbf{c}_1$, generates the boosts of the Lorentz group $SO(1, 1)$.

The Lie algebras of the Quadratic Spaces of Null Vectors (QSNV) of the Universal Null Substrate (UNS) are orthogonal. It follows that the Lie bracket of bivectors, generators of the Lie Groups at the origin, chosen in different substrates are all zero. Consequently, all Lie algebras and Lie groups of any geometric algebra $\mathbb{G}_{p,q}$ are completely determined by their orthogonal parts.

7.2 The Symmetric Group in $\mathbb{G}_{1,n}$

A *Null Cone Simplex* $\mathcal{S}_N^{\mathbf{s}} \subset \mathbb{G}_{1,n}^1$, with *characteristic* $\mathbf{S_N^s} := \mathbf{s_1 s_2} \cdots \mathbf{s_N}$,, is defined by its $1 \le N \le n+1$ null vector vertices $\{\mathbf{s_1}, \ldots \mathbf{s_N}\}$, satisfying the condition that for $i \neq j$, $s_i \cdot s_j = h_{ij} > 0$. The condition $h_{ij} > 0$, for distinct i and j, guarantees that the vertices of S_N are on the $(n+1)$-dimensional Minkowski null cone in $\mathbb{G}_{1,n}$, and that $\mathcal{S}_N^{\mathbf{s}}$ is *non-degenerate*, $\mathbf{s_1} \wedge \cdots \wedge \mathbf{s_N} \neq \mathbf{0}$.

We now define the Standard Null Simplex $\mathcal{S}_{n+1}^{\mathbf{c}} \subset \mathbb{G}_{1,n}^1$.

Definition 6. The *Standard Null Simplex* $\mathcal{S}_{n+1}^{\mathbf{c}} := \{\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}\} = V_{n+1}^{\mathbf{c}} \subset \mathbb{G}_{1,n}^1$, with characteristic

$$\mathbf{S}_{n+1}^{\mathbf{c}} := \mathbf{c}_1 \mathbf{c}_2 \cdots \mathbf{c}_{n+1} = \sum_{m=0}^{n+1} \langle \mathbf{c}_1 \cdots \mathbf{c}_{n+1} \rangle_m, \tag{28}$$

is constructed from the null vectors in $\mathbf{c}_i \in V^{\mathbf{c}}$, [16]. The SNS $\mathcal{S}_{n+1}^{\mathbf{c}}$ is said to be *even* or *odd* if $n+1$ is even or odd, respectively. If the null simplex $\mathcal{S}_{n+1}$ is even, or odd, then the sum in (28) need only be taken over even indices $0, 2, \ldots, n+1$, or odd indicies $1, 3, \ldots, n+1$, respectively.

For $\mathbb{G}_{1,n} = \mathcal{G}(V_{n+1}^{\mathbf{c}})$ in its locally dual null vector basis (6), its characteristic $\mathbf{S_{n+1}^c}$ in its natural order is

$$\mathbf{S_{n+1}^c} = \mathbf{c_1} \cdots \mathbf{c_i} \cdots \mathbf{c_j} \cdots \mathbf{c_{n+1}}.$$

The *permutation* (ij), acting on $\mathbf{S_{n+1}}$, gives

$$(ij)\mathbf{S_{n+1}^c} := \mathbf{c_1} \cdots \mathbf{c_j} \cdots \mathbf{c_i} \cdots \mathbf{c_{n+1}},$$

only interchanging $\mathbf{c}_i$ and $\mathbf{c}_j$, leaving the order of the other factors unchanged.

Theorem 2. The 2-cycle (ij) acting on $\mathbf{S_{n+1}^c}$ gives

$$(ij)\mathbf{S_{n+1}^c} = (-\mathbf{1})^{\mathbf{n}}(\mathbf{c_i} - \mathbf{c_j})\mathbf{S_{n+1}^c}(\mathbf{c_i} - \mathbf{c_j}).$$

Proof. Note that for any three distinct indices i, j, k, $(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_k = -\mathbf{c}_k(\mathbf{c}_i - \mathbf{c}_j)$,

$$(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_i(\mathbf{c}_i - \mathbf{c}_j) = -\mathbf{c}_j\mathbf{c}_i(\mathbf{c}_i - \mathbf{c}_j) = \mathbf{c}_j, \text{ and } (\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_j(\mathbf{c}_i - \mathbf{c}_j) = \mathbf{c}_i\mathbf{c}_j(\mathbf{c}_i - \mathbf{c}_j) = \mathbf{c}_i.$$

The proof is completed by noting that when $(\mathbf{c}_i - \mathbf{c}_j)$ is moved through $\mathbf{S_{n+1}^c}$, it will anti-commute with n factors and interchange the positions of $\mathbf{c}_i$ and $\mathbf{c}_j$. For example, for $n = 2$,

$$(-1)^2(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{S_3^c}(\mathbf{c_1} - \mathbf{c_2}) = (\mathbf{c_1} - \mathbf{c_2})\mathbf{c_1c_2c_3}(\mathbf{c_1} - \mathbf{c_2}) = -(\mathbf{c_1} - \mathbf{c_2})\mathbf{c_1c_2}(\mathbf{c_1} - \mathbf{c_2})\mathbf{c_3}$$
$$= \big((\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_1(\mathbf{c}_1 - \mathbf{c}_2)\big)\big((\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_2(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_3\big) = \mathbf{c}_2\mathbf{c}_1\mathbf{c}_3 = (12)\mathbf{S_3^c}.$$

□

This theorem demonstrates that the discrete symmetric group acting on $\mathbf{S_{n+1}^c}$ is generated by a set of discrete reflections within the Geometric Algebra $\mathbb{G}_{1,n}$. The vectors $(\mathbf{c}_i - \mathbf{c}_j)$ are the *roots* of the algebra, and the bivectors $\mathbf{c}_i \wedge \mathbf{c}_j$ generate the continuous transformations of the null cone that preserve the simplex's structure. This provides a direct path from the Universal Number Substrate, defined by null vectors, to the classical Lie groups SO(1,n).

Whereas the Caylley-Hamilton Theorem of a linear transformation is defined by its scalar invariants, the discrete invariants of the symmetric group acting on $\mathbf{S_{n+1}^c}$ in $\mathbb{G}_{1,n}$ is invariant up to the parity of the permutation acting on the m-vector parts,

$$(ij)\mathbf{S_{n+1}^c} = (-\mathbf{1})^{\mathbf{n}}(\mathbf{c_i} - \mathbf{c_j})\mathbf{S_{n+1}^c}(\mathbf{c_i} - \mathbf{c_j}) = (-\mathbf{1})^{\mathbf{n}} \sum_{\mathbf{m=0}}^{\mathbf{n+1}} (\mathbf{c_i} - \mathbf{c_j})\langle \mathbf{c_1} \cdots \mathbf{c_{n+1}} \rangle_{\mathbf{m}}(\mathbf{c_i} - \mathbf{c_j}).$$

8 Comprehensive Classification Table

Table 9 below utilizes Lounesto's notation [11, p.217], where ${}^2\mathbb{K}(n)$ denotes $\mathbb{K}(n) \oplus \mathbb{K}(n)$.

Table 9: **Comprehensive Table of QSNV Algebras** ($p+q \leq 8$)

p	q	Algebra	$p-q$ (mod 8)	Type	Lounesto Notation	Classification
0	0	$\mathbb{R}$	0	$\mathbb{R}$	$\mathbb{R}$	Scalar
1	0	$\mathbb{G}_{1,0}$	1	$^2\mathbb{R}$	$^2\mathbb{R}$	Hyperbolic
0	1	$\mathbb{G}_{0,1}$	7	$\mathbb{C}$	$\mathbb{C}$	Complex
1	1	$\mathbb{G}_{1,1}$	0	$\mathbb{R}$	$\mathbb{R}(2)$	Zero Residue
2	0	$\mathbb{G}_{2,0}$	2	$\mathbb{R}$	$\mathbb{R}(2)$	Euclidean
0	2	$\mathbb{G}_{0,2}$	6	$\mathbb{H}$	$\mathbb{H}$	Quaternion
2	1	$\mathbb{G}_{2,1}$	1	$^2\mathbb{R}$	$^2\mathbb{R}(2)$	Zero Residue
1	2	$\mathbb{G}_{1,2}$	7	$\mathbb{C}$	$\mathbb{C}(2)$	Zero Residue
3	0	$\mathbb{G}_{3,0}$	3	$\mathbb{C}$	$\mathbb{C}(2)$	Pauli
0	3	$\mathbb{G}_{0,3}$	5	$^2\mathbb{H}$	$^2\mathbb{H}$	Split-Bi-Quaternion
2	2	$\mathbb{G}_{2,2}$	0	$\mathbb{R}$	$\mathbb{R}(4)$	Zero Residue
3	1	$\mathbb{G}_{3,1}$	2	$\mathbb{R}$	$\mathbb{R}(4)$	Anti-Zero (Majorana)
1	3	$\mathbb{G}_{1,3}$	6	$\mathbb{H}$	$\mathbb{H}(2)$	Anti-Zero (Spacetime)
4	0	$\mathbb{G}_{4,0}$	4	$\mathbb{H}$	$\mathbb{H}(2)$	Euclidean
0	4	$\mathbb{G}_{0,4}$	4	$\mathbb{H}$	$\mathbb{H}(2)$	Quaternion
3	2	$\mathbb{G}_{3,2}$	1	$^2\mathbb{R}$	$^2\mathbb{R}(4)$	Zero Residue
2	3	$\mathbb{G}_{2,3}$	7	$\mathbb{C}$	$\mathbb{C}(4)$	Zero Residue
4	1	$\mathbb{G}_{4,1}$	3	$\mathbb{C}$	$\mathbb{C}(4)$	Anti-Zero
1	4	$\mathbb{G}_{1,4}$	5	$^2\mathbb{H}$	$^2\mathbb{H}(2)$	Anti-Zero
3	3	$\mathbb{G}_{3,3}$	0	$\mathbb{R}$	$\mathbb{R}(8)$	Zero Residue
4	2	$\mathbb{G}_{4,2}$	2	$\mathbb{R}$	$\mathbb{R}(8)$	Zero Residue
2	4	$\mathbb{G}_{2,4}$	6	$\mathbb{H}$	$\mathbb{H}(4)$	Zero Residue
5	1	$\mathbb{G}_{5,1}$	4	$\mathbb{H}$	$\mathbb{H}(4)$	Anti-Zero
1	5	$\mathbb{G}_{1,5}$	4	$\mathbb{H}$	$\mathbb{H}(4)$	Anti-Zero
4	3	$\mathbb{G}_{4,3}$	1	$^2\mathbb{R}$	$^2\mathbb{R}(8)$	Zero Residue
3	4	$\mathbb{G}_{3,4}$	7	$\mathbb{C}$	$\mathbb{C}(8)$	Zero Residue
6	1	$\mathbb{G}_{6,1}$	5	$^2\mathbb{H}$	$\mathbb{C}(8)$	Anti-Zero
1	6	$\mathbb{G}_{1,6}$	3	$\mathbb{C}$	$\mathbb{C}(8)$	Anti-Zero
5	2	$\mathbb{G}_{5,2}$	3	$\mathbb{C}$	$\mathbb{C}(8)$	Anti-Zero
2	5	$\mathbb{G}_{2,5}$	5	$^2\mathbb{H}$	$^2\mathbb{H}(4)$	Anti-Zero

9 Conclusion and Future Work

The Quadratic Space of Null Vectors (QSNV) offers a substrate-first approach to geometric algebras, inverting the traditional view that the metric signature is primary [12]. The Universal Null Substrate reveals a recursive structure that classifies all $\mathbb{G}_{p,q}$ algebras into two distinct regimes, Zero Residue (perfectly factorable into null simplexes) and Anti-Zero Residue (requiring metric residues). This clarifies the unique position of physical algebras. Spacetime Algebra $\mathbb{G}_{1,3}$ and the Majorana Algebra $\mathbb{G}_{3,1}$ sit on the boundary of the stability band. This suggests their role as holographic interfaces, the mathematical boundary where the purely null substrate of information is projected into the rigid Euclidean geometry of an observer's 3-dimensional experience. See further comments in the Appendix.

Table 9 shows that neutral geometric algebras $\mathbb{G}_{p,q}$, with $p=q$, characterize the concept of global duality. In tensor analysis this is equivalent to the concept of a vector and its dual covector. A *scalar angle* in local duality becomes secondary to the conformally invariant concept of the direction of a null vector when expressed in the locally dual bases of the Lorentz geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$. In the locally dual bases of these algebras, the only scalar angles are those whose inner products are the constants $\frac{1}{2}, -\frac{1}{2}$, with a 0 angle between null vectors in the different orthogonal blocks $V_r^{\mathbf{c}}, V_s^{\mathbf{w}}$, and $V_k^{\mathbf{n}}$. The geometric algebras $\mathbb{G}_{n,1}$ and $\mathbb{G}_{1,n}$ have been widely employed in the Conformal Model, together with ideas from projective geometry, to characterize properties of points, lines, planes, and circles. Timothy Havel has shown how the Conformal Model of $\mathbb{G}_{4,1}$ can be beautifully used to generalize Heron's formula for the area of a planar triangle in terms of the lengths of it three sides, to the volume of the tetradedron in terms of the areas its faces, and to higher dimensional simplices [7].

The Zero Residue conditions ($p \leq q \leq 2p$ or $q \leq p \leq 2q$) mirror the Bott Periodicity of classical unitary, orthogonal and symplectic groups. The factorization of high-dimensional algebras into 2×2 block matrices of $\mathbb{G}_{1,1}, \mathbb{G}_{1,2}$, and $\mathbb{G}_{2,1}$, reduces the computational cost of geometric products to simpler index arithmetic. Raoul Bott considered *networks* to be discrete versions of harmonic theory. He viewed the flow of electricity in a network as a discrete analog to the continuous problems of harmonic analysis and Hodge theory [10, 4].

The QSNV and UNS framework provides a natural algebraic language for the Holographic Principle of and the Covariant Entropy Bound, and Discrete Max-Focusing [2, 3]. There is another interesting connection. Though the Deng-Hani-Ma paper is primarily a masterpiece of kinetic theory and PDEs, its relevance to the "discrete/harmonic" discussion lies in the mathematical tools used to bridge these two worlds. The singular value and Jordan normal form decompositions of a linear operator become the basic tools of the finite dimensional Hurwitz Matrix Equation, leading to infinite dimensional Hurwitz Stability, the Lyapunov Equation of a dynamical system, and harmonic analysis [5]. Future work could extend this *null-sorting* algorithm to the higher dimensional cases, offering a discrete geometric foundation for representation theory [6].

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A QSNV and the Holographic Principle

The QSNV framework provides a natural algebraic language for the Holographic Principle and the Covariant Entropy Bound. This appendix explores the primary correspondences proposed between the Null Substrate and modern gravitational theory, as well as their historical-philosophical precursors in Dante Alighieri's *Paradiso* (1310 AD), and the much earlier (375 BCE) Greek Cave Allegory in Plato's "The Republic" [19].

A.1 The Null Simplex as a Holographic Pixel

In the Bekenstein-Hawking entropy formula, $S = \frac{A}{4G\hbar}$, information is discretized into Planck-area units. We propose that a fundamental null simplex in $V_{r,s,k}$ represents an *Information Pixel.* While standard geometry treats the interior of a volume as having more degrees of freedom than the surface, the Zero Residue Factorization (ZRF) mirrors the holographic realization that "Bulk" degrees of freedom are strictly limited by the boundary null-vector configurations.

A.2 Lightsheets and the ZRF Band

Raphael Bousso's lightsheet $L(B)$ is a 3D volume generated by non-expanding null geodesics orthogonal to a surface B. The QSNV identifies a "stability band" where $p \leq q \leq 2p$. We hypothesize the following physical correspondences:

- **Zero Residue Algebras:** Represent spaces where the information content (the bits) can be perfectly mapped onto the null substrate without "metric residue." This corresponds to a non-singular holographic encoding.
- **The Anti-Zero Residue Boundary:** The case $q = 2p$ represents the "saturated" holographic limit. Beyond this point, the algebra requires an irreducible Euclidean core (metric residue), which may correspond physically to the emergence of massive, non-null particles.

A.3 Dante's "Punto" as the Null Substrate

In *Paradiso* XXVIII (41–42), Dante describes the source of the universe: *"Da quel punto depende il cielo e tutta la natura"* (From that point depend the heavens and all of nature). In the QSNV context, this *punto* can be modeled as the ultimate Null Simplex ($p = 1, q = 1$).

- **Causal Dependency:** The verb *depende* reflects the ZRF's recursive nature—the complex manifold (Nature) is algebraically "suspended" from a singular, null-dimensional origin.

- **Information Singularity:** Dante's description of the point as infinitesimal yet containing the entire power of the universe anticipates the holographic concept that the most fundamental "pixel" governs the global bulk.

A.4 The Inversion Paradox and Metric Residue

A striking parallel to the Bousso Bound appears in lines 67–75 of Canto XXVIII. Dante observes that while physical spheres grow in power as they grow in size, the spiritual circles (Angelic Hierarchies) grow in power as they approach the center.

- **The Law of Virtute:** Beatrice explains that the *ampiezza* (physical extension) is a function of the *virtute* (algebraic intensity). This mirrors the QSNV requirement that as the purity of the Null Substrate (p) is diluted, a larger "Metric Residue" (spatial volume) is required to maintain the same information density.
- **Algebraic Agreement:** The *mirabil congruenza* (marvelous agreement) Beatrice describes between the point and the spheres represents the stability of the **Recursive Orthogonalization Theorem**. Spacetime is stable only if the outward "Bulk" copy remains consistent with the inner "Null" pattern.

A.5 Quantum Focusing and Algebra Stability

The Quantum Focusing Conjecture (QFC) states that the generalized expansion Θ of a lightsheet never increases: $\frac{d\Theta}{d\lambda} \leq 0$. In the QSNV framework, the stability of the recursive process is the algebraic analog to the QFC: if the null substrate were to "lose focus" (i.e., if the inner product rules became inconsistent), the resulting geometric algebra would fail to represent a stable physical spacetime.