Geometric Matrices and the Symmetric Group

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Abstract

We construct $2^n \times 2^n$ real and complex matrices in terms of Kronecker products of a Witt basis of $2n$ null vectors in the geometric algebra $G_{n,n}$ over the real and complex numbers. In this basis, every matrix is represented by a unique sum of products of null vectors. The complex matrices of $G_{n,n+1}$ provide a direct matrix representation for geometric algebras $G_{p,q}$, where $p + q \leq 2n + 1$. Properties of irreducible representations of the symmetric group are presented in this geometric setting.

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0 Introduction

A matrix is traditionally just a table of real or complex numbers. As eloquently put by Tobias Danzig,

\[ \ldots \text{a theory in which a whole array of elements is regarded as a number-individual. These filing cabinets are added and multiplied, and a whole calculus of matrices has been established which may be regarded as a continuation of the algebra of complex numbers.} \quad [1, \text{p.212}] \]

What is magical about matrices is contained in the definition of the multiplication of matrices. We give a geometrical construction of $2^n \times 2^n$ matrix algebras $Mat_{2^n}(F)$ over the real and complex numbers, when $F = \mathbb{R}$, or $F = \mathbb{C}$, respectively, in terms of the Kronecker Product of $2 \times 2$-blocks of anti-commuting null vectors.

Whereas this rather restricted class of matrix algebras grow in size quite rapidly,

\[ Mat_1(F), Mat_2(F), Mat_4(F), \ldots, Mat_{2^n}(F), \ldots \]
they have unique geometric properties which call for special attention. Indeed, it is the geometric properties of the matrix algebras $Mat_2(\mathbb{C})$ and $Mat_4(\mathbb{C})$, that early in the 20th Century was key to the development of the fundamental theories of relativity and quantum mechanics. We explain in detail how the structure of the geometric algebras $G_{p,q}$ are embedded in the matrix algebras $Mat_{2^n}(\mathbb{C})$. The comprehensive geometric interpretation of matrices of these sizes has many ramifications, justifying the terminology geometric matrices [2]. One promising area of application is sparse random matrices, such as used in error correcting codes [3].

The permutation on $n$ letters is a good place to start a study of the symmetric group $S_n$, and the corresponding natural permutation representation of the rows of a square $n \times n$ matrix [4]. Such matrices are easily represented in terms of the geometric matrices of our approach. Representation theory of finite groups has provided new insight into the structure of finite groups, with many applications in other areas of mathematics and science. More generally, the study of finite dimensional representations of Lie groups and Lie algebra, [5, 6], is a powerful tool in the study of quantum mechanics and atomic structure [7]. Aside from its many important applications, it is a beautiful mathematical structure that has developed over the last 150 years, and is worthy of study on its own merits.

1 Geometric matrices

Real or complex matrices of size $2^n \times 2^n$ are built up, block by block, by considering pairs of null vectors satisfying the following two simple rules:

N1) $a^2 = 0 = b^2$ The non-zero vectors $a$ and $b$ are nilpotents.
N2) $ab + ba = 1$ The sum of $ab$ and $ba$ is 1.

Since we are assuming product associativity, the second property easily implies that $aba = a$. We take $k$ such pairs of vectors $a_1, b_1, \ldots, a_k, b_k$, and assume that the null vectors with distinct indexes are pair-wise anti-commutative. That is,

$$a_ia_j = -a_ja_i, \quad b_ib_j = -b_jb_i, \quad a_ib_j = -b_ia_i$$

for $i \neq j$.

If $M = \{m_{ij}\}$ and $N = \{n_{jk}\}$ are matrices over geometric numbers, we define the left and right Kronecker product of $M$ and $N$ by

$$M \otimes N := \{M_{mn}\}, \quad M \overrightarrow{\otimes} N := \{m_{ij}N\}.$$

As an example, consider the pairs of null vectors $a_i$ and $b_i$ satisfying N1) and N2) above. Then

$$\begin{pmatrix} 1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ b_2 \end{pmatrix} := \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_{12} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ a_2 \end{pmatrix} \overrightarrow{\otimes} \begin{pmatrix} 1 \\ a_1 \end{pmatrix} := \begin{pmatrix} 1 & a_1 & a_2 & a_{21} \end{pmatrix}.$$
Defining $B_i^T := (1 \ b_i)$, and $A_i^T := (1 \ a_i)$, the above relationships take the form

$$B_1 \bigotimes B_2 = \begin{pmatrix} B_1 \\ B_1 b_2 \end{pmatrix} \quad \text{and} \quad A_2 \bigotimes A_1^T = (A_1^T \quad a_2 A_1^T).$$

Shortly, we will use these expressions and their generalizations.

In a previous paper [8], I showed that the geometric algebra defined by

$$G_{1,1} := \{ R(a, b) | a^2 = b^2 = 0, \ ab + ba = 1 \},$$

is specified by its spectral basis

$$G_{1,1} = Bab A^T = \begin{pmatrix} 1 \\ b \end{pmatrix} ab \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} ab \\ b \end{pmatrix} (ab \quad a) = \begin{pmatrix} ob \\ a \end{pmatrix} = \begin{pmatrix} ob \\ a \end{pmatrix}.$$

The matrix representation of a general element

$$g = g_{11}ab + g_{12}a + g_{21}b + g_{22}ba \in G_{1,1}$$

is given by

$$g = B^T u A g B^T u A = (1 \ b) u \begin{pmatrix} 1 \\ a \end{pmatrix} g (1 \ b) u \begin{pmatrix} 1 \\ a \end{pmatrix} = (1 \ b) u \begin{pmatrix} g \quad gb \\ ag \quad agb \end{pmatrix} u \begin{pmatrix} 1 \\ a \end{pmatrix} = B^T u [g] A, \quad (1)$$

where $u = ab$ and $[g] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ is the real matrix of the geometric number $g \in G_{1,1}$ with respect to this spectral basis. The matrix algebra $\text{Mat}_2(\mathbb{R})$ is algebraically isomorphic, and thus fully equivalent to the geometric algebra $G_{1,1}$, and inherits a comprehensive geometric interpretation.

By using the same spectral basis over the complex numbers, any geometric number $g \in G_{1,2}$ can be represented by a $2 \times 2$ matrix $[g]_C$ over the complex numbers. That is

$$g = (1 \ b) ab [g]_C \begin{pmatrix} 1 \\ a \end{pmatrix} = B^T u [g]_C A,$$

where the generating unit vectors $e_1, f_1, f_2$ of $G_{1,2}$ are defined by

$$e_1 = a + b, \ f_1 = a - b, \ \text{and} \ f_2 := e_1 f_1 i,$$

and $i := e_1 f_{12}$.

Calculating the conjugation operations [9, p.60], of reversion $g^\dagger$ and grade inversion $g^\sim$, gives

$$g^\dagger = A^T u [g]^T B = B^T u \begin{pmatrix} g_{22} & g_{12} \\ g_{21} & g_{11} \end{pmatrix}_C A, \quad (2)$$
\[ g^* = (A^T)^{-1} u [g]^T B^* = B^T u \left( \begin{array}{cc} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{array} \right) c \] 

Using the operation of reverse-inversion, the *determinant* 

\[ \det[g] := gg^* = B^T u [g][g^*] A = g_{11} g_{22} - g_{12} g_{21}. \] 

Before generalizing to higher dimensional geometric algebras \( \mathbb{G}_{p,q} \), consider \( \mathbb{G}_{2,2} \). The spectral basis of \( \mathbb{G}_{2,2} \) is 

\[ \mathbb{G}_{2,2} = B_1 \otimes B_2 u_{12} A_2^T \otimes A_1^T = \left( \begin{array}{cc} B_1 & A_2^T \\ B_1 & A_2^T \end{array} \right) u_{12} \left( \begin{array}{cc} A_1 & a_2 A_1^T \\ -A_2 & a_2 A_2^T \end{array} \right) \]

\[ = \left( \begin{array}{cc} B_1 u_1 A_1^T & B_1 u_2 A_2^T \\ B_1 u_1 A_1^T & B_1 u_2 A_2^T \end{array} \right) \]

\[ = \left( \begin{array}{cc} 1 \\ b_1 \end{array} \right) \otimes \left( \begin{array}{cc} 1 \\ b_2 \end{array} \right) u_{12} \left( \begin{array}{cc} 1 & a_2 \\ a_1 & 1 \end{array} \right) \]

\[ = \left( \begin{array}{cc} u_{12} & a_2 u_1 \\ b_2 u_1 & b_1 u_2 \end{array} \right) \]

where \( u_i := a_i b_i, u_1^i = b_i a_i, u_{12} := u_1 u_2, \) and \( u_{12}^i = u_1^i u_2^i \).

For \( g \in \mathbb{G}_{2,2} = \mathbb{G}_{1,1} \times \mathbb{G}'_{1,1} \), where \( \mathbb{G}_{1,1} := \mathbb{R}(a_1, b_1) \) and \( \mathbb{G}'_{1,1} := \mathbb{R}(a_2, b_2) \),

\[ g = B_1^T \otimes B_2^T u_{12} A_2 \otimes A_1 = B_1^T u_1 \left( \begin{array}{cc} [g]_{11} & [g]_{12} \\ [g]_{21} & [g]_{22} \end{array} \right) A_2 \]

\[ = B_1^T u_1 \left( \begin{array}{cc} g_1' & g_2' \\ g_3' & g_4' \end{array} \right) A_1 \]

where \([g]\) is the real 4 \times 4 matrix of \( g \) with respect to the spectral basis (6), \([g]_{kl}\) are its 2 \times 2 blocks, and \( g_1', g_2', g_3', g_4' \) are the elements in \( \mathbb{G}'_{1,1} \) represented by each of these blocks, respectively. Generalizing (7) for \( \mathbb{G}_{n+1,n+1} = \mathbb{G}_{1,1} \times \mathbb{G}'_{n,n} \), gives for \( g \in \mathbb{G}_{n+1,n+1} \)

\[ g = B_1^T u_1 \left( \begin{array}{cc} g_1' & g_2' \\ g_3' & g_4' \end{array} \right) A_1, \]
for \( g_1', g_2', g_3', g_4' \in \mathbb{G}_{n,n}' \).

For \( g \in \mathbb{G}_{2,2} = \mathbb{G}_{1,1} \times \mathbb{G}_{1,1}' \), the reverse of \( g \) is

\[
\langle g \rangle = A_1^T \otimes A_2^T \bigotimes u_{12}^\dagger [g]^T B_2^\dagger \otimes B_1 = ( A_1^T \ A_2^T a_2 ) u_{12}^\dagger [g]^T \left( \begin{array}{c} B_1 \\ b_2 B_1 \end{array} \right).
\]

We also have

\[
[g] = \left( \begin{array}{cccc}
g_{11} & g_{12} & g_{13} & g_{14} \\
g_{21} & g_{22} & g_{23} & g_{24} \\
g_{31} & g_{32} & g_{33} & g_{34} \\
g_{41} & g_{42} & g_{43} & g_{44}
\end{array} \right), \quad [g^\dagger] = \left( \begin{array}{cccc}
g_{44} & g_{34} & -g_{24} & -g_{14} \\
g_{43} & g_{33} & -g_{23} & -g_{13} \\
-g_{42} & -g_{32} & g_{22} & g_{12} \\
-g_{41} & -g_{31} & g_{21} & g_{11}
\end{array} \right),
\]

and

\[
[g^*] = \left( \begin{array}{cccc}
g_{44} & -g_{34} & g_{24} & -g_{14} \\
-g_{43} & g_{33} & -g_{23} & g_{13} \\
g_{42} & -g_{32} & g_{22} & -g_{12} \\
-g_{41} & g_{31} & -g_{21} & g_{11}
\end{array} \right).
\]

The matrices \([a_1], [b_1], [a_2], [b_2]\) of the null vectors \( a_1, b_1, a_2, b_2 \), are

\[
[a_1] = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array} \right), \quad [a_2] = \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right)
\]

\[
[b_1] = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array} \right), \quad [b_2] = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array} \right).
\]

Note that in the special case when

\[
g = g_{11} a_1 b_1 + g_{12} a_1 + g_{21} b_1 + g_{22} b_1 a_1 \in \mathbb{G}_{1,1} \subset \mathbb{G}_{1,1} \times \mathbb{G}_{1,1}',
\]

then the \( 4 \times 4 \)-matrix \([g]\) of \( g \) is

\[
[g] = \left( \begin{array}{cccc}
g_{11} & g_{12} & 0 & 0 \\
g_{21} & g_{22} & 0 & 0 \\
0 & 0 & g_{11} & g_{12} \\
0 & 0 & g_{21} & g_{22}
\end{array} \right).
\]

Alternatively, for \( g' = g_{11}' a_2 b_2 + g_{12}' a_2 + g_{21}' b_2 + g_{22}' b_2 a_2 \in \mathbb{G}_{1,1}' \), then

\[
[g] = \left( \begin{array}{cccc}
g_{11}' & 0 & g_{12}' & 0 \\
0 & g_{11}' & 0 & -g_{12}' \\
g_{21}' & 0 & g_{22}' & 0 \\
0 & -g_{21}' & 0 & g_{22}'
\end{array} \right).
\]

The standard basis of \( \mathbb{G}_{n,n} \) is defined by

\[
\mathbb{G}_{n,n} := \mathbb{R}(e_1, \ldots, e_n, f_1, \ldots, f_n) = \text{gen}_{\mathbb{R}} \{e_i, f_i | e_i := a_i + b_i, f_i = a_i - b_i\},
\]

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for $1 \leq i \leq n$. For the complexified geometric algebra

$$G_{n,n+1} := \mathbb{C}(e_1, \cdots, e_n, f_1, \cdots, f_n) = \text{gen}_{\mathbb{C}} \{ e_i, f_i \mid e_i := a_i + b_i, f_i = a_i - b_i \},$$

with $i := e_{n-1}f_1 \cdots n+1$. Defined in this way, the $2n+1$-vector $i$, playing the roll of the imaginary unit, is in the center of $G_{n,n+1}$, and $i^2 = -1$.

Consider now the geometric algebra $G_{p+1,q+1} = G_{p,q}' \times G_{1,1}$. Using the spectral basis of $G_{1,1} = \text{gen}_{\mathbb{R}} \{ a_1, b_1 \}$, a general element $g \in G_{p+1,q+1}$ can be written

$$g = h_1 a_1 b_1 + h_2 a_1 + h_3 b_1 + h_4 b_1 a_1 = a_1 b_1 h_1 + a_1 h_2^- + b_1 h_3^- + b_1 a_1 h_4,$$

where $h_1, h_2, h_3, h_4 \in G_{p,q}'$. Then, for $u_1 = a_1 b_1$,

$$g = (1 b_1) u_1 \left( \frac{1}{a_1} \right) g \left( 1 b_1 \right) u_1 \left( \frac{1}{a_1} \right) = B^T u_1 \left( \frac{h_1}{h_2} \frac{h_3}{h_4} \right) A. \quad (9)$$

Thus, the elements of $G_{p+1,q+1}$ can be expressed as $2 \times 2$ matrices over $G_{p,q}'$.

The spectral basis referred to in [9, p.206] is equivalent to

$$(1 a_1 a_2 a_{21})^T b_1 a_1 b_2 a_2 (1 b_1 b_2 b_{12})$$

The reverse $\dagger$ of this basis is

$$(1 b_1 b_2 b_{21})^T a_1 b_1 a_2 b_2 (1 a_1 a_2 a_{12}),$$

which is not equivalent to the spectral basis (6), meaning there is no inner automorphism which will take this spectral basis into the spectral basis (6), and at the same time leave invariant of the null vectors $a_i$ and $b_i$.

2 The permutation group algebra of $S_n$

The symmetric group $S_n$ consists of all permutations on $n$ letters. We use the usual cycle notation. For example,

$$S_3 = \{ 1, (12), (13), (23), (123), (132) \},$$

with group multiplication of cycles. An example cycle multiplication, from right to left, is

$$(23) = (12)(13)(12).$$

The permutation representation of $S_n$ is generated by real $n \times n$ matrices of the form

$$(1k) = 1_{(1k)},$$

where the matrix $1_{(1k)}$ is obtained by interchanging the first and the $k^{th}$ rows of the $n \times n$ of the identity matrix $1$. The generators of the permutation matrix group algebra $S_3$ are

$$S_3 = \text{gen} \{ (12), (13) \} = \text{gen} \{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \}.$$
In studying the permutation representation, and the closely related standard representation of $S_n$, it is expedient to introduce the all ones matrix

$$A_n := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

and the Casimir matrix [10],

$$C_n := A_n - 1_n,$$

where $1_n$ is the $n \times n$-identity matrix. For example

$$C_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Not surprisingly, both the $n$-all ones matrix and the $n$-Casimir matrix commutes with all the regular symmetric matrices $S_n$. It is also easy to verify the simple relationships

$$A_n^2 = nA_n \quad \text{and} \quad C_n^2 = (n-2)C_n + (n-1)1_n,$$

implying that

$$\min(A_n) := x(x-1), \quad \text{and} \quad \min(C_n) := (x+1)(x-(n-1)),$$

are, respectively, the minimal polynomials of $A_n$ and $C_n$.

3 The geometric group algebra of $S_n$

The geometric algebras $G_{n,n}$ and $G_{n,n+1}$ provide a comprehensive geometric interpretation for the elements of the group algebra of the symmetric group $S_n$, and give a new way of studying its properties. One problem of this approach is that these geometric group algebras only exist for dimensions $2^n$, corresponding to the matrix representations using $2^n \times 2^n$ matrices over the real or complex numbers. However, our approach provides both new tools and a geometric interpretation of the results.

We begin by examining the geometric group algebras of the smaller symmetric groups, and then generalize these results to $S_n$. The group algebra of $S_2$ is generated by the single element $(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding generating element in the geometric group algebra is $(12) = a_1 + b_1 \in \mathbb{G}_{1,1}$. Thus, the permutation representation of $S_2$ is

$$S_2 = \text{span}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} = \text{span}_\mathbb{R}\{1, a_1 + b_1\} = \text{gen}_\mathbb{R}\{a_1 + b_1\}.$$
To get the permutation geometric group algebra of $S_3$, we must go to the larger geometric algebra $G_{2,2}$. Using the spectral basis (6), we find the geometric permutation representation

$$S_3 = \text{gen}_\mathbb{R}\{(12), (13)\} = \text{gen}_\mathbb{R}\{1 + (a_1 + b_1 - 1)u_2, 1 + (a_2 + b_2 - 1)u_1\}. \quad (11)$$

For $S_4$, we have

$$S_4 = S_3 \cup \{(14)\} = S_3 \cup \{1 + (a_{21} + b_{12} - u_{12} - u_{12}^\dagger)\}. \quad (12)$$

In a different approach, [9, p.201-222], a special twisted product was developed, but has not yet been fully explored.

There is an irreducible representation, called the standard representation, which is easily obtained from the permutation representation of $S_n$. We illustrate the general method first for $S_5$ in $G_{2,2}$. The minimal polynomial (10) for $C_4$ is

$$\text{min}(C_4) = (x + 1)(x - 3).$$

The spectral basis [9, p.125], [13, 14], for this minimal polynomial is

$$s_1 = C_4 - 3 = \frac{A_4 - 4}{-4}, \quad \text{and} \quad s_2 = C_4 + 1 = \frac{A_4 - 4}{4},$$

where $s_1$ and $s_2$ are mutually annihilating idempotents with the property that $s_1 + s_2 = 1$.

Using (6), from the matrices $A_4$ and $C_4$, we calculate the corresponding geometric numbers $A_4$ and $C_4$ in $G_{2,2}$, getting

$$A_4 = 1 + a_1 + b_1 + (a_2 + b_2)\left((a_1 - b_1) + 2a_1 \land b_1\right) = A_2 \left(1 + 2(a_2 + b_2)\right)$$

$$= A_2(1 + 2(a_2 + b_2))a_1 \land b_1 = 1 + C_4.$$ 

where $A_2 := (1 + a_1 + b_1) \in G_{1,1} \subset G_{2,2}$.

More generally, for the minimal polynomials (10), the spectral basis of $C_2^n \in G_{n,n}$ is given by

$$s_1 = \frac{C_{2^n} - 2^n + 1}{-2^n} = \frac{A_{2^n} - 2^n}{-2^n}, \quad \text{and} \quad s_2 = \frac{C_{2^n} + 1}{2^n} = \frac{A_{2^n}}{2^n}. \quad (13)$$

We have the useful recursive relation

$$A_{2^n+1} = A_{2^n} \left(1 + 2^n(a_{n+1} + b_{n+1})(a_1 \land b_1) \cdots (a_n \land b_n)\right) = 1 + C_{2^n+1} \in G_{n+1,n+1},$$

where $A_{2^n} \in G_{n,n}$ for $n \geq 1$. For $n = 1$, $A_1 = 1$ and $C_1 = 0$.

Using the spectral basis (13),

$$C_{2^n} = (-1)s_1 + (2^n - 1)s_2.$$
Since the Casimir geometric number $C_{2^n}$ commutes with the permutation representations of $S_{2^n}$, it follows that the inner automorphism of the geometric number

$$g_c = s_1(1 - u_{12}^\dagger) + s_2 u_{12}^\dagger$$

will diagonalize $C_n$, meaning that

$$[g_c^{-1}C_n g_c] = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 2^{n-1} \end{pmatrix}.$$  

If we now apply the inner automorphism of $g_c$ to the permutation representation of $S_{2^n}$ in $G_{n,n}$, we get the standard irreducible representation of $S_{2^n}$ in $G_{n,n}$.

The explicit calculations for the standard irreducible representation of $S_4$ in $G_{2,2}$ are as follows:

$$C_4 = (-1)s_1 + 3s_2$$

for $s_1 = \frac{C_4 - 3}{4}$, and $s_2 = \frac{C_4 + 1}{4}$. Next, we perform surgery, removing the last column of $s_1$, and replacing it with the last column of $s_2$, giving

$$g_c = s_1(1 - u_{12}^\dagger) + s_2 u_{12}^\dagger,$$

which is a matrix whose column vectors are the eigenvectors of $C_4$. Checking,

$$[g_c^{-1}C_4 g_c] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

showing the $g_c$ diagonalizes $C_4$ as expected.

Applied to the permutation representations (12), (13) and (14) of $S_4$ in $G_{2,2}$, given in (11) and (12), $g_c$ block diagonalizes the matrix generators of these permutations, giving the standard irreducible representation of $S_4$,

$$ (12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [1 + (a_1 + b_1 - 1)u_2], $$

$$ (13) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [1 + (a_2 + b_2 - 1)u_1]$$

and

$$ (14) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [1 - (2 + b_1 + b_2)u_{12}]. $$
Note, in this representation the permutation matrices of (12) and (13) are unchanged, but the representation of (14) is now a 3 × 3 sub-block.

From what we have learned, it is easy to write down a standard representation of \( S_5 \), completing the permutation representations for the 2-cycles (12), (13), (14), with

\[
(15) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix} = [1 - (2 + b_1 + b_2 + b_{12})u_{12}].
\]

4 Group characters

In studying a particular group algebra, defined by its permutation representation as a subgroup of the symmetric group, it is very useful to find a matrix which commutes with all the conjugacy classes of the symmetric group, because these elements are one way of finding the respective irreducible representations [11]. Here we study the conjugacy classes of the group algebra of \( S_4 \).

There are five conjugacy classes of \( S_4 \): The group identity 1, the six 2-cycles, the 8 3-cycles, the 6 4-cycles, and 3 double 2-cycles, making up the 24 group elements of \( S_4 \). The general matrix that commutes with the generating 2-cycles (12), (13), and (14), of \( S_4 \) will necessarily commute with all of the elements of \( S_4 \). The matrix \([g_{ij}]\) that commutes with \( S_4 \) is

\[
g_{all} := \begin{pmatrix}
s & t & t & t \\
t & s & t & t \\
t & t & s & t \\
t & t & t & s
\end{pmatrix},
\]

(16)
dependent on the two independent parameters \( s, t \). The characteristic and minimal polynomials of \( g_{all} \) are \((\lambda - (s - t))^3(\lambda - (3t + s))\) and

\[
min(g_{all}) = (\lambda - (s - t))(\lambda - (3t + s)),
\]

respectively.

The matrix \( g_{alt} \) which commutes with the conjugacy class of the alternating group, or even subgroup of \( S_4 \) is

\[
g_{alt} = \begin{pmatrix}
s_1 & t_1 & s_2 & t_2 \\
t_1 & s_1 & t_2 & s_2 \\
s_2 & t_2 & s_1 & t_1 \\
t_2 & s_2 & t_1 & s_1
\end{pmatrix},
\]

(17)
dependent on the four parameters \( s_1, t_1, s_2, t_2 \). The minimal polynomial of \( g_{alt} \) is

\[
min(g_{alt}) = (\lambda - (t_2 - s_2 - t_1 + s_1))(\lambda - (-t_2 + s_2 - t_1 + s_1))
\]
\[ \left( \lambda - (-t_2 - s_2 + t_1 + s_1) \right) \left( \lambda - (t_2 + s_2 + t_1 + s_1) \right). \]

Clearly, the condition (17) of commuting with all even-cycles is less restrictive than the condition (16) of commuting with all of \( S_4 \).

Representation theory is built up on an inner product defined on the traces of matrices which define the characters of a representation. Given an element \( g \in \mathbb{G}_{n,n+1} \), together with its formally complex matrix \( [g] \),

\[
\text{tr}[g] := 2^n \langle g \rangle_{0+(2n+1)}, \quad \text{and} \quad \text{tr}[g] := 2^n \langle g^- \rangle_{0+(2n+1)},
\]

where \( i := e_{1\ldots n} f_{1\ldots n}^{\dagger} \). A representation of a group \( G \) in the geometric algebra \( \mathbb{G}_{n,n+1} \) is a mapping \( f : G \rightarrow \mathbb{G}_{n,n+1} \) of the group \( G \) into the geometric algebra \( \mathbb{G}_{n,n+1} \) which preserves the group operation, and for which \( f_e = 1 \), where \( e \in G \) is the group identity. It follows that \( f_{gh} = f_g f_h \) for all \( g, h \in G \). Let \( f \) be a representation of the group \( G \) in \( \mathbb{G}_{n,n+1} \). The group character \( \psi_f : G \rightarrow \mathbb{C} \)

is defined by \( \psi_f(g) := \text{tr}(f_g) \) for each \( g \in G \). A comprehensive introduction to representation theory is found in [6].

5 New tools

Consider an arbitrary geometric number \( g \in \mathbb{G}_{2,2} \), with the matrix \([g_{ij}]\) in the basis (6). We can perform “surgery” on this matrix. For example,

\[
[g - gu_{2}^{\dagger} - u_{2}^{\dagger}g] = \begin{pmatrix}
    g_{11} & g_{12} & 0 & 0 \\
    g_{21} & g_{22} & 0 & 0 \\
    0 & 0 & -g_{33} & 0 \\
    0 & 0 & 0 & g_{44}
\end{pmatrix}
\]

and

\[
[g - gu_{12}^{\dagger} - u_{12}^{\dagger}g] = \begin{pmatrix}
    g_{11} & g_{12} & g_{13} & 0 \\
    g_{21} & g_{22} & g_{23} & 0 \\
    g_{31} & g_{32} & g_{33} & 0 \\
    0 & 0 & 0 & -g_{44}
\end{pmatrix}
\]

The same such surgery was performed on \( s_1 \) and \( s_2 \) in (14) and (15) to obtain the respective automorphisms defined by \( g_c \). We can also directly extract rows or columns of the matrix \([g]\). For example,

\[
[gb_{1}u_{2}] = \begin{pmatrix}
    g_{12} & 0 & 0 & 0 \\
    g_{22} & 0 & 0 & 0 \\
    g_{32} & 0 & 0 & 0 \\
    g_{42} & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad [gb_{12}] = \begin{pmatrix}
    g_{14} & 0 & 0 & 0 \\
    g_{24} & 0 & 0 & 0 \\
    g_{34} & 0 & 0 & 0 \\
    g_{44} & 0 & 0 & 0
\end{pmatrix}
\]

Direct representations of matrix algebras by Clifford algebras \( \mathbb{G}_{n,n} \) of neutral signature restricts the classes of matrices considered to sizes of \( 2^n \times 2^n \). However,
the operation of surgery ameliorates this restriction by allowing us to extract matrices of smaller sizes. Because of their geometric significance, matrices over the complex numbers of size $2 \times 2$, the Pauli Matrices, and of size $2^2 \times 2^2$, the Dirac Matrices, have proven themselves to be of immense importance in physics. It is this extra geometric structure that geometric algebra makes explicit in its many different guises [12].

The standard spectral basis of the geometric algebras $G_{n,n}$ and $G_{n,n+1} := G_{n,n}(\mathbb{C})$ are easily constructed, starting with the primitive idempotent $u_1 \cdots u_n := u_1 \cdots u_n$ where each $u_i = a_i b_i$. The first column is then written down, followed by successive columns defined by the Kronecker products, with the dual blocks in reverse order,

$$G_{n,n} := B_1 \otimes \cdots \otimes B_n u_1 \cdots n A_n^T \otimes \cdots \otimes A_1^T.$$  

We have already demonstrated the method for $n = 2$ in (6); we now give the standard spectral basis for $G_{3,3}$.

The standard spectral basis defining $G_{3,3}$ is

$$G_{3,3} := B_1 \otimes B_2 \otimes B_3 u_{123} A_3 \otimes A_2 \otimes A_1.$$  

In expanded form, $G_{3,3} =$

$$\begin{pmatrix}
  u_{123} & a_1 u_{23} & a_2 u_{13} & a_3 u_{12} & a_3 u_{12} & a_3 u_{12} & a_3 u_{12} \\
  b_1 u_{23} & b_2 u_{13} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} \\
  b_2 u_{13} & b_2 u_{13} & -b_2 u_{13} & -b_2 u_{13} & -b_2 u_{13} & -b_2 u_{13} & -b_2 u_{13} \\
  b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} & b_3 u_{12} \\
  b_1 u_{12} & b_1 u_{12} & b_1 u_{12} & b_1 u_{12} & b_1 u_{12} & b_1 u_{12} & b_1 u_{12} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
  b_{123} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} & b_{231} \\
\end{pmatrix}$$

We can now easily construct the geometric permutation representation of $S_9$ in $G_{3,3}$. Several examples of the geometric permutations are

$$(12) = 1 + (a_1 + b_1 - 1) u_{23}, \quad (13) = 1 + (a_2 + b_2 - 1) u_{13},$$

$$(16) = 1 + (a_3 + b_3 - u_{13} - u_{13}^+) u_2$$

and

$$(19) = 1 - u_{123} - (1 + b_1)(1 + b_2)(1 + b_3) u_{123}.$$  

It is interesting to write down the matrix for the 9-cycle (123456789). We have

$$(123456789) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

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\[ b_1 + b_2 a_1 + b_3 a_21 - (a_{321} + a_{32} + a_3 + a_{21} - a_2 + a_1 + 1)u_{123}, \]

which is a 9th root of unity.

Let us use our new tools to decompose the regular representation of \( S_3 \) in \( G_{3,3} \) into the sum of its irreducible parts \([5, p.127]\). We begin by writing

\[ X = x_0 + x_1(18) + x_2(19) + x_3(89) + x_4(189) + x_5(198), \]

represented by an 8 \times 8 matrix \([X]\). We have used the generating 2-cycles (18) and (19) to take advantage of the irreducible standard representation of \( S_3 \) in \( \text{Mat}_2(\mathbb{R}) \). Letting \( x_{0:5} = x_0 + \cdots + x_5 \), the matrix \([X]\) of \( X \) is

\[
\begin{pmatrix}
    x_0 - x_2 + x_3 - x_5 & 0 & 0 & 0 & 0 & 0 & x_1 - x_3 - x_4 + x_5 \\
    -x_2 - x_5 & x_{0:5} & 0 & 0 & 0 & 0 & 0 \\
    -x_2 - x_5 & 0 & x_{0:5} & 0 & 0 & 0 & 0 \\
    -x_2 - x_5 & 0 & 0 & x_{0:5} & 0 & 0 & 0 \\
    -x_2 - x_5 & 0 & 0 & 0 & x_{0:5} & 0 & 0 \\
    -x_2 - x_5 & 0 & 0 & 0 & 0 & x_{0:5} & 0 \\
    x_1 - x_2 + x_4 - x_5 & 0 & 0 & 0 & 0 & 0 & x_0 + x_2 - x_3 - x_4 \\
\end{pmatrix}
\]

Except for the strange arrangement of 2 \times 2 block, the block diagonal matrix \([X]\) has desired form. The matrix \([X]\) can be fully diagonalized, however this would not give the desired irreducible decomposition of \([X]\). Instead, noting that \( x_{0:5} \) is an eigenvalue of the remaining 2-block, we block diagonalize \( X \) with the eigenvectors of the matrix \([X]_{x_0 \rightarrow 1, x_1 \rightarrow 2, \ldots, x_5 \rightarrow 6} \), getting the decomposition of \([X]\) into the sum of its irreducible representations

\[
\begin{pmatrix}
    x_{0:5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & x_{0:5} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x_{0:5} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & x_{0:5} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & x_{0:5} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & x_{0:5} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & x_0 + x_1 - x_3 - x_4 & x_2 - x_3 - x_4 + x_5 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 + x_1 + x_2 + x_4 - x_5 \\
\end{pmatrix}
\]

Since we are representing the group algebra of \( S_6 \) in \( G_{3,3} \), we can reduce the size of our representation by acting on the 6-dimensional column vector

\[ x = (0 \ 0 \ x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5)^T, \]

which essentially eliminates the first two rows and columns of the matrix.

**Appendix: Lower dimensional geometric algebras**

Recall that the standard basis of the geometric algebras \( G_{n,n} \) and \( G_{n,n+1} \) are defined by

\[ G_{n,n} := \mathbb{R}(e_1, \ldots, e_n, f_1, \ldots, f_n) \equiv \text{Mat}_2^n(\mathbb{R}) \]
and
\[ G_{n,n+1} := \mathbb{C}(e_1, \ldots, e_n, f_1, \ldots, f_n) \cong \text{Mat}_{2^n}(\mathbb{C}), \]
where \( e_i := a_i + b_i \) and \( f_i := a_i - b_i \) for \( i = 1, \ldots, n \). The geometric algebra \( G_{n,n+1} \) can be obtained as a real geometric algebra,
\[ G_{n,n+1} := \mathbb{R}(e_1, \ldots, e_n, f_1, \ldots, f_n, f_{n+1}), \quad (19) \]
where \( f_{n+1} := e_1 \cdots e_n f_1 \cdots n i \).
If \( p + q \leq 2n \), the \( 2^n \) dimensional geometric algebras \( G_{p,q} \) can be considered as the real geometric algebras,
\[ G_{p,q} := \mathbb{R}(e_1, \ldots, e_p, f_1, \ldots, f_q), \]
by noting that \((ie_i)^2 = -1\) and \((if_i)^2 = 1\). If \( p > n \) or \( q > n \), we simply change the required number of basis vectors \( f_i \) or \( e_i \), to \( e_i \)'s or \( f_i \)'s, respectively, by multiplying the \( f_i \) or \( e_i \) by \( i \). Additional \( 2^{2n+1} \)-dimensional real geometric algebras can be obtained from (19), by replacing an even number of the generating vectors by \( i \) times that generator. For example,
\[ G_{n-2,n+3} := \mathbb{R}(e_1, \ldots, e_{n-2}, ie_1, ie_2, f_1, \ldots, f_{n+1}) \cong \text{Mat}_2(2^n). \]

Below is a list of lower dimensional \( 2^{2n+1} \) geometric algebras that can be represented in terms of the matrix algebras \( \text{Mat}_2(\mathbb{C}) \), \( \text{Mat}_{22}(\mathbb{C}) \), and \( \text{Mat}_{23}(\mathbb{C}) \), or vice-versa.
\[ G_{1,2} := \mathbb{C}(e_1, f_1) \cong \text{Mat}_2(\mathbb{C}), \]
\[ G_{1,2} = \mathbb{R}(e_1, f_1, f_2), \quad G_{3,0} = \mathbb{R}(e_1, if_1, if_2), \]
\[ G_{2,3} := \mathbb{C}(e_1, e_2, f_1, f_2) \cong \text{Mat}_4(\mathbb{C}) \]
\[ G_{2,3} = \mathbb{R}(e_1, e_2, f_1, f_2, f_3), \quad G_{4,1} = \mathbb{R}(e_1, e_2, if_1, if_2, f_3), \]
\[ G_{0,5} = \mathbb{R}(ie_1, ie_2, f_1, f_2, f_3). \]
\[ G_{3,4} := \mathbb{C}(e_1, e_2, e_3, f_1, f_2, f_3, f_4) \cong \text{Mat}_8(\mathbb{C}) \]
\[ G_{3,4} = \mathbb{R}(e_1, e_2, e_3, f_1, f_2, f_3, f_4), \quad G_{5,2} = \mathbb{R}(e_1, e_2, e_3, if_1, if_2, f_3, f_4), \]
\[ G_{7,0} = \mathbb{R}(e_1, e_2, e_3, if_1, if_2, if_3, f_4), \quad G_{1,6} = \mathbb{R}(e_1, ie_2, ie_3, f_1, f_2, f_3, f_4). \]
A complete classification of geometric algebras is given in the paper, "Geometrization of the Real Number System" [12].
References


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