Notes on Plücker’s Relations in Geometric Algebra

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Abstract

Grassmannians are of fundamental importance in projective geometry, algebraic geometry, and representation theory. A vast literature has grown up utilizing using many different languages of higher mathematics, such as multilinear and tensor algebra, matroid theory, and Lie groups and Lie algebras. Here we explore the basic idea of the Plücker relations in Clifford’s geometric algebra. We discover that the Plücker relations can be fully characterized in terms of the geometric product, without the need for a confusing hodgepodge of many different formalisms and mathematical traditions found in the literature.

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0 Introduction

The standard treatments of the Plücker relations rely heavily on the concept of a vector space, its dual vector space, and generalizations. More elementary treatments express the basic ideas in terms of familiar matrices and determinants. Here we exploit a powerful formalism that was invented 140 years ago by William Kingdon Clifford. Clifford himself dubbed this formalism, a generalization of the works of H. Grassmann and W. Hamilton, “geometric algebra”, but it is often referred to today as “Clifford algebra”.

It is well known that Clifford’s geometric algebras are algebraically isomorphic to matrix algebras over the real or complex numbers. However, what is lacking is the recognition that whereas geometric algebras give a comprehensive geometrical interpretation to matrices, they also provide a way of unifying vectors and their duals into a rich algebraic structure that is missed in the standard
treatment of vector spaces and their dual spaces as separate objects. Our study
of the famous Plücker relations utilizes the powerful higher dimensional vector
analysis of geometric algebra. When the Plücker relations are satisfied, an ex-
licit factorization of an \( r \)-vector into the outer product of \( r \) vectors is found.
Also, a new characterization of the Plücker relations is given.

Let \( a \) and \( b \) be two non-zero null vector generators of a real or complex
associative algebra, \( \mathbb{R}(a, b) \) or \( \mathbb{C}(a, b) \), respectively, with the identity \( 1 \), and
satisfying the rules

\[
1) \ a^2 = 0 = b^2, \quad \text{and} \quad 2) \ ab + ba = 1. \quad (1)
\]

From these rules, a multiplication table for these elements is easily constructed,
given in Table 1.

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The null vectors \( a \) and \( b \) are dual to one another in the sense that the
symmetric inner product of \( a \) and \( b \), defined by

\[
a \cdot b := \frac{1}{2} (ab + ba) = \frac{1}{2}.
\]

In addition, the outer product of \( a \) and \( b \), gives the bivector

\[
a \wedge b := \frac{1}{2} (ab - ba),
\]

with the property

\[
(a \wedge b)^2 = \frac{1}{4} (ab - ba)^2 = \frac{1}{4} (ab + ba) = \frac{1}{4} \quad \text{so} \quad |a \wedge b| := \frac{1}{2}.
\]

Taking the sum of the above inner and outer products shows that the geometric (Clifford) product of the reciprocal null vectors \( a \) and \( b \) is the primitive
idempotent

\[
ab = a \cdot b + a \wedge b,
\]

already identified in Table 1. More details for the interested reader of the
construction of basic geometric algebras, obtained from \( \mathbb{R}(a, b) \) and \( \mathbb{C}(a, b) \), and
by taking the Kronecker products of such algebras, is deferred to the Appendix.

Geometric algebras are considered here to be the natural geometrization of
the real and complex number systems; an overview is provided in [9]. We use
exclusively the language of geometric algebra, developed in the books [6, 12].
Plücker’s coordinates are briefly touched upon on page 30 of [6], but Plücker’s relations are not discussed. These relations characterize when an $r$-vector $B$ in the geometric algebra $\mathbb{G}_n := \mathbb{G}(\mathbb{R}^n)$, or $\mathbb{G}_n(\mathbb{C}) := \mathbb{G}(\mathbb{C}^n)$, of an $n$-dimensional (real or complex) Euclidean vector space $\mathbb{R}^n$, or $\mathbb{C}^n$, respectively, can be expressed as an outer product of $r$-vectors. Several references to the relevant literature, and its more advanced applications, are [1, 2], [3, pp.227-231], [5], [7, pp.144-148], and [13].

1 Plücker’s relations in geometric algebra

A general geometric number $g \in \mathbb{G}_n$ is the sum of its $k$-vector parts,

$$g = \sum_{k=0}^{n} \langle g \rangle_k,$$

where the $k$-vector $\langle g \rangle_k \in \mathbb{G}_n^k$ is in the linear subspace $\mathbb{G}_n^k$ of $k$-vectors in $\mathbb{G}_n$, and the 0-vector part $\langle g \rangle_0 \in \mathbb{G}_n^0 \equiv \mathbb{R}$. A nonzero $r$-vector $B \in \mathbb{G}_n^r$ is said to be an $r$-blade if $B = v_1 \wedge \cdots \wedge v_r$ for vectors $v_1, \ldots, v_r \in \mathbb{G}_n^1$.

Let $B = \sum J_m B_{m, e_J m}$ be an $r$-vector expanded in the standard orthonormal basis of $r$-vector blades

$$e_J m := e_{j_1} e_{j_2} \cdots e_{j_r} \in \mathbb{G}_n^r,$$

ordered lexicographically, with $J_m = \{j_1, \ldots, j_r\}$ for $1 \leq j_1 < \cdots < j_r \leq n$, and let $(J_p \cap J_m)$ be the number of integers that the sequences $J_p$ and $J_m$ have in common. For example, if $J_p = \{12345\}$ and $J_m = \{45678\}$, then $(J_p \cap J_m) = 2$.

The reverse $g^\dagger$ of a geometric number $g \in \mathbb{G}_n$ is obtained by reversing the order of all of the geometric products of the vectors which define $g$.

**Lemma 1** In the standard orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{G}_n$, an $r$-vector $B \in \mathbb{G}_n^r$, expanded in this basis, is given by

$$B = \sum_J B_J m e_J m,$$

(2)

where $B_J k = B \cdot e_J k$.

**Proof:** In the orthonormal $r$-vector basis $\{e_J m\}$, the $r$-vector $B \in \mathbb{G}_n^r$ is given by

$$B = \sum_{k=1}^{n \choose r} B_J k e_J k$$

Dotting both sides of this equations with $e_J m^\dagger$, the reverse of $e_J m$, shows that $B_J m = B \cdot e_J m^\dagger$, since $e_J k \cdot e_J m^\dagger = \delta_{k,m}$.

\[\square\]
**Definition 1** A non-zero \( r \)-vector \( B \in \mathbb{G}_n^r \) is said to be divisible by a non-zero \( k \)-blade \( K \in \mathbb{G}_n^k \), where \( k \leq r \), if \( KB = \langle KB \rangle_{r-k} \). If \( k = r \), \( B \) is said to be totally decomposable by \( K \).

If \( B \) is divisible by the \( k \)-blade \( L \), then
\[
B = \frac{L(LB)}{L^2} = \frac{L \cdot B}{L^2}. \tag{3}
\]

If \( B \) is totally decomposable by \( L \), then
\[
B = \beta L \tag{4}
\]
for all \((r-1)\)-vectors \( A \in \mathbb{G}_n^{r-1} \). In particular, for all coordinate \((r-1)\)-vectors \( e_{K_p} \in \mathbb{G}_n^{r-1} \),
\[
(e_{K_p} \cdot B) \wedge B = 0.
\]

Since an \( r \)-blade trivially satisfies the Plücker relations, it follows that if there is a coordinate \((r-1)\)-blade \( e_K \), such that \((e_K \cdot B) \wedge B \neq 0\), then \( B \) is not an \( r \)-blade.

For every coordinate \((r-1)\)-blade \( e_{K_p} \), there is a coordinate \( r \)-blade \( e_{J_p} \), generally not unique, and a coordinate vector \( e_{j_p} \), satisfying
\[
e_{J_p} = \pm e_{j_p} e_{K_p} \iff e_{j_p} e_{J_p} = \pm e_{K_p}. \tag{5}\]

Indeed, when \( e_{K_p} \cdot B \neq 0 \), we always pick the coordinate vector \( e_{j_p} \), and rearrange the order of \( K_p = \pm K_{j_p} \), in such a way that \( B_{J_p} = (e_{j_p} \wedge e_{K_p})^\dagger \cdot B \neq 0 \). It follows that the Plücker relations (4) are equivalent to
\[
e_{j_p} \cdot \left[(e_{K_p} \cdot B) \wedge B\right] = B_{J_p} B - \left[(e_{K_p} \cdot B) \wedge (e_{j_p} \cdot B)\right] = 0,
\]
or equivalently,
\[
B = B_{J_p}^{-1} \left[(e_{K_p} \cdot B) \wedge (e_{j_p} \cdot B)\right]. \tag{6}
\]

For a given non-zero \( r \)-vector \( B \neq 0 \), and the set \( S_B \) defined in (7). Then \( r \leq \#_{\text{max}} S \leq n \), and \( \#_{\text{max}} = r \) only if \( B \) is an \( r \)-blade.

**Lemma 2** Given an \( r \)-vector \( B \neq 0 \), and the set \( S_B \) defined in (7). Then \( r \leq \#_{\text{max}} S \leq n \), and \( \#_{\text{max}} = r \) only if \( B \) is an \( r \)-blade.
Proof: Since $B \neq 0$, it follows that $B_{J_m} \neq 0$ for some

$$1 \leq m \leq \binom{n}{r}.$$ 

For each such $e_{J_m}$, there are $r$ possible choices for $e_{K_j}$ obtained by picking $K_j \subset J_m$, and the subset of $S_B$ generated by them are linearly independent. It follows that $r \leq \#_{\text{max}} S_B \leq n$.

Suppose now that $r < \#_{\text{max}} S_B$. To complete the proof of the Lemma, we show that there is a Plücker relation that is not satisfied for $B$, so it cannot be an $r$-blade. Let 

$$L = (e_{K_1} \cdot B) \land \cdots \land (e_{K_r} \cdot B) \neq 0,$$

be the outer product of the maximal number of linearly independent vectors from $S_B$. Then $k > r$ and $LB = L \cdot B \neq 0$.

Letting 

$$w = [(e_{K_1} \cdot B) \land \cdots \land (e_{K_{r+1}} \cdot B)] \cdot B \in \mathbb{G}_n^1,$$

it follows that $w \neq 0$, and

$$w^2 = [(e_{K_1} \cdot B) \land \cdots \land (e_{K_{r+1}} \cdot B)] \cdot (B \land w) \neq 0.$$ 

Since $w^2 \neq 0$, it follows that $w \land B \neq 0$. Using the identity

$$w = [(e_{K_1} \cdot B) \land \cdots \land (e_{K_{r+1}} \cdot B)] \cdot B$$

$$= (e_{K_1} \cdot B) [(e_{K_2} \cdot B) \land \cdots \land (e_{K_{r+1}} \cdot B)] \cdot B$$

$$- (e_{K_2} \cdot B) [(e_{K_1} \cdot B) \land \cdots \land (e_{K_{r+1}} \cdot B)] \cdot B$$

$$+ \cdots + (-1)^r (e_{K_{r+1}} \cdot B) [(e_{K_1} \cdot B) \land \cdots \land (e_{K_{r}} \cdot B)] \cdot B,$$

where $\lor^i$ means we are omitting the term $e_{K_i} \cdot B$ in the exterior product. It follows that

$$w \land B = \sum_{i=1}^{r} (-1)^{i+1} \alpha_i (e_{K_i} \cdot B) \land B \neq 0,$$

where $\alpha_i := [(e_{K_1} \cdot B) \land \cdots \lor^i \cdots \land (e_{K_{r+1}} \cdot B)] \cdot B \in \mathbb{R}$. But this implies that $\alpha_j (e_{K_j} \cdot B) \land B \neq 0$ for some $j$, so $B$ is not an $r$-blade.

On the other hand, if $\#_{\text{max}} S = r$, then for $L = (e_{K_1} \cdot B) \land \cdots \land (e_{K_r} \cdot B)$, $LB = L \cdot B \in \mathbb{R}$, and using (3),

$$B = \frac{L \cdot B}{L^2} L$$

is an $r$-blade. \hfill \Box

As a direct consequence of Lemma 2, we have

**Theorem 1** For $2 \leq r \leq n$, a non-zero $r$-vector $B \in \mathbb{G}_n^r$ is an $r$-blade iff the Plücker relations (4) are satisfied. When the Plücker relations are satisfied, then $B = \frac{L \cdot B}{L^2} L$ for the $r$-blade $L$ defined in the last paragraph of the proof of Lemma 2.
2 Examples

Everybodies’ favorite example is when \( r = 2 \) and \( n = 4 \). A general non-zero 2-vector \( B \in \mathbb{G}_4^2 \) has the form

\[
B = \sum_{m=1}^{6} B_{J_m} e_{J_m} \in \mathbb{G}_4^2.
\]

Since \( B \neq 0 \), at least one component \( B_{J_p} e_{ij} \neq 0 \). Without loss of generality, we can assume \( e_{ij} = e_{12} \). It follows that the Plücker relation (6) for this component is

\[
B = B_{12}^{-1} \left[ (e_{1} \cdot B) \wedge (e_{2} \cdot B) \right] = B_{12}^{-1} \left[ (B_{12} e_2 + B_{13} e_3 + B_{14} e_4) \wedge (-B_{12} e_1 + B_{23} e_3 + B_{24} e_4) \right].
\]

Since

\[
2(e_{i} \cdot B) \wedge B = e_{i} \cdot (B \wedge B) = (B_{12} B_{34} - B_{13} B_{24} + B_{14} B_{23}) e_{i} e_{1234},
\]

it follows that \( B \) is a 2-blade iff \( B \wedge B = 0 \), or equivalently,

\[
B_{12} B_{34} - B_{13} B_{24} + B_{14} B_{23} = 0.
\]

This occurs when \( B \) is totally decomposable, given in (8). More generally, a similar argument holds true for \( r = 2 \) and any \( n > 4 \), but with more Plücker relations (9) to be satisfied.

To get more insight into what is going on, let \( B = \sum_{m} B_{J_m} e_{J_m} \in \mathbb{G}_n^r \) be a general \( r \)-vector. Following [8], we calculate

\[
B^2 = \sum_{m,p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} = \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m \neq p} B_{J_m} B_{J_p} e_{J_m} e_{J_p}
\]

\[
= \sum_{m=p} B_{J_m} B_{J_p} e_{J_m} e_{J_p} + \sum_{m < p} B_{J_m} B_{J_p} \left( 1 + (-1)^{r-k} \right) e_{J_m} e_{J_p},
\]

where \( k = \#(J_m \cap J_p) \), showing that \( B^2 = \langle B^2 \rangle_0 \) if \( r \) and \( k \) have different parity for all values of \( m < p \).

Consider \( B = e_{123} + e_{456} \in \mathbb{G}_6^3 \). It is easy to show, using (4), that \( B \) is not a 3-blade, since

\[
(e_{12} \cdot B) \wedge B = e_3 \wedge B = e_{3456} \neq 0.
\]

Note, however, that \( B^2 = -2 \), as also follows from (10) since \( k \) and \( r \) have opposite parities. Clearly, the condition \( B^2 = \langle B^2 \rangle_0 = 0 \) is not sufficient to guarantee that \( B \) is an \( r \)-blade, but \( B^2 \neq \langle B^2 \rangle_0 \) guarantees that \( B \) is not an \( r \)-blade.

In [8], Nguyen expresses the Plücker Relations differently, and gives a different proof. The following Theorem shows that Nguyen’s Plücker Relations are equivalent to definition (4). Let \( e_K \) denote an arbitrary coordinate \( (r-1) \)-vector in \( \mathbb{G}_n^{r-1} \).
Theorem 2 A non-zero $r$-vector $B \in \mathbb{G}_n^r$ is an $r$-blade iff both the conditions

$$B^2 = B \cdot B, \quad \text{and} \quad BvB \in \mathbb{G}_n^1 \quad \text{for all} \quad v = e_K \cdot B \in \mathbb{G}_n^1,$$

are satisfied.

Proof: It is easy to show that if $B \in \mathbb{G}_n^r$ is an $r$-blade, then both of the conditions in the Theorem are true. To complete the proof we must only show that if both conditions are not satisfied, then $B$ is not an $r$-blade. If $B^2 \neq B \cdot B$, then $B$ is not an $r$-blade, so it is only left to show that if $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^1$, then $B$ is not an $r$-blade.

The following identity is needed:

$$BvB = B(v \cdot B + v \wedge B) = B(v \cdot B) + B(v \wedge B) + \langle B(vB) \rangle_{3,5,\ldots} \quad \text{(12)}$$

Suppose now that $B^2 = B \cdot B$ and $BvB \notin \mathbb{G}_n^1$ for some $v = e_K \cdot B$. The identity (12) implies that $\langle B(vB) \rangle_{3,5,\ldots} \neq 0$. We also know that

$$2B \wedge v = Bv + (-1)^r vB \iff Bv = 2B \wedge v - (-1)^r vB,$$

so that

$$\langle B(vB) \rangle_{3,5,\ldots} = \langle (2B \wedge v - (-1)^r vB)B \rangle_{3,5,\ldots} = 2\langle (B \wedge v)B \rangle_{3,5,\ldots} \neq 0,$$

since $B^2 = B \cdot B$. But this implies that $B \wedge (e_K \cdot B) \neq 0$, so by (4), $B$ is not an $r$-blade. \qed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Shows $S_m$ and $S_p$ in the case for odd $r = 9$, with an odd overlap $k = 5$. In this case we choose $e_K = e_{j_2\cdots j_9}$, the $j_2, \cdots j_9$ are the last 8 digits of $S_m$. A similar construction applies when $r$ and $k$ are even.}
\end{figure}

Note that (4) is fully equivalent to the conditions in the Theorem 2. If $B^2 \neq B \cdot B$ then there is at least one $e_K \in \mathbb{G}_n^{r-1}$ such that $(e_K \cdot B) \wedge B \neq 0$. This follows from the coordinate expansion of $B$ given in (10), because the parity condition allows us to find at least one non-zero $B_{J_m}B_{J_p}e_{J_m}e_{J_p}$, with overlap $k = \#(S_{J_m} \cap S_{J_p})$, such that $(e_K \cdot B) \wedge B \neq 0$. We simply pick $e_K$ in such a way that $K \subset S_m$, but $\#(K \cap J_p) \leq r - 2$, see Figure 1. In the example (11), $B$ is not a 3-blade because even though $B^2 = B \cdot B$, for

$$e_{12} \cdot B = -e_3, \quad Be_3B = 2e_{12456} \in \mathbb{G}_6^5.$$
It is also interesting to note that just as \( B^2 = B \cdot B \) whenever \( B \) is an \( r \)-vector satisfying the Plücker relations (4), \( BvB = (-1)^{r+1}(B \cdot B)v \) for all \( v = e_K \cdot B \) whenever \( B \) is an \( r \)-vector satisfying the Plücker Relations (4). This follows from the identity

\[
BvB = B[2v \wedge B + (-1)^{r+1}Bv] = (-1)^{r+1}B^2v \iff v \wedge B = 0
\]

for all \( v = e_K \cdot B \), when \( B \) satisfies the Plücker relations (4).

Let us consider now some other examples. For the 3-vector \( B \in \mathbb{G}^3_5 \),

\[
B := -2e_{134} + e_{145} - 2e_{234} + e_{245} + e_{315} + e_{325} + e_{345},
\]

it is easily verified that the Plücker relations

\[
(e_{41} \cdot B) \wedge B = (2e_3 + e_5) \wedge B = 0, \quad e_{54} \cdot B = (e_1 + e_2 + e_3) \wedge B = 0,
\]

\[
e_{13} \cdot B = (2e_4 + e_5) \wedge B = 0,
\]

and that the vectors \((e_{41} \cdot B) = 2e_3 + e_5, (e_{54} \cdot B) = (e_1 + e_2 + e_3), e_{13} \cdot B = 2e_4 + e_5\) are linearly independent. Applying Theorem 1, with

\[
L = (e_1 + e_2 + e_3) \wedge (2e_3 + e_5) \wedge (2e_4 + e_5)
\]

\[
B = \frac{L \cdot B}{L^2} L = -\frac{1}{2} L.
\]

For

\[
B = e_{123} + e_{124} + e_{125} + e_{126} + e_{345} + e_{346} + e_{356} + e_{456} \in \mathbb{G}^3_6,
\]

we find that

\[
(e_{21} \cdot B) \wedge B = (B_{123}B_{456} - B_{124}B_{356} + B_{125}B_{346} - B_{126}B_{345})e_{3456}
\]

\[
+ (B_{123}B_{145} - B_{124}B_{135} + B_{125}B_{134})e_{1345} + \cdots
\]

\[
+ (B_{124}B_{156} - B_{125}B_{146} + B_{126}B_{145})e_{1456} = 0,
\]

but

\[
(e_{13} \cdot B) \wedge B = e_2 \wedge B = e_{2456} + e_{2356} + e_{2346} + e_{2345} \neq 0.
\]

Since \((e_{21} \cdot B) \wedge B = 0\), it is divisible by \(e_{12} \cdot B\). Using (6), we find that

\[
B = (e_{21} \cdot B) \wedge (e_{3} \cdot B) = (e_3 + e_4 + e_5 + e_6) \wedge (e_{12} + e_{56} + e_{46} + e_{45}).
\]

But since it does not satisfy the second condition, it is not totally decomposable.

As a final example, consider \( B = (e_1 + e_2 + e_3 + e_4) \wedge (e_{23} + e_{34} + e_{46}) \in \mathbb{G}^3_6\). There are no non-zero vectors of the form \( e_{ij} \cdot B \) such that

\[
(e_{ij} \cdot B) \wedge B = 0.
\]

However, the 2-blade \( e_{36} - e_{46} \), satisfying

\[
(e_{36} - e_{46}) \cdot B = e_1 + e_2 + e_3 + e_4
\]

does divide \( B \), since \((S \cdot B) \wedge B = 0\).
Appendix: Geometric matrices

Let \([g] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\) be a real matrix in \(\text{Mat}_2(\mathbb{R})\). Since \(\mathbb{R}(a,b)\) is an associative algebra, the module \(\text{Mat}_2(\mathbb{R}(a,b))\) is well-defined. We use the matrix [g] to define the unique element \(g \in \mathbb{R}(a,b)\), specified by

\[
g := (ba \quad a) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} ba \\ b \end{pmatrix} = g_{11}ba + g_{12}b + g_{21}a + g_{22}ab.
\]

We say that [g] is the matrix of \(g \in \mathbb{R}(a,b)\) with respect to the Witt basis of null vectors, \(\{a, b\}\), over the real numbers \(\mathbb{R}\). It is easy to show that this establishes an algebraic isomorphism between \(\mathbb{R}(a,b)\) and \(\text{Mat}_2(\mathbb{R})\), [10].

The standard orthonormal basis \(\{e_1, f_1\}\) of the geometric algebra \(G_{1,1} := \mathbb{R}(e_1, f_1) \equiv \mathbb{R}(a,b)\)

is

\[
e_1 := a + b, \quad \text{and} \quad f_1 := a - b,
\]

and satisfies the rules

1) \(e_1^2 = 1 = -f_1^2\), and 2) \(e_1f_1 = -f_1e_1\). \quad (13)

The geometric algebra \(G_{1,1} := \mathbb{R}(e_1, f_1)\), defined by the rules (13), is nothing more than a change of basis, from the Witt basis of null vectors (1) to standard orthonormal basis (13). As a linear space over \(\mathbb{R}\),

\[
G_{1,1} = \text{span}_\mathbb{R}\{1, e_1, f_1, e_1f_1\}.
\]

In terms of the null basis, the bivector \(e_1f_1 = (a+b)(a-b) = ba - ab = 2b \wedge a\).

Letting \(u_1 = ba = \frac{1}{2}(1 + e_1f_1)\), the matrices of the standard basis vectors, are

\[
e_1 = (ba \quad a) [e_1] \begin{pmatrix} ba \\ a \end{pmatrix} = (1 \quad e_1) u_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix},
\]

and

\[
f_1 = (ba \quad a) [e_1] \begin{pmatrix} ba \\ a \end{pmatrix} = (1 \quad e_1) u_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix}.
\]

The matrix of the bivector \(e_1f_1\), is then easily found to be

\[
[e_1f_1] = [e_1][f_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

By complexifying the basis vector \(f_1 \in G_{1,1}\), we obtain the geometric algebra \(G_3\), a geometric interpretation of the famous Pauli matrices, and a geometric interpretation of the imaginary \(i = \sqrt{-1}\). We define

\[
G_3 := \mathbb{R}(e_1, e_2, e_3),
\]

\[
9
\]
where $e_2 := i f_1$ and $e_3 := e_1 f_1$. It then follows that $e_{123} = e_1 i f_1 e_1 f_1 = i$, giving imaginary unit $i$ the geometric interpretation of the unit trivector $e_{123} \in \mathbb{G}_3^2$. Note also the unit vector $e_3$, along the $z$-axis of $\mathbb{R}^3$ is identified with the bivector $e_1 f_1 \in \mathbb{G}_1^2$. The Pauli matrices $[e_k]$, satisfying

$$e_k = \begin{pmatrix} ba \\ a_k \end{pmatrix} \begin{pmatrix} ba \\ b \end{pmatrix},$$

are given by

$$[e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [e_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Geometric matrices of real or complex higher dimensional geometric algebras $\mathbb{G}_{n,n} := \mathbb{G}(\mathbb{R}^{n,n})$, or $\mathbb{G}_{n,n}(\mathbb{C}) := \mathbb{G}(\mathbb{C}^n)$, are obtained by taking directed Kronecker products of blocks of pairs of null vectors $\{a_k, b_k\}$, each block satisfying the rules (1), in addition to being anti-commutative. That is

$$a_i a_j = -a_j a_i, \quad b_i b_j = -b_j b_i, \quad a_i b_j = -b_j a_i,$$

for all $1 \leq i, j \leq n$ and $i \neq j$. For example, the geometric matrix $[g] \in \text{Mat}_4(\mathbb{R})$ defines the geometric number $g \in \mathbb{G}_{2,2}$, given by $g :=$

$$(1 \quad a_1) \boxtimes (1 \quad a_2) u_{12}[g] \begin{pmatrix} 1 \\ b_2 \end{pmatrix} \boxtimes \begin{pmatrix} 1 \\ b_1 \end{pmatrix} = (1 \quad a_1 \quad a_2 \quad a_{12}) u_{12}[g] \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_{21} \end{pmatrix},$$

where $u_{12} = u_1 u_2 = b_1 a_1 b_2 a_2$. Details of the general construction can be found in [11].

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