Nested Coordinate Systems in Geometric Algebra

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Abstract

A nested coordinate system is a reassigning of independent variables to take advantage of geometric or symmetry properties of a particular application. Polar, cylindrical and spherical coordinate systems are primary examples of such a regrouping that have proved their importance in the separation of variables method for solving partial differential equations. Geometric algebra offers powerful complimentary algebraic tools that are unavailable in other treatments.

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0 Introduction

Geometric algebra $\mathbb{G}_3$ is the natural generalization of Gibbs Heaviside vector algebra, but unlike the latter, it can be immediately generalized to higher dimensional geometric algebras $\mathbb{G}_{p,q}$ of a quadratic form. On the other hand, Clifford analysis, the generalization of Hamilton's quaternions, is also expressed in Clifford's geometric algebras [1]. The main purpose of this article is to formulate the concept of a *nested coordinate system*, a generalization of the well-known methods of orthogonal coordinate systems to apply to any coordinate system. We restrict ourselves to the geometric algebra $\mathbb{G}_3$ because of its close relationship to the Gibbs-Heaviside vector calculus [3]. This restriction also draws attention to the clear advantages of geometric algebra over the later, because of its powerful associative algebraic structure.

The idea of a nested rectangular coordinate system arises naturally when studying properties of polar coordinates in the 2 and 3-dimensional Euclidean vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. We begin by discussing the relationship between ordinary polar coordinates and the nested rectangular coordinate system $\mathcal{N}_{1,2}$, before going on to the higher dimensional nested coordinate system $\mathcal{N}_{1,2,3}$ utilized in the reformulation of cylindrical and spherical coordinates. A detailed discussion of the geometric algebra $\mathbb{G}_3$ is not given here, but results are often expressed in the closely related well-known Gibbs-Heaviside vector analysis for the benefit of the reader.

1 Polar and nested coordinate systems

Let $\mathbb{G}_2 := \mathbb{G}(\mathbb{R}^2)$ be the geometric algebra of 2-dimensional Euclidean space $\mathbb{R}^2$. An introductory treatment of the geometric algebras $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ is given in [4, 5, 6]. Most important in studying the geometry of the Euclidean plane is the position vector

$$\mathbf{x} := \mathbf{x}[x, \hat{x}] = x\hat{x} \tag{1}$$

expressed here as a product of its *Euclidean magnitude* x and its *unit direction*, the unit vector $\hat{x}$. In terms of rectangular coordinates $(x_1, x_2) \in \mathbb{R}^2$,

$$\mathbf{x} = \mathbf{x}[x_1, x_2] = x_1 e_1 + x_2 e_2, \tag{2}$$

for the orthogonal unit vectors e_1, e_2 along the x_1 and x_2 axis, respectively. The advantage of our notation is that it immediately generalizes to 3 and higher dimensional spaces of arbitrary signature (p, q) in any of the definite geometric algebras $\mathbb{G}_{p,q} := \mathbb{G}(\mathbb{R}^{p,q})$ of a quadratic form.

The *vector derivative*, or *gradient* in the Euclidean plane is defined by

$$\nabla := e_1\partial_1 + e_2\partial_2 \tag{3}$$

where $\partial_1 := \frac{\partial}{\partial x_1}$ and $\partial_2 := \frac{\partial}{\partial x_2}$ are partial derivatives [3, p.105]. Clearly,

$$e_1 = \partial_1 \mathbf{x} = e_1 \cdot \nabla \mathbf{x}, \quad e_2 = \partial_2 \mathbf{x} = e_2 \cdot \nabla \mathbf{x}.$$

Since ∇ is the usual 2-dimensional gradient, it has the well-known properties

$$\nabla \mathbf{x} = 2, \quad \text{and} \quad \nabla x = \hat{x}.$$[1]

With the help of the *product rule* for differentiation,

$$2 = \nabla \mathbf{x} = (\nabla x)\hat{x} + x(\nabla \hat{x}) = \hat{x}^2 + x(\nabla \hat{x}). \tag{4}$$

Since in geometric algebra $\mathbf{x}^2 = x^2$, it follows that $\hat{x}^2 = 1$, so that for $\mathbf{x} \in \mathbb{R}^2$,

$$\nabla \hat{x} = \frac{1}{x} \ \text{ and } \ e_1 \cdot \nabla x = e_1 \cdot \hat{x} = \frac{x_1}{x}, \quad e_2 \cdot \nabla x = e_2 \cdot \hat{x} = \frac{x_2}{x}. \tag{5}$$

Similarly, $\nabla \hat{x} = \frac{n-1}{x}$ for $\mathbf{x} \in \mathbb{R}^n$. This is the first of many demonstrations of the power of geometric algebra over standard vector algebra.

By a *nested* rectangular coordinate system $\mathcal{N}_{1,2}(x_1, x[x_1, x_2])$, we mean

$$\mathbf{x} = x\hat{x} = \mathbf{x}[x_1, x] = \mathbf{x}\big[x_1, x[x_1, x_2]\big].$$

[1] Note in geometric algebra, unlike in standard vector analysis, we need not write $\nabla \cdot \mathbf{x} = 2$. This has many important consequences in the development of the subject.

The grouping of the variables allows us to consider x_1 and $x := \sqrt{x_1^2 + x_2^2}$ to be independent. The partial derivatives with respect to these independent variables is denoted by $\hat{\partial}_1 := \frac{\hat{\partial}}{\partial x_1}$ and $\hat{\partial}_x := \frac{\hat{\partial}}{\partial x}$, the hat on the partial derivatives indicating the new choice of independent variables.

For polar coordinates $(x, \theta) \in \mathbb{R}^2$, for $x := \sqrt{x_1^2 + x_2^2} \geq 0$, $0 \leq \theta < 2\pi$, and $\mathbf{x} := \mathbf{x}[x, \theta]$,

$$\mathbf{x} = x\hat{x}[\theta] = x(e_1 \frac{x_1}{x} + e_2 \frac{x_2}{x}) = x(e_1 \cos\theta + e_2 \sin\theta), \tag{6}$$

where $\cos\theta := \frac{x_1}{x}$ and $\sin\theta := \frac{x_2}{x}$. Using (5),

$$\nabla\hat{x} = \nabla\hat{x}[\theta] = (\nabla\theta)\frac{\partial\hat{x}}{\partial\theta} = \frac{1}{x} \quad \iff \quad \nabla\theta = \frac{1}{x}\frac{\partial\hat{x}}{\partial\theta}, \ \nabla^2\theta = 0, \tag{7}$$

since

$$\nabla\hat{x} = (\nabla\theta)\partial_\theta(e_1 \cos\theta + e_2 \sin\theta) = (\nabla\theta)(-e_1 \sin\theta + e_2 \cos\theta),$$

and

$$\nabla^2\theta = -\frac{\hat{x}}{x^2}\partial_\theta\hat{x} + \frac{1}{x}(\nabla\theta)\partial_\theta^2\hat{x} = -2\Big(\frac{\hat{x}}{x^2} \cdot (\partial_\theta\hat{x})\Big) = 0.$$

The $\iff$ follows by multipling both sides of the first equation by the unit vector $\partial_\theta\hat{x}$, which is allowable in geometric algebra. Note also the use of the famous geometric algebra identity $2\mathbf{a} \cdot \mathbf{b} = (\mathbf{ab} + \mathbf{ba})$ for vectors $\mathbf{a}$ and $\mathbf{b}$, [4, p.26].

The 2-dimensional gradient ∇,

$$\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} = e_1\partial_1 + e_2\partial_2 \tag{8}$$

already defined in (3), and the Laplacian ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \partial_1^2 + \partial_2^2. \tag{9}$$

In polar coordinates,

$$\hat{\nabla} = (\nabla x)\hat{\partial}_x + (\nabla\theta)\hat{\partial}_\theta = \hat{x}\,\hat{\partial}_x + \frac{1}{x}(\hat{\partial}_\theta x)\hat{\partial}_\theta \tag{10}$$

for the gradient where $\hat{\partial}_\theta := \frac{\hat{\partial}}{\partial\theta}$, and since $\hat{\nabla}^2\theta = 0$,

$$\begin{aligned}\hat{\nabla}^2 = \hat{\nabla}\big(\hat{x}\,\hat{\partial}_x + (\hat{\nabla}\theta)\,\hat{\partial}_\theta\big) &= \Big(\hat{\nabla}\hat{x} + \hat{x} \cdot \hat{\nabla}\Big)\hat{\partial}_x + \Big(\hat{\nabla}^2\theta + (\hat{\nabla}\theta) \cdot \hat{\nabla}\Big)\hat{\partial}_\theta \\ &= \hat{\partial}_x^2 + \frac{1}{x}\hat{\partial}_x + \frac{1}{x^2}\hat{\partial}_\theta^2. \end{aligned} \tag{11}$$

for the Laplacian. The decomposition of the Laplacian (11), directly implies that Laplace's differential equation is separable in polar coordinates.

When expressed in nested rectangular coordinates $\mathcal{N}_{1,2}(x_1, x)$, the gradient $\nabla \equiv \hat{\nabla}$ takes the form

$$\hat{\nabla} := (\nabla x_1)\frac{\hat{\partial}}{\partial x_1} + (\nabla x)\frac{\hat{\partial}}{\partial x} = e_1\hat{\partial}_1 + \hat{x}\hat{\partial}_x. \tag{12}$$

Dotting equations (8) and (12) on the left by e_1 and $\hat{x}$ gives the transformation rules

$$\partial_1 = \hat{\partial}_1 + \frac{x_1}{x}\hat{\partial}_x, \quad \hat{x}\cdot\hat{\nabla} = \frac{x_1}{x}\hat{\partial}_1 + \hat{\partial}_x = \cos\theta\,\partial_1 + \sin\theta\,\partial_2.$$

Using these formulas the nested Laplacian takes the form

$$\hat{\nabla}^2 = \hat{\partial}_1^2 + 2\frac{x_1}{x}\hat{\partial}_x\hat{\partial}_1 + \frac{1}{x}\hat{\partial}_x + \hat{\partial}_x^2 = -\hat{\partial}_1^2 + 2\partial_1\hat{\partial}_1 + \frac{1}{x}\hat{\partial}_x + \hat{\partial}_x^2. \tag{13}$$

The unusual feature of the nested Laplacian is that it is defined in terms of both the ordinary partial derivative ∂_1 and the nested partial derivative $\hat{\partial}_1$. Whereas partial derivatives generally commute, partial derivatives of different types do not. For example, it is easily verified that

$$\partial_1\hat{\partial}_1 x_1 x^2 = 2x_1, \quad \text{whereas} \quad \hat{\partial}_1\partial_1 x_1 x^2 = 4x_1.$$

Because the mixed partial derivatives $\hat{\partial}_x\hat{\partial}_1$ occurs in (13), Laplace's differential equation in the real rectangular coordinate system $\mathcal{N}_{1,2}(x_1, x)$ is not, in general, separable. Indeed, suppose that a harmonic function F is separable, so that $F = X_1X$ for $X_1 = X_1[x_1], X = X[x]$. Using (13),

$$\frac{\hat{\nabla}^2 F}{X_1 X} = \frac{\partial_1^2 X_1}{X_1} + \frac{\left(\partial_x^2 X + \frac{1}{x}\partial_x\right)X}{X} + 2\Big(\frac{x_1\partial_1 X_1}{X_1}\Big)\Big(\frac{\partial_x X}{xX}\Big) = 0. \tag{14}$$

The last term on the prevents F in general from being separable. However, it is easily checked that $F = k\frac{x_1}{x^2}$ is harmonic and a solution of (13). When $X_1[x_1] = kx_1$, it is easily checked that $\frac{x_1\partial_1 X_1}{X_1} = 1$. Letting $F = kx_1X[x]$, and requiring $\hat{\nabla}^2 F = 0$, leads to the differential equation for $X[x]$,

$$3\partial_x X + x\partial_x^2 X = 0,$$

with the solution $X[x] = c_1\frac{1}{x^2} + c_2$. The simplest example of a harmonic function $F = X_1X$ is when $X_1 = x_1$ and $X = \frac{1}{x^2}$. A graph of this function is shown in Figure 1.

2 Special harmonic functions in nested coordinates

Consider the real nested rectangular coordinate system (x_1, x_p, x), defined by

$$\mathcal{N}_{1,2,3} := \{(x_1, x_p, x)|\ \mathbf{x} = x\hat{x} = x_1e_1 + x_p\hat{x}_p + x\hat{x}\},$$

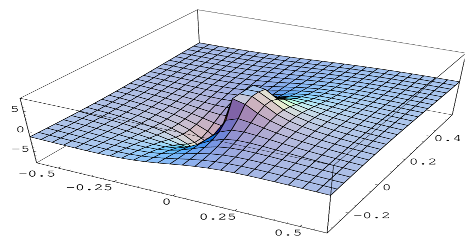

Figure 1: The harmonic 2-dimensional function $F = \frac{x_1}{x_1^2+x_2^2}$ is shown.

where $x_p = \sqrt{x_1^2+x_2^2} \geq 0,\ x = \sqrt{x_1^2+x_2^2+x_3^2} \geq 0$. In nested coordinates, the gradient $\nabla = e_1\partial_1 + e_2\partial_2 + e_3\partial_3$ takes the form

$$\hat\nabla = (\hat\nabla x_1)\hat\partial_1 + (\hat\nabla x_p)\hat\partial_p + (\hat\nabla x)\hat\partial_x = e_1\hat\partial_1 + \hat x_p\hat\partial_p + \hat x\hat\partial_x, \tag{15}$$

where $\hat\partial_p := \frac{\hat\partial}{\partial x_p}$. Formulas relating the gradients ∇ and $\hat\nabla$ easily follow:

$$\partial_1 = \hat\partial_1 + e_1\cdot\hat x_p\,\hat\partial_p + e_1\cdot\,\hat x\,\hat\partial_x = \hat\partial_1 + \frac{x_1}{x_p}\,\hat\partial_p + \frac{x_1}{x}\,\hat\partial_x \tag{16}$$

$$\partial_2 = e_2\cdot\hat x_p\,\hat\partial_p + e_2\cdot\,\hat x\,\hat\partial_x = \frac{x_2}{x_p}\,\hat\partial_p + \frac{x_2}{x}\,\hat\partial_x \tag{17}$$

and

$$\partial_3 = e_3\cdot\,\hat x\,\hat\partial_x = \frac{x_3}{x}\,\hat\partial_x. \tag{18}$$

For the Laplacian ∇^2 in nested coordinates, with the help of (15),

$$\hat\nabla^2 = \hat\nabla(e_1\hat\partial_1 + \hat x_p\partial_{x_p} + \hat x\hat\partial_x) = e_1\cdot\hat\nabla\,\hat\partial_1 + \hat\nabla\cdot\hat x_p\,\hat\partial_p + \hat\nabla\cdot\hat x\,\hat\partial_x$$

$$= e_1 \cdot \hat{\nabla}\,\hat{\partial}_1 + \hat{\nabla} \cdot \hat{x}_p\,\hat{\partial}_p + \hat{\nabla} \cdot \hat{x}\,\hat{\partial}_x$$

$$= \Big(\hat{\partial}_1 + \frac{x_1}{x_p}\,\hat{\partial}_p + \frac{x_1}{x}\,\hat{\partial}_x\Big)\hat{\partial}_1 + \Big(\frac{1}{x_p} + \frac{x_1}{x_p}\hat{\partial}_1 + \hat{\partial}_p + \frac{x_p}{x}\hat{\partial}_p\Big)\hat{\partial}_p$$

$$+\Big(\frac{2}{x} + \frac{x_1}{x}\hat{\partial}_1 + \frac{x_p}{x}\hat{\partial}_p + \hat{\partial}_x\Big)\hat{\partial}_x$$

$$= \hat{\partial}_1^2 + \hat{\partial}_p^2 + \hat{\partial}_x^2 + 2\left(\frac{x_1}{x_p}\hat{\partial}_1\hat{\partial}_p + \frac{x_1}{x}\hat{\partial}_1\hat{\partial}_x + \frac{x_p}{x}\hat{\partial}_p\hat{\partial}_x\right) + \frac{1}{x_p}\hat{\partial}_p + \frac{2}{x}\hat{\partial}_x. \tag{19}$$

Another expression for the Laplacian in mixed coordinates is obtained with the help of (16),

$$\hat{\nabla}^2 = -\hat{\partial}_1^2 + \hat{\partial}_p^2 + \hat{\partial}_x^2 + 2\left(\partial_1\hat{\partial}_1 + \frac{x_p}{x}\hat{\partial}_p\hat{\partial}_x\right) + \frac{1}{x_p}\hat{\partial}_p + \frac{2}{x}\hat{\partial}_x. \tag{20}$$

Suppose $F = F[x_1, x_p, x]$. In order for F to be harmonic, $\hat{\nabla}^2 F = 0$. Assuming that F is separable, $F = X_1[x_1]X_p[x_p]X_x[x]$, and applying the Laplacian (20) to F gives

$$\hat{\nabla}^2 F = (\hat{\partial}_1^2 X_1)X_p X_x + X_1\Big((\hat{\partial}_p^2 + \frac{1}{x_p}\hat{\partial}_p)X_p\Big)X_x + X_1 X_p\Big(\frac{2}{x}\hat{\partial}_x X_x\Big)$$

$$+2\bigg(\big(x_p\partial_p X_p\big)\big(\frac{1}{x}\hat{\partial}_x X_x\big)X_1 + \big(\partial_1 X_p\big)X_x + X_p\big(\partial_1 X_x\big)\bigg). \tag{21}$$

We now calculate the interesting expression

$$\frac{\big(x_p\partial_p X_p\big)\big(\frac{1}{x}\hat{\partial}_x X_x\big)X_1 + \big(\partial_1 X_p\big)X_x + X_p\big(\partial_1 X_x\big)}{X_1 X_p X_x}$$

$$= \Big(x_p\big(\partial_p \log X_p\big)\Big)\Big(\big(\frac{1}{x}\partial_x \log X_x\big)\Big) + \frac{\partial_1 \log(X_p X_x)}{X_1}.$$

In general, because of the last term in (21), a function $F = X_1 X_p X_x$ will not be separable. However, just as in the two dimensional case, there are 3-dimensional harmonic solutions of the form $F = x_1^k x_p^m x^n$. Taking the Laplacian (19) of F, with the help of [7], gives

$$\hat{\nabla}^2 F = (2km + m^2)x^n x_1^k x_p^{m-2} + (-k + k^2)x^n x_1^{k-2} x_p^m$$

$$+(2kn + 2mn + n(1+n))x^{n-2} x_1^k x_p^m = 0.$$

This last expression vanishes when the system of three equations,

$$\{2km + m^2 = 0, \quad -k + k^2 = 0, \quad \text{and} \quad 2kn + 2mn + n(1+n) = 0\}.$$

All of the distinct non-trivial harmonic solutions $F = x_1^k x_p^m x^n$ are listed in the following Table

k	m	n
1	0	0
0	0	-1
1	-2	0
1	0	-3
1	-2	1

(22)

3 Cylindrical and spherical coordinates

Cylindrical and spherical coordinates are examples of nested coordinates $\mathcal{N}_{1,2}(\mathbb{R})$, and $\mathcal{N}_{2,3}(\mathbb{R})$, respectively. For the first,

$$\mathbf{x} = \mathbf{x}[x_p, \theta, x_3] = \mathbf{x}_p[x_p, \theta] + \mathbf{x}_3[x_3], \tag{23}$$

where $\mathbf{x}_p = x_p \hat{x}_p[\theta]$, $x_p = \sqrt{x_1^2 + x_2^2}$, and $\mathbf{x}_3 = x_3 e_3$. Cylindrical coordinates $(x_p, \theta, x_3) \in \mathbb{R}^3 = \mathbb{R}^2 \cup \mathbb{R}^1$ are a decomposition of $\mathbb{R}^3$ into the polar coordinates $(x_p, \theta) \in \mathbb{R}^2$, already studied in Section 1, and $x_3 \in \mathbb{R}^1$. For spherical coordinates, $\mathbf{x}_p = x_p \hat{x}_p[\theta]$ the same as in cylindrical and polar coordinates, and

$$\mathbf{x} = \mathbf{x}[x, \theta, \varphi] = x\hat{x}[\theta, \varphi]) = x\Big(e_3 \cos\varphi + \hat{x}_p[\theta] \sin\varphi\Big), \tag{24}$$

where

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2},\ \ \hat{x}[\theta, \varphi] = e_3 \cos\varphi + \hat{x}_p[\theta] \sin\varphi,\ \ \hat{x}_p[\theta] = e_1 \cos\theta + e_2 \sin\theta.$$

The basic quantities that define both cylindrical and spherical coordinates are shown in Figure 2.

The gradient $\hat{\nabla}$ and Laplacian $\hat{\nabla}^2$ for cylindrical coordinates are easily calculated. With the help of (7), (10), and (11),

$$\hat{\nabla} = (\hat{\nabla} x_p)\hat{\partial}_p + (\hat{\nabla}\theta)\hat{\partial}_\theta + (\hat{\nabla} x_3)\hat{\partial}_3$$

for the cylindrical gradient, and

$$\hat{\nabla}^2 = \hat{\nabla}\Big(\hat{x}_p\, \hat{\partial}_p + (\hat{\nabla}\theta)\, \hat{\partial}_\theta + e_3 \hat{\partial}_3\Big) = \hat{\partial}_p^2 + \frac{1}{x_p}\hat{\partial}_p + \frac{1}{x_p^2}\hat{\partial}_\theta^2 + \hat{\partial}_3^2 \tag{25}$$

for the cylindrical Laplacian. Letting $F[\mathbf{x}] = X_p[x] X_\theta[\theta] X_3[x_3]$, the resulting equation is easily separated and solved by standard methods, resulting in three second order differential equations with solutions,

$$X_p[x_p] = k_1 J_n[\beta x_p] + k_2 Y_n[\beta x_p,]$$

$$X_\theta[\theta] = k_3 \cos n\theta + k_4 \sin n\theta$$

$$X_3[x_3] = k_5 \cosh(\alpha(m - x_3)) + k_6 \sinh(\alpha(m - x_3)),$$

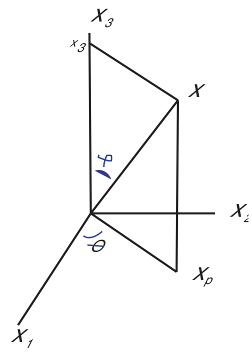

Figure 2: For cylindrical coordinates, $\mathbf{x} = x_p \hat{x}_p[\theta] + x_3 e_3$. For spherical coordinates, $\mathbf{x} = x(e_3 \cos\varphi + \hat{x}_p[\theta]\sin\varphi)$.

where J_n and Y_n are Bessel functions of the first and second kind . The constants are determined by the various boundary conditions that must be satisfied in different applications [8, p.254].

Turning to spherical coordinates $(x, \theta, \varphi) \in \mathbb{R}^3$, the spherical gradient

$$\hat{\nabla} = (\hat{\nabla} x)\hat{\partial}_x + (\hat{\nabla}\theta)\hat{\partial}_\theta + (\hat{\nabla}\varphi)\hat{\partial}_\varphi = \hat{x}\hat{\partial}_x + \frac{1}{x_p}(\hat{\partial}_\theta \hat{x}_p)\hat{\partial}_\theta + \frac{1}{x}(\hat{\partial}_\varphi \hat{x})\hat{\partial}_\varphi, \tag{26}$$

where from previous calculations for polar and cylindrical coordinates,

$$\hat{\nabla}\theta = \frac{1}{x_p}(\hat{\partial}_p \hat{x}_p), \;\; (\hat{\nabla}\theta)^2 = \frac{1}{x_p^2}, \quad \hat{\nabla}^2\theta = 0, \quad \hat{\nabla}\varphi = \frac{1}{x}\hat{\partial}_\varphi \hat{x}, \quad (\hat{\nabla}\varphi)^2 = \frac{1}{x^2}. \tag{27}$$

Furthermore, since $\hat{x} = \hat{x}[\theta, \varphi] = e_3 \cos\varphi + \hat{x}_p[\theta]\sin\varphi$

$$\frac{2}{x} = \hat{\nabla}\hat{x} = (\hat{\nabla}\theta)(\hat{\partial}_\theta \hat{x}) + (\hat{\nabla}\varphi)(\hat{\partial}_\varphi \hat{x}) = \frac{1}{x_p}(\hat{\partial}_\theta \hat{x})(\hat{\partial}_\varphi \hat{x}) + \frac{1}{x},$$

it follows that

$$(\hat{\partial}_\theta \hat{x})(\hat{\partial}_\varphi \hat{x}) = \frac{x_p}{x} = \sin\varphi, \quad \text{and} \quad \hat{\nabla}^2\varphi = \frac{x_3}{x^2 x_p}.$$

That $\hat{\nabla}^2\varphi = \frac{x_3}{x^2 x_p}$ follows using (26) and (27),

$$\hat{\nabla}^2\varphi = \hat{\nabla}\big(\frac{1}{x}\hat{\partial}_\varphi \hat{x}\big) = \Big(-\frac{\hat{x}}{x^2} + \frac{1}{x}\hat{\nabla}\Big)\hat{\partial}_\varphi \hat{x}$$

$$= -\frac{\hat{x}}{x^2}\hat{\partial}_\varphi \hat{x} - \frac{1}{x}\Big((\hat{\nabla}x)\hat{\partial}_x\hat{\partial}_\varphi\hat{x} + (\hat{\nabla}\theta)\hat{\partial}_\theta\hat{\partial}_\varphi\hat{x} + (\hat{\nabla}\varphi)\hat{\partial}_\varphi^2\hat{x}\Big)$$

$$= -\Big(\frac{\hat{x}}{x^2}\hat{\partial}_\varphi\hat{x} + \frac{1}{x^2}(\hat{\partial}_\varphi\hat{x})\hat{x}\Big) + \frac{1}{xx_p}(\hat{\partial}_\theta\hat{x}_p)(\hat{\partial}_\varphi\hat{\partial}_\theta x) = \frac{x_3}{x^2x_p},$$

since partial derivatives commute, $\hat{\partial}_x\hat{x} = 0$, and $\hat{\partial}_\varphi^2\hat{x} = -\hat{x}$.

For the spherical Laplacian, using (26) and (27),

$$\hat{\nabla}^2 = \hat{\nabla}\Big(\hat{x}\hat{\partial}_x + (\hat{\nabla}\theta)\hat{\partial}_\theta + (\hat{\nabla}\varphi)\hat{\partial}_\varphi\Big)$$

$$= \Big(\frac{2}{x} + \hat{\partial}_x\Big)\hat{\partial}_x + (\hat{\nabla}\theta)\cdot\hat{\nabla}\hat{\partial}_\theta + \Big(\hat{\nabla}^2\varphi + (\hat{\nabla}\varphi)\cdot\hat{\nabla}\Big)\hat{\partial}_\varphi$$

$$= \Big(\hat{\partial}_x + \frac{2}{x}\Big)\hat{\partial}_x + \frac{1}{x_p^2}\hat{\partial}_\theta^2 + \Big(\frac{x_3}{x^2x_p} + \frac{1}{x^2}\hat{\partial}_\varphi\Big)\hat{\partial}_\varphi,$$

equivalent to the usual expression for the Laplacian in spherical coordinates [8, p.256].

Just as in cylindrical coordinates, the solution of Laplace's equation in spherical coordinates is separable, $F = X_x[x]X_\theta[\theta]X_\varphi[\varphi]$, resulting in three second order differential equations with solutions

$$X_x[x] = k_1x^\beta + k_2x^{-(\beta+1)},$$

$$X_\theta[\theta] = k_3\cos n\theta + k_4\sin n\theta,$$

$$X_\varphi[\varphi] = k_5P_n^m(\cos\varphi) + k_6Q_n^m(\cos\varphi),$$

where P_n^m and Q_n^m are the Legendre functions of the first and second kind, respectively [8, p.258].

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